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Article

Existence and uniqueness results for nonlinear hybrid implicit Caputo-Hadamard fractional differential equations

Abdelouaheb Ardjouni[1,2,*], Adel Lachouri[2] and Ahcene Djoudi[2]

[1] Department of Mathematics and Informatics, University of Souk Ahras, P.O. Box 1553, Souk Ahras, 41000, Algeria.
[2] Applied Mathematics Lab, Faculty of Sciences, Department of Mathematics, University of Annaba, P.O. Box 12, Annaba 23000, Algeria.
* Correspondence: abd_ardjouni@yahoo.fr

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Abstract: In this paper, we use the Banach fixed point theorem to obtain the existence, interval of existence and uniqueness of solutions for nonlinear hybrid implicit Caputo-Hadamard fractional differential equations. We also use the generalization of Gronwall's inequality to show the estimate of the solutions.

Keywords: Implicit fractional differential equations, Caputo-Hadamard fractional derivatives, fixed point theorems, existence, uniqueness.

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1. Introduction

The concept of fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non integer order. Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1–17] and the references therein.

Recently, Ahmad and Ntouyas [3] discussed the existence of solutions for the hybrid Hadamard differential equation

$$\begin{cases} {}^{H}D^{\alpha}\left(\frac{x(t)}{g(t,x(t))}\right) = f\left(t, x\left(t\right)\right), \ t \in \left[1, T\right], \\ {}^{H}I^{\alpha}x(t)\big|_{t=1} = \eta, \end{cases}$$

where ${}^{H}D^{\alpha}$ is the Hadamard fractional derivative of order $0 < \alpha \leq 1$. By employing the Dhage fixed point theorem, the authors obtained existence results.

In [4], Ardjouni and Djoudi studied the existence, interval of existence and uniqueness of solutions for nonlinear implicit Caputo-Hadamard fractional differential equations with nonlocal conditions

$$\begin{cases} \mathfrak{D}_1^{\alpha}\left(x\left(t\right)\right) = f\left(t, x\left(t\right), \mathfrak{D}_1^{\alpha}\left(x\left(t\right)\right)\right), \ t \in \left[1, T\right], \\ x\left(1\right) + g\left(x\right) = x_0, \end{cases}$$

where $f : [1, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : C\left([1, T], \mathbb{R}\right) \to \mathbb{R}$ are nonlinear continuous functions and $\mathfrak{D}_1^{\alpha}$ denotes the Caputo-Hadamard fractional derivative of order $0 < \alpha < 1$.

Motivated by these works, we study the existence, interval of existence and uniqueness of solution for the following nonlinear hybrid implicit Caputo-Hadamard fractional differential equation

$$\begin{cases} \mathfrak{D}_1^\alpha\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right) = h\left(t, x\left(t\right), \mathfrak{D}_1^\alpha\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right)\right), \ t \in [1,T], \\ x\left(1\right) = \theta g\left(1, x(1)\right) + f\left(1, x(1)\right), \ \theta \in \mathbb{R}, \end{cases} \tag{1}$$

where $f : [1,T] \times \mathbb{R} \to \mathbb{R}$, $g : [1,T] \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ and $h : [1,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are nonlinear continuous functions and $\mathfrak{D}_1^\alpha$ denotes the Caputo-Hadamard fractional derivative of order $0 < \alpha < 1$. To show the existence, interval of existence and uniqueness of solutions, we transform (1) into an integral equation and then use the Banach fixed point theorem. Further, by the generalization of Gronwall's inequality we obtain the estimate of solutions of (1).

2. Preliminaries

In this section, we present some basic definitions, notations and results of fractional calculus [2,7,12,15] which are used throughout this paper.

Definition 1 ([12])**.** The Hadamard fractional integral of order $\alpha > 0$ for a continuous function $x : [1, +\infty) \to \mathbb{R}$ is defined as

$$\mathfrak{I}_1^\alpha x\left(t\right) = \frac{1}{\Gamma\left(\alpha\right)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} x\left(s\right) \frac{ds}{s}, \ \alpha > 0.$$

Definition 2 ([12])**.** *The Caputo-Hadamard fractional derivative of order α* for a continuous function $x : [1, +\infty) \to \mathbb{R}$ is defined as

$$\mathfrak{D}_1^\alpha x\left(t\right) = \frac{1}{\Gamma\left(n-\alpha\right)} \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \delta^n\left(x\right)\left(s\right) \frac{ds}{s}, \ n-1 < \alpha < n,$$

where $\delta^n = \left(t\frac{d}{dt}\right)^n$, $n = [\alpha] + 1$.

Lemma 3 ([12])**.** *Let $\alpha > 0$, $n \in \mathbb{N}$. Suppose $x \in C^{n-1}\left([1, +\infty)\right)$ and $\delta^{(n)}x$ exists almost everywhere on any bounded interval of $[1, +\infty)$. Then*

$$\mathfrak{I}_1^\alpha\left[\mathfrak{D}_1^\alpha x\right](t) = x(t) - \sum_{k=0}^{n-1} \frac{\delta^{(k)}x\left(1\right)}{\Gamma\left(k+1\right)} \left(\log t\right)^k.$$

In particular, when $0 < \alpha < 1$, $\mathfrak{I}_1^\alpha\left[\mathfrak{D}_1^\alpha x\right](t) = x(t) - x(1)$.

Lemma 4 ([12])**.** *For all $\mu > 0$ and $v > -1$, then*

$$\frac{1}{\Gamma\left(\mu\right)} \int_1^t \left(\log \frac{t}{s}\right)^{\mu-1} \left(\log s\right)^v ds = \frac{\Gamma\left(v+1\right)}{\Gamma\left(\mu+v+1\right)} \left(\log t\right)^{\mu+v}.$$

The following generalization of Gronwall's lemma for singular kernels plays an important role in obtaining our main results.

Lemma 5 ([15])**.** *Let $x : [1,T] \to [0,\infty)$ be a real function and w is a nonnegative locally integrable function on $[1,T]$. Assume that there is a constant $a > 0$ such that for $0 < \alpha < 1$*

$$x(t) \leq w(t) + a \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} x(s) \frac{ds}{s}.$$

Then, there exists a constant $k = k(\alpha)$ such that

$$x(t) \leq w(t) + ka \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} w(s) \frac{ds}{s},$$

for every $t \in [1,T]$.

3. Main results

In this section, we give the equivalence of the initial value problem (1) and prove the existence,interval of existence, uniqueness and estimate of solution of (1).

The proof of the following lemma is close to the proof Lemma 6.2 given in [7].

Lemma 6. *If the functions $f:[1,T]\times\mathbb{R}\to\mathbb{R}$, $g:[1,T]\times\mathbb{R}\to\mathbb{R}\backslash\{0\}$ and $h:[1,T]\times\mathbb{R}^2\to\mathbb{R}$ are continuous, then the initial value problem (1) is equivalent to the nonlinear fractional Volterra integro-differential equation*

$$x(t) = f(t,x(t))+\theta g(t,x(t))+\frac{g(t,x(t))}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}h\left(s,x(s),\mathfrak{D}_1^{\alpha}\left(\frac{x(s)-f(s,x(s))}{g(s,x(s))}\right)\right)\frac{ds}{s},$$

for $t\in[1,T]$.

Theorem 7. *Let $T>0$. Assume that the continuous functions $f:[1,T]\times\mathbb{R}\to\mathbb{R}$, $g:[1,T]\times\mathbb{R}\to\mathbb{R}\backslash\{0\}$ and $h:[1,T]\times\mathbb{R}^2\to\mathbb{R}$ satisfy the following conditions*
(H1) There exists $M_g\in\mathbb{R}^+$ such that

$$|g(t,u)|\le M_g,$$

for all $u\in\mathbb{R}$ and $t\in[1,T]$.
(H2) There exists $M_h\in\mathbb{R}^+$ such that

$$|h(t,u,v)|\le M_h,$$

for all $u,v\in\mathbb{R}$ and $t\in[1,T]$.
(H3) There exist $K_1,K_2,K_3\in\mathbb{R}^+,K_4\in(0,1)$ with $K_1+K_2|\theta|\in(0,1)$ such that

$$\begin{aligned}|f(t,u)-f(t,u^*)| &\le K_1|u-v|,\\ |g(t,u)-g(t,u^*)| &\le K_2|u-v|,\end{aligned}$$

and

$$|h(t,u,v)-h(t,u^*,v^*)|\le K_3|u-u^*|+K_4|v-v^*|,$$

for all $u,v,u^*,v^*\in\mathbb{R}$ and $t\in[1,T]$.

Let

$$1<b<\min\left\{T,\exp\left(\frac{((1-(K_1+K_2|\theta|))(1-K_4)\Gamma(\alpha+1))}{(M_hK_2(1-K_4)+M_gK_3)}\right)^{\frac{1}{\alpha}}\right\}. \tag{2}$$

Then (1) has a unique solution $x\in C([1,b],\mathbb{R})$.

Proof. Let

$$\mathfrak{D}_1^{\alpha}\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right)=z_x(t),\ x(1)=\theta g(1,x(1))+f(1,x(1)),$$

then by Lemma 6, we have

$$x(t)=f(t,x(t))+\theta g(t,x(t))+\frac{g(t,x(t))}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}z_x(s)\frac{ds}{s},$$

where

$$z_x(t)=h(t,f(t,x(t))+\theta g(t,x(t))+g(t,x(t))\mathfrak{I}_1^{\alpha}z_x(t),z_x(t)).$$

That is $x(t)=f(t,x(t))+\theta g(t,x(t))+g(t,x(t))\mathfrak{I}_1^{\alpha}z_x(t)$. Define the mapping $P:C([1,b],\mathbb{R})\to C([1,b],\mathbb{R})$ as follows

$$(Px)(t)=f(t,x(t))+\theta g(t,x(t))+\frac{g(t,x(t))}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}z_x(s)\frac{ds}{s}.$$

It is clear that the fixed points of P are solutions of (1). Let $x, y \in C([1,b], \mathbb{R})$, then we have

$$\begin{aligned}
|(Px)(t) - (Py)(t)| &= \Bigg| f(t,x(t)) + \theta g(t,x(t)) + \frac{g(t,x(t))}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} z_x(s) \frac{ds}{s} \\
&\quad - f(t,y(t)) + \theta g(t,y(t)) - \frac{g(t,y(t))}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} z_y(s) \frac{ds}{s} \Bigg| \\
&\leq |f(t,x(t)) - f(t,y(t))| + |\theta| \, |g(t,x(t)) - g(t,y(t))| \\
&\quad + |g(t,x(t)) - g(t,y(t))| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |z_x(s)| \frac{ds}{s} \\
&\quad + |g(t,y(t))| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |z_x(s) - z_y(s)| \frac{ds}{s} \\
&\leq K_1 |x(t) - y(t)| + K_2 |\theta| \, |x(t) - y(t)| + K_2 |x(t) - y(t)| \frac{M_h}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\
&\quad + \frac{M_g}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |z_x(s) - z_y(s)| \frac{ds}{s},
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
|z_x(t) - z_y(t)| &\leq |h(t,x(t),z_x(t)) - h(t,x(t),z_y(t))| \\
&\leq K_3 |x(t) - y(t)| + K_4 |z_x(t) - z_y(t)| \\
&\leq \frac{K_3}{1-K_4} |x(t) - y(t)| .
\end{aligned} \tag{4}$$

By replacing (4) in the inequality (3), we get

$$\begin{aligned}
|(Px)(t) - (Py)(t)| &\leq K_1 |x(t) - y(t)| + K_2 |\theta| \, |x(t) - y(t)| + K_2 |x(t) - y(t)| \frac{M_h}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\
&\quad + \frac{M_g}{\Gamma(\alpha)} \frac{K_3}{1-K_4} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |x(s) - y(s)| \frac{ds}{s} \\
&\leq K_1 \|x - y\| + K_2 \left(|\theta| + \frac{M_h (\log t)^\alpha}{\Gamma(\alpha+1)} \right) \|x - y\| + \frac{K_3}{1-K_4} \left(\frac{M_g (\log t)^\alpha}{\Gamma(\alpha+1)} \right) \|x - y\| \\
&\leq (K_1 + K_2 |\theta| + \left(M_h K_2 + \frac{M_g K_3}{1-K_4} \right) \frac{(\log t)^\alpha}{\Gamma(\alpha+1)}) \|x - y\| .
\end{aligned}$$

Since $t \in [1,b]$, Then

$$\|Px - Py\| \leq \beta \|x - y\| ,$$

where

$$\beta = K_1 + K_2 |\theta| + \left(M_h K_2 + \frac{M_g K_3}{1-K_4} \right) \frac{(\log b)^\alpha}{\Gamma(\alpha+1)}.$$

That is to say the mapping P is a contraction in $C([1,b], \mathbb{R})$. Hence, by the Banach fixed point theorem, P has a unique fixed point $x \in C([1,b], \mathbb{R})$. Therefore, (1) has a unique solution. □

Theorem 8. *Assume that $f : [1,T] \times \mathbb{R} \to \mathbb{R}$, $g : [1,T] \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $h : [1,T] \times \mathbb{R}^2 \to \mathbb{R}$ satisfy (H1), (H2) and (H3). If x is a solution of (1), then*

$$|x(t)| \leq \left(\frac{(1-K_4)(1-(K_1+K_2|\theta|))\Gamma(\alpha+1) + M_g K_3 K (\log T)^\alpha}{(1-K_4)(1-(K_1+K_2|\theta|))\Gamma(\alpha+1)} \right) \left(Q_1 + |\theta| Q_2 + \frac{M_g Q_3 (\log T)^\alpha}{(1-K_4)\Gamma(\alpha+1)} \right),$$

where $Q_1 = \sup_{t\in[1,T]} |f(t,0)|$, $Q_2 = \sup_{t\in[1,T]} |g(t,0)|$, $Q_3 = \sup_{t\in[1,T]} |h(t,0,0)|$ and $K \in \mathbb{R}^+$ is a constant.

Proof. Let

$$\mathfrak{D}_1^{\alpha}\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right)=z_x(t),\ x(1)=\theta g(1,x(1))+f(1,x(1)),$$

then by Lemma 6, $x(t)=f(t,x(t))+\theta g(t,x(t))+g(t,x(t))\mathfrak{I}_1^{\alpha}z_x(t)$. Then by (H1), (H2) and (H3), for any $t\in[1,T]$ we have

$$\begin{aligned}|x(t)| &\leq |f(t,x(t))|+|\theta|\,|g(t,x(t))|+|g(t,x(t))|\,|\mathfrak{I}_1^{\alpha}z_x(t)|\\ &\leq |f(t,x(t))-f(t,0)|+|f(t,0)|+|\theta|\left(|g(t,x(t))-g(t,0)|+|g(t,0)|\right)+M_g\,|\mathfrak{I}_1^{\alpha}z_x(t)|\\ &\leq K_1|x(t)|+Q_1+|\theta|\left(K_2|x(t)|+Q_2\right)+M_g\mathfrak{I}_1^{\alpha}|z_x(t)|.\end{aligned}$$

On the other hand, for any $t\in[1,T]$ we get

$$\begin{aligned}|z_x(t)|=|h(t,x(t),z_x(t))| &\leq |h(t,x(t),z_x(t))-h(t,0,0)|+|h(t,0,0)|\\ &\leq K_3|x(t)|+K_4|z_x(t)|+|h(t,0,0)|\\ &\leq \frac{K_3}{1-K_4}|x(t)|+\frac{Q_3}{1-K_4}.\end{aligned}$$

Therefore

$$|x(t)|\leq K_1|x(t)|+Q_1+|\theta|\left(K_2|x(t)|+Q_2\right)+M_g\mathfrak{I}_1^{\alpha}\left(\frac{K_3}{1-K_4}|x(t)|+\frac{Q_3}{1-K_4}\right).$$

Thus

$$\begin{aligned}\left(1-(K_1+K_2|\theta|)\right)|x(t)| \leq\ & Q_1+|\theta|Q_2+\frac{M_gQ_3(\log T)^{\alpha}}{(1-K_4)\Gamma(\alpha+1)}+\left(\frac{M_gK_3}{(1-K_4)\left(1-(K_1+K_2|\theta|)\right)}\right)\\ &\times\left(\mathfrak{I}_1^{\alpha}\left\{\left(1-(K_1+K_2|\theta|)\right)|x(t)|\right\}\right).\end{aligned}$$

By Lemma 5, there is a constant $K=K(\alpha)$ such that

$$\begin{aligned}\left(1-(K_1+K_2|\theta|)\right)|x(t)| &\leq Q_1+|\theta|Q_2+\frac{M_gQ_3(\log T)^{\alpha}}{(1-K_4)\Gamma(\alpha+1)}+\left(\frac{M_gK_3K(\log T)^{\alpha}}{(1-K_4)\left(1-(K_1+K_2|\theta|)\right)\Gamma(\alpha+1)}\right)\\ &\quad\times\left(Q_1+|\theta|Q_2+\frac{M_gQ_3(\log T)^{\alpha}}{(1-K_4)\Gamma(\alpha+1)}\right)\\ &\leq \left(\frac{(1-K_4)\left(1-(K_1+K_2|\theta|)\right)\Gamma(\alpha+1)+M_gK_3K(\log T)^{\alpha}}{(1-K_4)\left(1-(K_1+K_2|\theta|)\right)\Gamma(\alpha+1)}\right)\\ &\quad\times\left(Q_1+|\theta|Q_2+\frac{M_gQ_3(\log T)^{\alpha}}{(1-K_4)\Gamma(\alpha+1)}\right).\end{aligned}$$

Hence

$$|x(t)|\leq\left(\frac{(1-K_4)\left(1-(K_1+K_2|\theta|)\right)\Gamma(\alpha+1)+M_gK_3K(\log T)^{\alpha}}{(1-K_4)\left(1-(K_1+K_2|\theta|)\right)\Gamma(\alpha+1)}\right)\left(Q_1+|\theta|Q_2+\frac{M_gQ_3(\log T)^{\alpha}}{(1-K_4)\Gamma(\alpha+1)}\right).$$

This completes the proof. □

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