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Article

Global existence, uniqueness, and asymptotic behavior of solution for the Euler-Bernoulli viscoelastic equation

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Abstract: We study the global existence and uniqueness of a solution to an initial boundary value problem for the Euler-Bernoulli viscoelastic equation $u_{tt} + \Delta^2 u - g_1 * \Delta^2 u + g_2 * \Delta u + u_t = 0$. Further, the asymptotic behavior of solution is established.

Keywords: Euler-Bernoulli viscoelastic equation, global existence, asymptotic behavior, memory.

MSC: 35G16, 74Dxx, 35B40.

1. Introduction

This work is concerned with the global existence, uniqueness, and asymptotic behavior of solution for the Euler-Bernoulli viscoelastic equation

$$\begin{cases} u_{tt} + \Delta^2 u - g_1 * \Delta^2 u + g_2 * \Delta u + u_t = 0, \quad x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u = 0, \quad \frac{\partial u}{\partial \nu} = 0, x \in \partial\Omega,\ t > 0, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and ν is the unit outer normal on $\partial\Omega$. Here g_1 and g_2 are positive functions satisfying some conditions to be specified later, and

$$g_i * \chi(t) = \int_0^t g_i(t-\tau)\chi(\tau)d\tau, \quad i = 1, 2.$$

The Euler-Bernoulli equation

$$u_{tt}(x,t) + \Delta^2 u(x,t) + h(u_t) = f(u), \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \tag{2}$$

describes the deflection $u(x,t)$ of a beam (when $n = 1$) or a plate (when $n = 2$), where $\Delta^2 u := \Delta(\Delta u) = \sum_{j=1}^n \left(\sum_{i=1}^n u_{x_i} u_{x_i}\right)_{x_j x_i}$, and h and f represent the friction damping and the source respectively.

Lange and Menzala [1] considered

$$u_{tt}(x,t) + \Delta^2 u(x,t) + a(t) u_t(x,t) = 0 \tag{3}$$

where $x \in \mathbb{R}^n$, $t \geq 0$, $a(t) = m(\|\nabla v(\cdot,t)\|^2_{L^2(\mathbb{R}^n)})$ and the real-valued function $m : [0,+\infty) \to [1,+\infty)$ will be assumed to be of class C^1 satisfying the condition $m(s) \geq 1 + s$ for all $s \geq 0$. They remarked that the imaginary part of the solutions of Schrödinger's equation

$$iw_t = \Delta w + im\left(\|\nabla(Imw)\|^2_{L^2(\mathbb{R}^n)}\right) Rew = 0,$$

are precisely the solutions for (3). Then, using Fourier transform, the existence of global classical solutions and algebraic decay rate were proved for initial data whose regularity depends on the spacial dimension n. Messaoudi [2] studied the equation

$$u_{tt}(x,t) + \Delta^2 u(x,t) + a|u_t|^{m-2}u_t = b|u|^{p-2}u, \tag{4}$$

where $a, b > 0$, $p, m > 2$. He established an existence result for (4) and showed that the solution continued to exist globally if $m \geq p$. If we take the viscoelastic materials into consideration, the model (2) becomes

$$u_{tt}(x,t) + \Delta^2 u(x,t) - \int_0^t g(t-s)\Delta^2 u(x,s)ds + h(u_t) = f(u), \tag{5}$$

where g is so-called viscoelastic kernel. The term $\int_0^t g(t-s)\Delta^2 u(x,s)ds$ describes the hereditary properties of the viscoelastic materials [3]. It expresses the fact that the stress at any instant t depends on the past history of strains which the material has undergone from time 0 up to t. Tatar [4] obtained the property of the energy decay of the model (5) for $h = f = 0$ and from this, we know that the term $\int_0^t g(t-s)\Delta^2 u(x,s)ds$, similar to the friction damping, can cause the inhibition of the energy.

Messaoudi and Mukiawa [5] studied the fourth-order viscoelastic plate equation

$$u_{tt}(x,t) + \Delta^2 u(x,t) - \int_0^t g(t-s)\Delta^2 u(x,s)ds = 0,$$

in the bounded domain $\Omega = (0,\pi) \times (-l,l) \subset \mathbb{R}^2$ with nontraditional boundary conditions. The authors established the well-posedness of the solution and a decay result.
Rivera *et al.* [6] investigated the plate model:

$$u_{tt} + \Delta^2 u - \sigma\Delta u_{tt} + \int_0^t g(t-s)\Delta^2 u(s)ds = 0,$$

in the bounded domain $\Omega \subset \mathbb{R}^2$ with mixed boundary condition and suitable geometrical hypotheses on $\partial\Omega$. They established that the energy decays to zero with the same rate of the kernel g such as exponential and polynomial decay. To do so in the second case they made assumptions on g, g' and g'' which means that $g \simeq (1+t)^{-p}$ for $p > 2$. Then they obtained the same decay rate for the energy. However, their approach can not be applied to prove similar results for $1 < p \leq 2$.
Cavalcanti et al. [7] investigated the global existence, uniqueness and stabilization of energy of

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)ds + a(t)u_t = 0$$

where

$$a(t) = M\left(\int_\Omega |\nabla u(x,t)|^2 dx\right) \quad \text{with} \quad M \in C^1([0,+\infty)).$$

By taking a bounded or unbounded open set Ω where $M(s) > m_0 > 0$ for all $s \geq 0$, the authors showed in [7] that the energy goes to zero exponentially, provided that g goes to zero at the same form.

The aim of this work is to study the global existence of regular and weak solutions of problem (1) for the bounded domain, then for $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ a increasing C^2 function such that

$$\xi(0) = 0, \quad \xi'(0) > 0, \quad \lim_{t\to+\infty} \xi(t) = +\infty, \quad \xi''(t) < 0 \quad \forall t \geq 0. \tag{6}$$

the solution features the asymptotic behavior

$$E(t) \leq E(0)e^{-\kappa\xi(t)}, \quad \forall t \geq 0,$$

where $E(t)$ is defined in (38) and κ is a positive constant independent of the initial energy $E(0)$.

2. Preliminaries and main results

We begin by introducing some notation that will be used throughout this work. For functions $u(x,t)$, $v(x,t)$ defined on Ω, we introduce

$$(u,v) = \int_{\Omega} u(x)v(x)dx \text{ and } \|u\|_2 = \left(\int_{\Omega} |u(x)|^2 dx\right)^{\frac{1}{2}}.$$

Define

$$X = \left\{ u \in H_0^2(\Omega); \Delta^2 u \in L^2(\Omega) \right\}$$

Then, X is a Hilbert space endowed with the natural inner product

$$(u,v)_X = (u,v)_{H_0^2} + (\Delta^2 u, \Delta^2 v).$$

Now let us precise the hypotheses on g_1 and g_2.

(H1) $g_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded function satisfying

$$g_1(t) \in C^2(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad g_1(0) > 0.$$

(H2) There exist positive constants α_1, α_2 and α_3 such that

$$-\alpha_1 g_1(t) \le g_1'(t) \le -\alpha_2 g_1(t), \quad \forall t \ge 0,$$

(H3)

$$0 \le g_1''(t) \le \alpha_3 g_1(t), \quad \forall t \ge 0,$$

(H4) $g_2 : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded function satisfying

$$g_2(t) \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad g_2(0) > 0.$$

(H5) There exist positive constants η_1 and η_2 such that

$$-\eta_1 g_2(t) \le g_2'(t) \le -\eta_2 g_2(t), \quad \forall t \ge 0,$$

(H6)

$$1 - \int_0^t \left(g_1(s) + \lambda_1^{-1} g_2(s) \right) ds = l > 0,$$

where $\lambda_1 > 0$ is the first eigenvalue of the spectral Dirichlet problem

$$\Delta^2 u = \lambda_1 u \quad \text{in } \Omega, \quad u = \frac{\partial u}{\partial \nu} = 0 \text{ in } \partial\Omega,$$

$$\|\nabla u\|_2 \le \frac{1}{\sqrt{\lambda_1}} \|\Delta u\|_2.$$

Lemma 1. *For $\phi, \psi \in C^1([0,+\infty[, \mathbb{R})$ we have*

$$2\int_0^t \int_{\Omega} \phi(t-s)\psi\psi' dxds = -\frac{d}{dt}\left((\phi \Box \psi)(t) - \int_0^t \phi(s)ds \|\psi\|_2^2 \right) + (\phi' \Box \psi)(t) - \phi(t)\|\phi\|_2^2,$$

where

$$(\phi \Box \psi)(t) = \int_0^t \phi(t-s)\|\psi(t) - \psi(s)\|_2^2 ds.$$

Theorem 2. *Assume that $(H1)-(H6)$ hold, and that $\{u_0, u_1\}$ belong to $H_0^2(\Omega) \times L^2(\Omega)$. Then, Problem (1) admits a unique weak solution u in the class*

$$u \in C^0([0,\infty); H_0^2(\Omega)) \cap C^1([0,\infty); L^2(\Omega)).$$

Moreover, for $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ a increasing C^2 function satisfying (6) and, if $\|g_1\|_{L^1(0,\infty)}$ is sufficiently small, we have for $\kappa > 0$

$$E(t) \le E(0)e^{-\kappa\xi(t)}, \quad \forall t \ge 0.$$

3. Existence of Solutions

In this section we first prove the existence and uniqueness of regular solutions to Problem (1). Then, we extend the same result to weak solutions using density arguments.

3.1. Regular solutions

Let (w_j) be a Galerkin basis in X, and let V_m be the subspace generated by the first m vectors $w_1, ..., w_m$. We search for a function

$$u_m(t) = \sum_{i=1}^{m} k_{im}(t)w_i(x), \ m = 1, 2, ...$$

satisfying the approximate Cauchy problem

$$\begin{aligned} &(u_m''(t), v) + (\Delta u_m(t), \Delta v) - \int_0^t g_1(t-s)(\Delta u_m(s), \Delta v)ds \\ &- \int_0^t g_2(t-s)(\nabla u_m(s), \nabla v)ds + (u_m'(t), v) = 0, \quad \forall v \in V_m, \end{aligned} \tag{7}$$

$$u_m(0) = u_{0m} \longrightarrow u_0 \ \text{ in } \ X \ \text{ and } \ u_m'(0) = u_{1m} \longrightarrow u_1 \ \text{ in } \ H_0^2(\Omega). \tag{8}$$

By standard methods in differential equations, we can prove the existence of solutions to the problem (5) $-$ (6) on $[0, t_m)$ with $0 < t_m < T$. In order to extend the solution of (7) $-$ (8) to the whole $[0, T]$, we need the following priori estimate.
Estimate 1. Taking $v = 2u_m'(t)$ in (7), we have

$$\begin{aligned} &\frac{d}{dt}\left[\|u_m'(t)\|_2^2 + \|\Delta u_m(t)\|_2^2\right] + 2\|u_m'(t)\|_2^2 - 2\int_0^t g_1(t-s)\Delta u_m(s) \cdot \Delta u_m'(t)dxds \\ &-2\int_0^t g_2(t-s)\nabla u_m(s) \cdot \nabla u_m'(t)dxds = 0. \end{aligned} \tag{9}$$

Using Lemma 1, we obtain

$$\begin{aligned} &-2\int_0^t g_1(t-s)\int_\Omega \Delta u_m(s) \cdot \Delta u_m'(t)dxds \\ &= \frac{d}{dt}\left\{(g_1\Box\Delta u_m)(t) - \left(\int_0^t g_1(s)ds\right)\|\Delta u_m(t)\|_2^2\right\} - (g_1'\Box\Delta u_m)(t) + g_1(t)\|\Delta u_m(t)\|_2^2, \end{aligned} \tag{10}$$

and

$$\begin{aligned} &-2\int_0^t g_2(t-s)\int_\Omega \nabla u_m(s) \cdot \nabla u_m'(t)dxds \\ &= \frac{d}{dt}\left\{(g_2\Box\nabla u_m)(t) - \left(\int_0^t g_2(s)ds\right)\|\nabla u_m(t)\|_2^2\right\} - (g_2'\Box\nabla u_m)(t) + g_2(t)\|\nabla u_m(t)\|_2^2, \end{aligned} \tag{11}$$

Inserting Equations (10) and (11) into Equation (9) and integrating over $[0, t] \subset [0, T]$, we obtain

$$\begin{aligned} &\|u_m'(t)\|_2^2 + \left(1 - \int_0^t g_1(s)ds\right)\|\Delta u_m(t)\|_2^2 + (g_1\Box\Delta u_m)(t) - \left(\int_0^t g_2(s)ds\right)\|\nabla u_m(t)\|_2^2 + (g_2\Box\nabla u_m)(t) \\ &+2\int_0^t \|u_m'(s)\|_2^2 ds - \int_0^t (g_1'\Box\Delta u_m)(s)ds + \int_0^t\int_\Omega g_1(s)|\Delta u_m(s)|^2 dxds - \int_0^t (g_2'\Box\nabla u_m)(s)ds \\ &+\int_0^t\int_\Omega g_2(s)|\nabla u_m(s)|^2 dxds = \|u_{1m}(t)\|_2^2 + \|\Delta u_{0m}\|_2^2. \end{aligned} \tag{12}$$

By using the fact that

$$(g_1\Box\Delta u_m)(t)+(g_2\Box\nabla u_m)(t)-\int_0^t(g_1'\Box\Delta u_m)(s)ds-\int_0^t(g_2'\Box\nabla u_m)(s)ds$$
$$+\int_0^t\int_\Omega g_1(s)|\Delta u_m(s)|^2dxds+\int_0^t\int_\Omega g_2(s)|\nabla u_m(s)|^2dxds\geq 0,$$

and

$$\begin{aligned}\left(1-\int_0^t g_1(s)ds\right)\|\Delta u_m(t)\|_2^2-\left(\int_0^t g_2(s)ds\right)\|\nabla u_m(t)\|_2^2 &\geq \left(1-\int_0^t\left[g_1(s)+\lambda_1^{-1}g_2(s)\right]ds\right)\|\Delta u_m(t)\|_2^2\\ &\geq l\|\Delta u_m(t)\|_2^2,\end{aligned}$$

Equation (12) yields

$$\|u_m'(t)\|_2^2+l\|\Delta u_m(t)\|_2^2+2\int_0^t\|u_m'(s)\|_2^2ds\leq\|u_{1m}(t)\|_2^2+\|\Delta u_{0m}\|_2^2. \tag{13}$$

Taking the convergence of Equation (8) into consideration, we arrive at

$$\|u_m'(t)\|_2^2+l\|\Delta u_m(t)\|_2^2+2\int_0^t\|u_m'(s)\|_2^2ds\leq L_1. \tag{14}$$

where $L_1=\|u_1\|_2^2+\|\Delta u_0\|_2^2$.

Estimate 2. Firstly, we obtain an estimate for $u_m''(0)$ in the L^2 norm. indeed, setting $v=u_m''(0)$ and $t=0$ in Equation (7), we obtain

$$\|u_m''(0)\|_2^2\leq\left[\|\Delta^2u_{0m}\|_2+\|u_{1m}\|_2\right]\|u_m''(0)\|_2. \tag{15}$$

From Equations (8), (14) and (15), it follows that

$$\|u_m''(0)\|_2\leq L_2,\quad\forall m\in\mathbb{N}, \tag{16}$$

where L_2 is a positive constant independent of $m\in\mathbb{N}$. Differentiating Equation (7) with respect to t, and setting $v=u_m''(t)$, we obtain

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\|u_m''(t)\|_2^2+\frac{1}{2}\|\Delta u_m'(t)\|_2^2\right]+\|u_m''(t)\|_2^2\\
=\;&-g_1(0)\int_\Omega\Delta^2u_m(t)u_m''(t)dx-\int_\Omega\int_0^t g_1'(t-s)\Delta^2u_m(s)u_m''(t)dsdx\\
&-g_2(0)\int_\Omega\Delta u_m(t)u_m''(t)dx-\int_\Omega\int_0^t g_2'(t-s)\Delta u_m(s)u_m''(t)dsdx\\
&=-g_2(0)\int_\Omega\Delta u_m(t)u_m''(t)dx-\int_\Omega\int_0^t g_2'(t-s)\Delta u_m(s)u_m''(t)dsdx\\
&-g_1(0)\|\Delta u_m'(t)\|_2^2+g_1(0)\frac{d}{dt}\int_\Omega\Delta u_m(t)\cdot\Delta u_m'(t)dx\\
&+\frac{d}{dt}\left\{\int_0^t g_1'(t-s)\int_\Omega\Delta u_m(t)\cdot\Delta u_m'(t)dxds\right\}-g_1'(0)\int_\Omega\Delta u_m(t)\cdot\Delta u_m'(t)dx\\
&-\int_0^t g_1''(t-s)\int_\Omega\Delta u_m(s)\cdot\Delta u_m'(t)dxds.\end{aligned} \tag{17}$$

By (H5), Hölder's inequality and Young's inequality give

$$-\int_\Omega\int_0^t g_2'(t-s)\Delta u_m(s)u_m''(t)dsdx\leq\frac{1}{2}\|u_m''(t)\|_2^2+\frac{\eta_1^2\|g_2\|_{L^1}}{2}\int_0^t g_2(t-s)\|\Delta u_m(s)\|_2^2ds. \tag{18}$$

From Equation (14) we obtain

$$-g_2(0)\int_\Omega \Delta u_m(t)u_m''(t)dx \le \frac{1}{2}\|u_m''(t)\|_2^2 + \frac{[g_2(0)]^2L_1}{2l} \tag{19}$$

and

$$\begin{aligned} -g_1'(0)\int_\Omega \Delta u_m(t)\cdot\Delta u_m'(t)dx &\le \frac{|g_1'(0)|}{2}\left(\|\Delta u_m(t)\|_2^2 + \|\Delta u_m'(t)\|_2^2\right) \\ &\le \frac{|g_1'(0)|L_1}{2l} + \frac{|g_1'(0)|}{2}\|\Delta u_m'(t)\|_2^2. \end{aligned} \tag{20}$$

From (H3), we deduce

$$\int_\Omega\int_0^t g_1''(t-s)\Delta u_m(s)\Delta u_m'(t)dsdx \le \frac{1}{2}\|\Delta u_m'(t)\|_2^2 + \frac{\alpha_3^2\|g_1\|_{L^1}}{2}\int_0^t g_1(t-s)\|\Delta u_m(s)\|_2^2ds \tag{21}$$

Inserting Equations (18)-(21) in Equation (17), we get

$$\begin{aligned} \frac{1}{2}\|u_m''(t)\|_2^2 + \frac{1}{2}\|\Delta u_m'(t)\|_2^2 &\le \|u_m''(0)\|_2^2 + \|\Delta u_{1m}'\|_2^2 + C_3 + g_1(0)\int_\Omega \Delta u_m(t)\cdot\Delta u_m'(t)dx \\ &+ \int_0^t g_1'(t-s)\int_\Omega \Delta u_m(t)\cdot\Delta u_m'(t)dxds + C_4\int_0^t \|\Delta u_m'(s)\|_2^2ds, \end{aligned} \tag{22}$$

where

$$C_3 = \left[\frac{\eta_1^2\|g_2\|_{L^1}}{2} + \frac{[g_2(0)]^2L_1}{2l} + \frac{|g_1'(0)|L_1}{2l}\right]T + \left[\frac{\eta_1^2\|g_2\|_{L^1(0,\infty)}\|g_2\|_{L^\infty(0,\infty)}}{2} + \frac{\alpha_1^2\|g_1\|_{L^1(0,\infty)}\|g_1\|_{L^\infty(0,\infty)}}{2}\right]\frac{L_1T}{l}$$

and

$$C_4 = \frac{|g_1'(0)|}{2} + \frac{1}{2}.$$

Using Hölder's inequality, we know that, for any $\delta > 0$,

$$\begin{aligned} &g_1(0)\int_\Omega \Delta u_m(t)\cdot\Delta u_m'(t)dx + \int_0^t g_1'(t-s)\int_\Omega \Delta u_m(t)\cdot\Delta u_m'(t)dxds \\ &\le 2\delta\|\Delta u_m'(t)\|_2^2 + \frac{[g_1(0)]^2}{4\delta}\|\Delta u_m(t)\|_2^2 + \frac{\alpha_1^2}{4\delta}\|g_1\|_{L^1(0,\infty)}\|g_1\|_{L^\infty(0,\infty)}\int_0^t \|\Delta u_m(s)\|_2^2ds \\ &\le 2\delta\|\Delta u_m'(t)\|_2^2 + C_5, \end{aligned} \tag{23}$$

where

$$C_5 = \left[\frac{[g_1(0)]^2}{4\delta} + \frac{\alpha_1^2}{4\delta}\|g_1\|_{L^1(0,\infty)}\|g_1\|_{L^\infty(0,\infty)}T\right]\frac{L_1}{l}.$$

Combining Equation (22) and Equation (23), we get

$$\frac{1}{2}\|u_m''(t)\|_2^2 + \left(\frac{1}{2} - 2\delta\right)\|\Delta u_m'(t)\|_2^2 \le \|u_m''(0)\|_2^2 + \|\Delta u_{1m}'\|_2^2 + C_3 + C_5 + C_4\int_0^t \|\Delta u_m'(s)\|_2^2ds, \tag{24}$$

Fixing $\delta > 0$, sufficiently small, so that $\frac{1}{2} - 2\delta > 0$ in Equation (24), and taking into account Equations (8) and (16), we get from Gronwall's Lemma the second estimate,

$$\|u_m''(t)\|_2^2 + \|\Delta u_m'(t)\|_2^2 \le L_3, \tag{25}$$

where L_3 is a positive constant independent of $m \in \mathbb{N}$ and $t \in [0, T]$.

Estimate 3. Let $m_1 \ge m_2$ be two natural numbers, and consider $z_m = u_{m_1} - u_{m_2}$. Then, applying the same way as in the estimate 1 and observing that $\{u_{0m}\}$ and $\{u_{1m}\}$ are Cauchy sequence in X and $H_0^2(\Omega)$, respectively, we deduce

$$\|z_m'(t)\|_2^2 + \|\Delta z_m(t)\|_2^2 + 2\int_0^t \|z_m'(s)\|_2^2 ds \to 0, \text{ as } n \to +\infty, \tag{26}$$

for all $t \in [0,T]$.
Therefore, from Equations (24), (25) and (26), we deduce that there exist a subsequence $\{u_\mu\}$ of $\{u_m\}$ and u such that

$$u_\mu' \to u' \text{ strongly in } C^0([0,T];L^2(\Omega)), \tag{27}$$

$$u_\mu \to u \text{ strongly in } C^0([0,T];H_0^2(\Omega)), \tag{28}$$

$$u_\mu'' \to u'' \text{ weakly star in } L^\infty(0,T;L^2(\Omega)). \tag{29}$$

The above convergences (27)-(29) are enough to pass to the limit in Equation (7), to obtain

$$\begin{array}{l} u'' + \Delta^2 u - \int_0^t g_1(t-s)\Delta^2 u(s)ds + \int_0^t g_2(t-s)\Delta u(s)ds + u' = 0 \text{ in } L^\infty(0,\infty;L^2(\Omega)), \\ u(0) = u_0, \quad u'(0) = u_1. \end{array}.$$

Next, we want to show the uniqueness of solution of (7)-(8). Let $u^{(1)}$, $u^{(2)}$ be two solutions of (7)-(8). Then $z = u^{(1)} - u^{(2)}$ satisfies

$$(z''(t),v) + (\Delta z(t),\Delta v) - \int_0^t g_1(t-s)(\Delta z(s),\Delta v)ds - \int_0^t g_2(t-s)(\nabla z(s),\nabla v)ds + (z'(t),v) = 0, \quad \forall v \in H_0^2(\Omega), \tag{30}$$

$$z(x,0) = z'(x,0) = 0, \quad x \in \Omega,$$

$$z = 0, \quad \frac{\partial z}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0.$$

Setting $v = 2z'(t)$ in (30), then as in deriving (14), we see that

$$\|z'(t)\|_2 = \|\Delta z(t)\|_2 = 0 \text{ for all } t \in [0,T]. \tag{31}$$

Therefore, we have the uniqueness.

3.2. Weak solutions

Let $(u_0,u_1) \in H_0^2(\Omega) \times L^2(\Omega)$. Then, since $X \times H_0^2(\Omega)$ is dense in $H_0^2(\Omega) \times L^2(\Omega)$ there exists $(u_{0\mu},u_{1\mu}) \subset X \times H_0^2(\Omega)$ such that

$$u_{0\mu} \to u_0 \text{ in } H_0^2(\Omega) \text{ and } u_{1\mu} \to u_1 \text{ in } L^2(\Omega). \tag{32}$$

Then, for each $\mu \in \mathbb{N}$, there exists a unique regular solution u_μ of Problem (1) in the class

$$u_\mu \in L^\infty(0,\infty;H_0^2(\Omega)), \quad u_\mu' \in L^\infty(0,\infty;H_0^2(\Omega)), \quad u_\mu'' \in L^\infty(0,\infty;L^2(\Omega)). \tag{33}$$

In view of Equation (33) and using an analogous argument to that in Estimate 1 and Estimate 3, we find a sequence $\{u_\mu\}$ of solutions to Problem (1) such that

$$u_\mu' \to u' \text{ weak star in } L^\infty(0,T;L^2(\Omega)), \tag{34}$$

$$u_\mu \to u \text{ weak star in } L^\infty(0,T;H_0^2(\Omega)), \tag{35}$$

$$u_\mu \to u \text{ strongly in } C^0([0,T];H_0^2(\Omega)), \tag{36}$$

$$u_\mu' \to u' \text{ strongly in } C^0([0,T];L^2(\Omega)), \tag{37}$$

The convergences (33)-(36) are sufficient to pass to the limit in order to obtain a weak solution of Problem (1), which satisfies

$$\begin{array}{l} u'' + \Delta^2 u - \int_0^t g_1(t-s)\Delta^2 u(s)ds + \int_0^t g_2(t-s)\Delta u(s)ds + u' = 0 \text{ in } L^2(0,\infty; H^{-2}(\Omega)), \\ u(0) = u_0, \quad u'(0) = u_1. \end{array}.$$

The uniqueness of weak solutions requires a regularization procedure and can be obtained using the standard method of Visik-Ladyzhenskaya, c.f. Lions and Magenes [8, Chap. 3, Sec. 8.2.2].

4. Asymptotic Behaviour

In this section, we discuss the asymptotic behavior of the above-mentioned weak solutions. Let us define the energy associated to Problem (1) as

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g_1(s)ds\right)\|\Delta u(t)\|_2^2 + \frac{1}{2}(g_1\Box\Delta u)(t) \\ &\quad -\frac{1}{2}\left(\int_0^t g_2(s)ds\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g_2\Box\nabla u)(t). \end{aligned} \tag{38}$$

To demonstrate our decay result, the lemmas below are essential.

Lemma 3. *For any $t > 0$*

$$0 \le E(t) \le \frac{1}{2}\Big[\|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2 + (g_1\Box\Delta u)(t) + (g_2\Box\nabla u)(t)\Big].$$

Proof. Using the fact that $\|\nabla u(t)\|_2^2 \le \lambda_1^{-1}\|\Delta u(t)\|_2^2$, we have

$$\begin{aligned} &\left(1 - \int_0^t g_1(\tau)d\tau\right)\|\Delta u(t)\|_2^2 - \left(\int_0^t g_2(\tau)d\tau\right)\|\nabla u(t)\|_2^2 \\ &\ge \left(1 - \int_0^t \Big[g_1(s) + \lambda_1^{-1}g_2(s)\Big]ds\right)\|\Delta u(t)\|_2^2 \end{aligned}$$

and according to $(H6)$ we have $E(t) \ge 0$,
and

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\Delta u(t)\|_2^2 + \frac{1}{2}(g_1\Box\Delta u)(t) + \frac{1}{2}(g_2\Box\nabla u)(t) \\ &\quad -\frac{1}{2}\left\{\left(\int_0^t g_1(s)ds\right)\|\Delta u(t)\|_2^2 + \left(\int_0^t g_2(s)ds\right)\|\nabla u(t)\|_2^2\right\} \\ &\le \frac{1}{2}\Big[\|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2 + (g_1\Box\Delta u)(t) + (g_2\Box\nabla u)(t)\Big]. \end{aligned}$$

□

Lemma 4. *The energy $E(t)$ satisfies*

$$E'(t) \le -\|u_t(t)\|_2^2 - \frac{1}{2}\alpha_2(g_1\Box\Delta u)(t) - \frac{1}{2}\eta_2(g_2\Box\nabla u)(t) - \frac{1}{2}\left[g_1(0) - \alpha_1\|g_1\|_{L^1(0,\infty)}\right]\|\Delta u(t)\|_2^2 \le 0. \tag{39}$$

Proof. Multiplying the first equation in (1) by u_t and integrating over Ω, we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\Delta u(t)\|_2^2\right] + \|u_t(t)\|_2^2 = \int_0^t g_1(t-\tau)\Delta u(\tau)\cdot\Delta u_t(t)dxd\tau + \int_0^t g_2(t-\tau)\nabla u(\tau)\cdot\nabla u_t(t)dxd\tau.$$

Exploiting (10)-(11) and by $(H1)-(H5)$, we deduce

$$\begin{aligned} E'(t) &= -\|u_t(t)\|_2^2+\frac{1}{2}(g_1'\Box\Delta u)(t)-\frac{1}{2}g_1(t)\|\Delta u(t)\|_2^2+\frac{1}{2}(g_2'\Box\nabla u)(t)-\frac{1}{2}g_2(t)\|\nabla u(t)\|_2^2 \\ &\le -\|u_t(t)\|_2^2-\frac{1}{2}\alpha_2(g_1\Box\Delta u)(t)-\frac{1}{2}\eta_2(g_2\Box\nabla u)(t)-\frac{1}{2}g_1(t)\|\Delta u(t)\|_2^2. \end{aligned} \tag{40}$$

From assumptions $(H2)$ and since $\int_0^t g_1'(\tau)d\tau = g_1(t)-g_1(0)$, we obtain

$$\begin{aligned} -\frac{1}{2}g_1(t)\|\Delta u(t)\|_2^2 &= -\frac{1}{2}g_1(0)\|\Delta u(t)\|_2^2-\frac{1}{2}\left(\int_0^t g_1'(s)ds\right)\|\Delta u(t)\|_2^2 \\ &\le -\frac{1}{2}g_1(0)\|\Delta u(t)\|_2^2+\frac{\alpha_1}{2}\|g_1\|_{L^1(0,\infty)}\|\Delta u(t)\|_2^2 \\ &= -\frac{1}{2}\left[g_1(0)-\alpha_1\|g_1\|_{L^1(0,\infty)}\right]\|\Delta u(t)\|_2^2. \end{aligned} \tag{41}$$

Combining Equation (40) and Equation (41), we conclude that

$$\begin{aligned} E'(t) &\le -\|u_t(t)\|_2^2-\frac{1}{2}\alpha_2(g_1\Box\Delta u)(t)-\frac{1}{2}\eta_2(g_2\Box\nabla u)(t) \\ &\quad -\frac{1}{2}\left[g_1(0)-\alpha_1\|g_1\|_{L^1(0,\infty)}\right]\|\Delta u(t)\|_2^2 \le 0. \end{aligned}$$

Multiplying Equation (39) by $e^{\kappa\xi(t)}$ $(\kappa>0)$ and utilizing Lemma 3, we have

$$\begin{aligned} \frac{d}{dt}\left(e^{\kappa\xi(t)}E(t)\right) &\le -e^{\kappa\xi(t)}E(t)\|u_t(t)\|_2^2-\frac{1}{2}\alpha_2(g_1\Box\Delta u)(t)e^{\kappa\xi(t)}E(t)-\frac{1}{2}\eta_2(g_2\Box\nabla u)(t)e^{\kappa\xi(t)}E(t) \\ &\quad -\frac{1}{2}\left[g_1(0)-\alpha_1\|g_1\|_{L^1(0,\infty)}\right]e^{\kappa\xi(t)}E(t)\|\Delta u(t)\|_2^2+\kappa\xi'(t)e^{\kappa\xi(t)}E(t) \\ &\le -\frac{1}{2}\left[2-\kappa\xi'(t)\right]e^{\kappa\xi(t)}E(t)\|u_t(t)\|_2^2-\frac{1}{2}\left[\alpha_2-\kappa\xi'(t)\right]e^{\kappa\xi(t)}E(t)(g_1\Box\Delta u)(t) \\ &\quad -\frac{1}{2}\left[\eta_2-\kappa\xi'(t)\right]e^{\kappa\xi(t)}E(t)(g_2\Box\nabla u)(t) \\ &\quad -\frac{1}{2}\left[g_1(0)-\alpha_1\|g_1\|_{L^1(0,\infty)}-\kappa\xi'(t)\right]e^{\kappa\xi(t)}E(t)\|\Delta u(t)\|_2^2. \end{aligned} \tag{42}$$

Using the fact that ξ' is decreasing we arrive at

$$\begin{aligned} \frac{d}{dt}\left(e^{\kappa\xi(t)}E(t)\right) &\le -\frac{1}{2}\left[2-\kappa\xi'(0)\right]e^{\kappa\xi(t)}E(t)\|u_t(t)\|_2^2-\frac{1}{2}\left[\alpha_2-\kappa\xi'(0)\right]e^{\kappa\xi(t)}E(t)(g_1\Box\Delta u)(t) \\ &\quad -\frac{1}{2}\left[\eta_2-\kappa\xi'(0)\right]e^{\kappa\xi(t)}E(t)(g_2\Box\nabla u)(t) \\ &\quad -\frac{1}{2}\left[g_1(0)-\alpha_1\|g_1\|_{L^1(0,\infty)}-\kappa\xi'(0)\right]e^{\kappa\xi(t)}E(t)\|\Delta u(t)\|_2^2. \end{aligned} \tag{43}$$

Choosing $\|g_1\|_{L^1(0,\infty)}$ sufficiently small so that

$$g_1(0)-\alpha_1\|g_1\|_{L^1(0,\infty)} = L > 0,$$

and choosing κ sufficiently small in order to have

$$2-\kappa\xi'(0)>0,\quad \alpha_2-\kappa\xi'(0)>0,\quad \eta_2-\kappa\xi'(0)>,\quad L-\kappa\xi'(0)>0.$$

from Equation (43) we arrive at

$$\frac{d}{dt}\left(e^{\kappa\xi(t)}E(t)\right)\le 0,\quad t>0. \tag{44}$$

Integrating the above inequality over $(0,t)$, it follows that

$$E(t)\le E(0)e^{-\kappa\xi(t)},\quad t>0. \tag{45}$$

□

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