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Existence of positive periodic solutions of neutral nonlinear differential systems with variable delays

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Abstract: By using some mixed techniques of the Mawhin coincidence degree theory and the Krasnoselskii fixed point theorem, we obtained the existence of positive periodic solutions of the neutral nonlinear differential system. Also, sufficient conditions for the existence of positive periodic solutions to the system with feedback control are given. Our results substantially extend and improve existing results.

Keywords: Coincidence degree, fixed point theorem, positive periodic solutions, neutral nonlinear differential system, variable delay.

MSC: 54H25, 35B09, 35B10, 47H10.

1. Introduction

Functional differential equations are not only an extension of ordinary delay differential equations but also provide good models in many fields including Biology, Mechanics, Economics and bio-mathematics. For example, in population dynamics [1], since a growing population consumes more (or less) food than a matured one, depending on individual species, this leads to delay functional equations. Positive periodic solutions of differential equations have been studied extensively in recent times. We refer to the references [1–22] in this article and references therein for a wealth of information on this subject.

In this paper, we study the existence of positive periodic solutions of a system of neutral differential equations. The study on the functional differential equations is more intricate than ordinary delay differential equations. That is why comparing plenty of results on the existence of positive periodic solutions for various types of first-order or second-order ordinary delay differential equations or studies on positive periodic solutions for delay differential equations are relatively less, and most of them are confined to first-order delay differential equations, see [1] which are studied by using some techniques of the Mawhin coincidence degree theory.

In this paper, we consider the following class of nonlinear neutral differential system with several delays

$$\frac{dx\left(t\right)}{dt}=\beta x'\left(t-\tau\left(t\right)\right)+f\left(x\left(t-\tau\left(t\right)\right)\right)+g\left(u\left(t-\tau\left(t\right)\right)\right)+p\left(t\right), \tag{1}$$

$$\frac{du\left(t\right)}{dt}=-a\left(t\right)u\left(t\right)+\frac{d}{dt}F\left(t,u\left(t-\sigma\left(t\right)\right)\right)+c\left(t\right)G\left(t,x\left(t-\tau\left(t\right)\right),u\left(t-\sigma\left(t\right)\right)\right), \tag{2}$$

where $\beta>0$ is a parameter, $G\in C\left(\mathbb{R}^{3},\mathbb{R}\right)$, $F\in C\left(\mathbb{R}^{2},\mathbb{R}\right)$, $f\in C\left(\mathbb{R},\mathbb{R}\right)$ and $a,c,p\in C\left(\mathbb{R},\mathbb{R}^{+}\right)$. All of the above functions are continuous, T–periodic with $T>0$ is a constant. Here, we obtain various sufficient conditions for the existence of positive periodic solutions for the problem (1)–(2) by employing two available operators and by applying the coincidence degree theorem and the fixed point theorem. Special cases of (1)–(2) have been considered and investigated by many other authors. For example, very recently, in [2], Huo and Li discussed the existence of a positive periodic solutions of the delay differential system

$$\begin{cases} \dfrac{dx(t)}{dt} = \pm x(t)\, G(t, x(t-\tau_1(t)), ..., x(t-\tau_n(t)), u(t-\delta(t))), \\ \dfrac{du(t)}{dt} = -a(t)\, u(t) + b(t)\, x(t-\sigma(t)), \end{cases}$$

where all of the above functions are T–periodic functions with $T > 0$ is a constant and G satisfies some specific conditions. The main tool employed in their study is based on some techniques of the Mawhin coincidence degree. For details on the Mawhin technique, we refer the reader to Gaines and Mawhin [16].

2. Periodic solutions

Let us give some known notions and notations used in the theory of coincidence degree theorem which are taken from [16–18] and which we will apply in the present part. We seek conditions under which there exists a T–periodic function x which can be solution of (1) for all function $u \in X$. Otherwise speaking, our result here of existence of T–periodic solutions of equation (1) doesn't depend on the choice of $u \in X$. For that end some preparations and notations are needed. For that purpose, let $T > 0$ and let X be the set of all continuously differentiable scalar functions x, periodic in t of period T. Take

$$\begin{aligned} Z \ &: \ = \{x \in C(\mathbb{R}, \mathbb{R}) : x(t+T) = x(t)\}. \\ X \ &: \ = \left\{x \in C^1(\mathbb{R}, \mathbb{R}) : x(t+T) = x(t)\right\}, \end{aligned}$$

and denote

$$\begin{aligned} \|x\| &= \sup_{t\in\mathbb{R}} |x(t)|. \\ |x|_0 &= \max\{\|x\|, \|\dot{x}\|\}. \end{aligned}$$

Then, Z and X are Banach spaces when they are endowed with the norms $\|\cdot\|$, $|\cdot|_0$ respectively.

The method we use, for proving existence, in this paper involves the applications of the continuous theorem of coincidence degree (see Gaines and Mawhin [16]). This theorem needs some introduction. So, let X and Z be two Banach spaces. Consider the operator equation

$$Lx = \lambda N(x, \lambda),\ \lambda \in (0, 1),$$

where $L : X \cap DomL \to Z$ is a linear operator and λ is a parameter. Let P and Q denote two projectors such that

$$P : X \cap DomL \to \ker L \text{ and } Q : Z \to Z/ImL.$$

Recall that a linear mapping $L : X \cap DomL \to Z$ with $\ker L = L^{-1}(0)$ and $ImL = L(DomL)$, will be called a Fredholm mapping if the following two conditions hold;

(i) $\ker L$ has a finite dimension;

(ii) ImL is closed and has a finite codimension.

Recall also that the codimension of ImL is the dimension of Z/ImL, i.e., the dimension of the cokernel $co\ker L$ of L. When L is a Fredholm mapping, its index is the integer $Ind(L) = \dim\ker L - co\dim ImL$. We shall say that a mapping N is L–compact on Ω if the mapping $QN : \bar{\Omega} \to Z$ is continuous, $QN(\bar{\Omega})$ is bounded, and $K_P(I-Q)N : \bar{\Omega} \to X$ is compact, i.e., it is K_P is continuous and $K_P(I-Q)N(\bar{\Omega})$ is relatively compact, where $K_P : ImL \to DomL \cap \ker P$ is the inverse of the restriction L_P of L to $DomL \cap \ker P$, so that $LK_P = I$ and $K_PL = I - P$.

Now, we state the continuous theorem of coincidence degree (Gaines and Mawhin [16]) which enables us to prove the existence of periodic solutions to (1). For its proof we refer the reader to [16].

Lemma 1. *Let X and Z be two Banach spaces and L a Fredholm mapping of index zero. Assume that $N : \bar{\Omega} \times [0,1] \to Z$ is L–compact on $\bar{\Omega} \times [0,1]$ with Ω open bounded in X. Furthermore, we assume that*

(a) for each $\lambda \in (0,1)$, $x \in \partial\Omega \cap DomL$,
$$Lx \neq \lambda N(x,\lambda),$$

(b) for each $x \in \partial\Omega \cap \ker L$,
$$QNx \neq 0,$$

and
$$\deg\{QNx, \Omega \cap \ker L, 0\} \neq 0.$$

Then the equation $Lx = N(x,1)$ has at least one solution in $\bar{\Omega}$.

As a first case, we consider the following nonlinear neutral equation with delay

$$\frac{d}{dt}x(t) = \beta x'(t-\tau(t)) + f(x(t-\tau(t))) + g(u(t-\sigma(t))) + p(t), \; x \in X, \; t \in \mathbb{R}, \tag{3}$$

where
$$p(t+T) = p(t), \tag{4}$$

and
$$\tau(t+T) = \tau(t), \sigma(t+T) = \sigma(t). \tag{5}$$

All of the above functions are continuous, T–periodic functions and $T > 0$ is a constant. Here $\beta > 0$ is a parameter. Before we state the main results we make the following basic assumptions on the delay function $\tau(t)$ of (3).

(H0) The inverse of $t - \tau(t)$ exists and we denote it by $r(t)$ such that

$$\tau'(t) \neq 1 \text{ for } t \in [0,T].$$

Lemma 2. *Assume the condition (H0) holds. Suppose that in (1) the following conditions hold*

(H1) there exists constants K such that $|g(x)| \leq K$ for all $x \in \mathbb{R}$
(H2) $xf(x) > 0$ and there exists a constant $M > 0$ such that when $\|x\| \geq M$, then we have

$$|f(x)| > K + \|p\| + R\beta\|x\|,$$

with,
$$R = \sup_{t\in[0,T]} \left| \frac{1}{1-\tau'(t)} \right| \text{ and } R_0 = \sup_{t\in[-\tau(0),T-\tau(T)]} \frac{1}{[1-\tau'(r(t))]^2}.$$

(H3) There exists a positive constant $H > 0$ such that

$$\lim_{x\to-\infty} \frac{f(x)}{x} \leq H,$$

Then, the equation (1) has at least one T–periodic solution if $1 > 2\beta R_0 + 2H(T+1)$.

Proof. In order to apply Lemma 1. Set

$$Lx(t) = \frac{dx(t)}{dt} = \dot{x}(t), \; x \in X, \; t \in \mathbb{R},$$

$$N(x,\lambda) = \lambda\beta x'(t-\tau(t)) + f(x(t-\tau(t))) + \lambda g(u(t-\sigma(t))) + \lambda p(t),$$
$$\text{for all } x \in X \text{ and } t \in \mathbb{R},$$

and

$$Px = \frac{1}{T}\int_0^T x(t)\,dt,\ x \in X \text{ and } Qz = \frac{1}{T}\int_0^T z(t)\,dt,\ z \in Z.$$

Obviously, $\ker L = \{x \mid x \in X,\ x = \xi,\ \xi \in \mathbb{R}\}$, $ImL = \left\{y \mid y \in Z,\ \int_0^T y(t)\,dt = 0\right\}$ are closed in X and $\dim \ker L = co \dim ImL$. Hence, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_P : ImL \to \ker P \cap DomL$ has the form

$$K_P(x) = \int_0^t x(s)\,ds - \frac{1}{T}\int_0^T\int_0^t x(s)\,dsdt.$$

One has

$$(QN)(x,\lambda) = -\frac{1}{T}\int_0^T \left[\lambda\beta x'(t-\tau(t)) + f(x(t-\tau(t))) + \lambda g(u(t-\sigma(t))) + \lambda p(t)\right]dt,$$

and

$$\begin{aligned} K_P(I-Q)N(x,\lambda) = &-\frac{1}{T}\int_0^T \left[\lambda\beta x'(t-\tau(t)) + f(x(t-\tau(t))) + \lambda g(u(t-\sigma(t))) + \lambda p(t)\right]dt. \\ &+\frac{1}{T}\int_0^T\int_0^t \left[\lambda\beta x'(s-\tau(s)) + f(x(s-\tau(s))) + \lambda g(u(s-\sigma(s))) + \lambda p(s)\right]dsdt \\ &+\left(\frac{t}{T}-\frac{1}{2}\right)\int_0^T \left[\lambda\beta x'(t-\tau(t)) + f(x(t-\tau(t))) + \lambda g(u(t-\sigma(t))) + \lambda p(t)\right]dt. \end{aligned}$$

Clearly, QN and $K_P(I-Q)N$ are continuous and, moreover, $QN(\bar{\Omega}\times[0,1])$,

$K_P(I-Q)N(\bar{\Omega}\times[0,1])$ are relatively compact for any open bounded set $\Omega \subset X$. Hence, N is L–compact on $\bar{\Omega}$. Here Ω is any open bounded set in X. Now we reach the position to search for an appropriate open bounded subset X for the application of Lemma 1. The corresponding differential equation for the operator $Lx = \lambda N(x(t),\lambda)$, $\lambda \in (0,1)$, takes the form

$$\dot{x}(t) = \lambda^2\beta x'(t-\tau(t)) + \lambda f(x(t-\tau(t))) + \lambda^2 g(u(t-\sigma(t))) + \lambda^2 p(t) \tag{6}$$

Let $x \in X$ be a solution of (6) for a certain $\lambda \in (0,1)$. By integrating (6) over the interval $[0,T]$, we obtain

$$\int_0^T \left[\lambda\beta x'(t-\tau(t)) + f(x(t-\tau(t))) + \lambda g(u(t-\sigma(t))) + \lambda p(t)\right]dt = 0, \tag{7}$$

Thus, there is a point $\xi \in [0,T]$, such that

$$\lambda\beta x'(\xi-\tau(\xi)) + \lambda f(x(\xi-\tau(\xi))) + \lambda^2 g(u(\xi-\sigma(\xi))) + \lambda p(\xi) = 0. \tag{8}$$

Moreover, in view of (H1) and (8)

$$\begin{aligned} |f(x(\xi-\tau(\xi)))| &\le \beta\left|x'(\xi-\tau(\xi))\right| + \lambda^2|g(u(\xi-\sigma(\xi)))| + \lambda|p(\xi)| \\ &\le \beta\left[\sup_{\xi\in[0,T]}\frac{1}{|1-\tau'(\xi)|}\right]\left|(1-\tau'(\xi))x'(\xi-\tau(\xi))\right| \\ &\quad+\lambda^2|g(u(\xi-\sigma(\xi)))| + \lambda|p(\xi)| \\ &\le \beta R\,|\dot{x}(\xi-\tau(\xi))| + |g(u(\xi-\sigma(\xi)))| + |p(\xi)| \\ &\le \beta R\,\|\dot{x}\| + K + \|p\|, \end{aligned} \tag{9}$$

We shall prove that there is a point $t^* \in [0,T]$ such that

$$|x(t^*)| \le \|\dot{x}\|.$$

Otherwise, if $|x(\xi-\tau(\xi))| \geq M$ and any $u \in X$. Conditions (H1), (H2) and (9) ensure that

$$\beta R\,|x(\xi-\tau(\xi))| + K + \|p\| < |f(x(\xi-\tau(\xi)))| \leq \beta R\,\|\dot{x}\| + K + \|p\|.$$

So that,

$$|x(\xi-\tau(\xi))| \leq \|\dot{x}\|.$$

Denote $\xi-\tau(\xi) = t^* + kT$, $t^* \in [0,T]$ with k being an integer. Then,

$$|x(\xi-\tau(\xi))| = |x(t^*+kT)| = |x(t^*)| \leq \|\dot{x}\|, \tag{10}$$

so, from (10) we have

$$\|x\| \leq |x(t^*)| + \int_0^T |\dot{x}(t)|\,dt < \|\dot{x}\| + \int_0^T |\dot{x}(t)|\,dt < \|\dot{x}\|(T+1). \tag{11}$$

For such a small $\varepsilon > 0$, in view of assumption (H3), we find that there must be a constant $D > M$, which is independent of x, u and λ, such that

$$\frac{f(x)}{x} \leq (H+\varepsilon), \text{ for all } x \in X \text{ and } u \in X.$$

Now let

$$\begin{aligned} E_1 &= \{t : t \in [0,T],\ x(t-\tau(t)) > D\}, \\ E_2 &= \{t : t \in [0,T],\ x(t-\tau(t)) < -D\}, \\ E_3 &= \{t : t \in [0,T],\ |x(t-\tau(t))| \leq D\}, \end{aligned}$$

and

$$f_D = \sup\{f(x) : \|x\| \leq D\}.$$

From (7) and using condition (H1), we have

$$\begin{aligned} \int_0^T f(x(t-\tau(t)))\,dt &= \left(\int_{E_1} + \int_{E_2} + \int_{E_3}\right) f(x(t-\tau(t)))\,dt \\ &\leq \beta \int_0^T |x'(t-\tau(t))|\,dt + \int_0^T |g(u(t-\sigma(t)))|\,dt + \int_0^T |p(t)|\,dt \\ &\leq \beta \int_0^T \left|\frac{1}{1-\tau'(t)}\right| |\dot{x}(t-\tau(t))|\,dt + KT + T\|p\| \\ &\leq \beta \int_{-\tau(0)}^{T-\tau(T)} \left|\frac{1}{1-\tau'(r(s))}\right| |\dot{x}(s)|\,ds + KT + T\|p\| \\ &\leq \beta \left(\sup_{t\in[-\tau(0),T-\tau(T)]} \frac{1}{[1-\tau'(r(t))]^2}\right) \int_0^T |\dot{x}(t)|\,dt + KT + T\|p\| \\ &\leq \beta R_0 \int_0^T |\dot{x}(t)|\,dt + T(K+\|p\|), \end{aligned}$$

which implies that

$$\int_0^T f(x(t-\tau(t)))\,dt \leq \beta R_0 \int_0^T |\dot{x}(t)|\,dt + T(K+\|p\|). \tag{12}$$

One can deduce from (12)

$$\begin{aligned} \int_{E_1} |f(x(t-\tau(t)))|\,dt &\leq \left(\int_{E_2} + \int_{E_3}\right) f(x(t-\tau(t)))\,dt + \beta R_0 \int_0^T |\dot{x}(t)|\,dt + T(K+\|p\|) \\ &\leq T(H+\varepsilon)\|x\| + Tf_D + \beta R_0 \int_0^T |\dot{x}(t)|\,dt + T(K+\|p\|) \end{aligned} \tag{13}$$

Thus from (6), (11), (12) and (13), we have

$$\int_0^T |\dot{x}(t)|\,dt \le \beta \int_0^T |x'(t-\tau(t))|\,dt + \int_0^T |f(x(t-\tau(t)))|\,dt + \int_0^T |g(u(t-\tau(t)))|\,dt + \int_0^T |p(t)|\,dt$$
$$\le \beta R_0 \int_0^T |\dot{x}(t)|\,dt + \int_{E_1} |f(x(t-\tau(t)))|\,dt + \left(\int_{E_2} + \int_{E_3}\right) |f(x(t-\tau(t)))|\,dt + T(K+\|p\|)$$
$$\le \beta R_0 \int_0^T |\dot{x}(t)|\,dt + \int_{E_1} |f(x(t-\tau(t)))|\,dt + T(H+\varepsilon)\|x\| + Tf_D + T(K+\|p\|)$$
$$\le 2\beta R_0 \int_0^T |\dot{x}(t)|\,dt + 2\left[T(H+\varepsilon)\|x\| + Tf_D + T(K+\|p\|)\right],$$

we deduce that

$$\|\dot{x}\|\, T\left(1-2\beta R_0 - 2(H+\varepsilon)(T+1)\right) \le T\left[2f_D + K + \|p\|\right]$$

that is

$$\|\dot{x}\| \le \frac{[2f_D + K + \|p\|]}{(1-2[\beta R_0 + (H+\varepsilon)(T+1)])} := J_1. \tag{14}$$

Substituting (14) in (11), we obtain

$$\|x\| \le \frac{(T+1)[2f_D + K + \|p\|]}{(1-2[\beta R_0 + (H+\varepsilon)(T+1)])} := J_2. \tag{15}$$

Take

$$\begin{aligned} J &= \max\{J_1, J_2\} \\ \Omega_1 &= \{x \in X |\ |x|_0 \le J\}. \end{aligned}$$

Notice first that Ω_1 is a closed convex bounded subset of a Banach space. So Ω_1 satisfies the condition (a) of the Lemma 1. When $x \in \partial\Omega_1 \cap \ker L = \partial\Omega_1 \cap \mathbb{R}$, x is a constant in $\mathbb{R}$ with $|x|_0 = J$. Then,

$$\begin{aligned} (QN)(x,0) &= -\frac{1}{T}\int_0^T f(x(t-\tau(t)))\,dt \\ &= -\frac{1}{T}\int_0^T f(\pm J)\,dt \ne 0. \end{aligned}$$

Finally, consider the mapping

$$\Psi(x,\lambda) = \theta x + \frac{1-\theta}{T}\int_0^T f(x)\,dt,\ \theta \in [0,T].$$

Since for every $\theta \in [0,1]$ and x in the intersection of $\ker L$ and $\partial\Omega_1$, we have

$$x\Psi(x,\theta) = \theta x^2 + \frac{(1-\theta)}{T}\int_0^T x f(x)\,dt > 0.$$

It follows from the property of invariance under a homotopy that

$$\begin{aligned} \deg\{QN(x,0), \Omega_1 \cap \ker L, 0\} &= \deg\{-f(x), \Omega_1 \cap \ker L, 0\} \\ &= \deg\{-x, \Omega_1 \cap \ker L, 0\} \\ &= \deg\{-x, \Omega_1 \cap \mathbb{R}, 0\} \ne 0. \end{aligned}$$

We know that Ω_1 verifies all the requirements of Lemma 1. Then (1) has at least one T–periodic solution $x \in \Omega_1$. The proof is complete. □

Lemma 3. *Suppose the conditions (H1) (H2) of Lemma 2 and (H0) hold. Suppose further that*

(H4) there exists a positive constant $Q > 0$ such that

$$\lim_{x\to+\infty}\frac{f(x)}{x}\leq Q.$$

Then, the equation (1) has at least one T–periodic solution if $1>2\beta R_0+2H(T+1)$.

Proof. By straightforward modification of the proof of Lemma 2 we may apply the argument in the proof of Lemma 2 to obtain Lemma 3. □

It is obvious that the existence problem of T–periodic solution of (1)-(2) is equivalent to that of T–periodic solutions of the equation (2).

In fact equation (1) has a T–periodic solution for all T-periodic function $u\in X$. So in this connection we offer existence criteria for the periodic solutions of the (2).

Next recall that the problem of existence T–periodic solution with feedback control system (1)–(2), then from the results of the previous sections we derive what follows

$$\frac{d}{dt}u(t)=-a(t)u(t)+\frac{d}{dt}F(t,u(t-\sigma(t)))+c(t)G(t,x(t-\tau(t)),u(t-\sigma(t)))$$

In this part we use a different method which relies on the hybrid theorem of Krasnoselskii to establish the existence of periodic solutions of (2). To get round this, we will introduce some notations to simplify notations. We ask that $a(t)$ satisfies the average condition

$$\int_0^T a(v)\,dv>0, \tag{16}$$

Define the function Φ by

$$\Phi(t,s):=\frac{e^{-\int_s^t a(v)dv}}{1-e^{-\int_0^T a(v)dv}},\ t\in[0,T],s\in[0,t]. \tag{17}$$

In addition to (16)–(17), suppose

$$m:=\frac{\mu}{1-\mu}\leq\Phi(t,s)\leq\frac{1}{1-\mu}:=M, \tag{18}$$

where

$$\mu=\exp\left(-\int_0^T a(v)\,dv\right).$$

Assume also that the functions $a(t),c(t),\sigma(t),F(t,u)$ and $G(t,x,u)$ are continuous and periodic in t with period T, that is,

$$a(t+T)=a(t),\ c(t+T)=c(t),\ \ \sigma(t+T)=\sigma(t), \tag{19}$$

and

$$F(t,u)=F(t+T,u),G(t,x,u)=G(t+T,x,u). \tag{20}$$

Recall that the Equation (2) can be rewritten as

$$\begin{aligned}\frac{d}{dt}\left[u(t)-F(t,u(t-\sigma(t)))\right]&=-a(t)\left[u(t)-F(t,u(t-\sigma(t)))\right]-a(t)F(t,u(t-\sigma(t)))\\&\quad+c(t)G(t,x(t-\tau(t)),u(t-\sigma(t))).\end{aligned} \tag{21}$$

Let $u\in X$ be a solution of (2), multiply both sides of (21) with $e^{\int_0^t a(s)ds}$ and then integrate from $t-T$ to t to obtain

$$\begin{aligned}\int_{t-T}^{t}\left\{\left[u(s)-F(s,u(s-\sigma(s)))\right]e^{\int_0^s a(v)dv}\right\}'ds&=-\int_{t-T}^{t}e^{\int_0^s a(v)dv}\left[a(s)F(s,u(s-\sigma(s)))\right]ds\\&\quad+\int_{t-T}^{t}e^{\int_0^s a(v)dv}c(s)G(s,x(s-\tau(s)),u(s-\sigma(s)))\,ds.\end{aligned}$$

Which implies that

$$\begin{aligned}&[u(t)-F(t,u(t-\sigma(t)))]\left[e^{\int_0^t a(s)ds}-e^{\int_0^{t-T}a(s)ds}\right]\\&=\int_{t-T}^{t}e^{\int_0^s a(v)dv}\left[c(s)G(s,x(s-\tau(s)),u(s-\sigma(s)))-a(s)F(s,u(s-\sigma(s)))\right]ds.\end{aligned}$$

Dividing both sides of the above equation by $e^{\int_0^t a(s)ds}$ and due to the fact that $u(t)=u(t-T)$ and by conditions (19) and (20) we conclude that that the solution of (2) is given by

$$u(t)=F(t,u(t-\sigma(t)))+\int_{t-T}^{t}\Phi(t,s)\left[c(s)G(s,x(s-\tau(s)),u(s-\sigma(s)))-a(s)F(s,u(s-\sigma(s)))\right]ds. \tag{22}$$

Now, define a mapping A by

$$(A\varphi)(t)=F(t,\varphi(t-\sigma(t)))+\int_{t-T}^{t}\Phi(t,s)\left[c(s)G(s,x(s-\tau(s)),\varphi(s-\sigma(s)))-a(s)F(s,\varphi(s-\sigma(s)))\right]ds. \tag{23}$$

Due to the periodicity conditions (19) and (20), one can easily check that $(A\varphi)(t)$ is periodic of T. Lastly in this section, we state the Krasnoselskii Fixed point theorem which enables us to prove the existence of positive periodic solutions to (2). For the proof of the Krasnoselskii fixed point theorem we refer the reader to [22].

Theorem 4. *Let Ω be a closed bounded convex nonempty subset of a Banach space $(X,\|\cdot\|)$. Suppose that A_1 and A_2 map Ω into itself satisfying*

(i) $x,y\in\Omega$, implies $A_1x+A_2y\in\Omega$
(ii) A_1 is completely continuous,
(iii) A_2 is a contraction mapping.

Then there exists $z\in\Omega$ with $z=A_1z+A_2z$.

To apply Theorem 4, we need to define a Banach space X, a closed convex subset Ω_2 of X and construct two mappings, one is a contraction and the other is compact. So, we let $(X,\|\cdot\|)$ and $\Omega_2=\{\varphi\in X: l\le\varphi\le L\}$, where l is non-negative constant and L is positive constant. We express equation (23) as

$$(A\varphi)(t):=(A_1\varphi)(t)+(A_2\varphi)(t) \tag{24}$$

where $A_1,A_2:\Omega_2\to X$ are defined as follows

$$(A_1\varphi)(t):=\int_{t-T}^{t}\Phi(t,s)\left[c(s)G(s,x(s-\tau(s)),\varphi(s-\sigma(s)))-a(s)F(s,\varphi(s-\sigma(s)))\right]ds, \tag{25}$$

and

$$(A_2\varphi)(t):=F(t,\varphi(t-\sigma(t))). \tag{26}$$

Comparing (24) to (22), it is easy to see that the existence of periodic solutions for (2) is equivalent to the existence of solutions $u\in\Omega_2$ for the operator equation

$$u=A_1u+A_2u.$$

In this section we obtain the existence of a periodic solution of (2) by considering the two cases; $F(t,u)\ge 0$ and $F(t,u)\le 0$ for all $t\in\mathbb{R}$, $u\in\Omega_2$. We assume that function $F(t,u)$ is locally Lipschitz continuous in u. That is, there exists a positive constant α such that

$$|F(t,u)-F(t,y)|\le\alpha\|u-y\| \text{ for all } t\in[0,T],\, u,y\in\Omega_2. \tag{27}$$

In the case $F(t,u) \geq 0$, we assume that there exist a non-negative constant k_1 and positive constant k_2 such that

$$k_1 u \leq F(t,u) \leq k_2 u \text{ for all } t \in [0,T], u \in \Omega_2, \tag{28}$$

$$k_2 < 1, \tag{29}$$

and for all $u \in \Omega_2$, $\|x\| \leq J$

$$\frac{l(1-k_1)}{Tm} \leq c(t) G(t,x,u) - a(t) F(t,u) \leq \frac{L(1-k_2)}{MT} \text{ for all } t \in [0,T]. \tag{30}$$

Lemma 5. *Suppose that the conditions (19), (20) and (28)–(30) hold. Then $A_1 : \Omega_2 \to X$ is completely continuous.*

Proof. Let $\varphi \in \Omega_2$. Obviously, $A_1\varphi$ is continuous and it is easy to show that $(A_1\varphi)(t+T) = (A_1\varphi)(t)$. For $t \in [0,T]$ and for $\varphi \in \Omega_2$, we have

$$\begin{aligned} |(A_1\varphi)(t)| &= \left| \int_{t-T}^{t} \Phi(t,s) [c(s) G(s, x(s-\tau(s)), \varphi(s-\sigma(s))) - a(s) F(s, \varphi(s-\sigma(s)))] \, ds \right| \\ &\leq TM \frac{L(1-k_2)}{TM} = L(1-k_2). \end{aligned}$$

Thus from the estimation of $|(A_1\varphi)(t)|$ we have

$$|(A_1\varphi)(t)| \leq L(1-k_2).$$

This shows that $A_1(\Omega_2)$ is uniformly bounded. To show that $A_1(\Omega_2)$ is equicontinuous, let $\{\varphi_n\} \in \Omega_2$ where n is a positive integer. Next we calculate $\frac{d}{dt}(A_1\varphi_n)$ and show that it is uniformly bounded. By making use of (19) and (20) we obtain by taking the derivative in (25) that

$$\begin{aligned} \frac{d}{dt}(A_1\varphi_n)(t) &= -a(t)(A_1\varphi_n)(t) \\ &\quad + [\Phi(t,t) - \Phi(t,t-T)] [c(t) G(t, x(t-\tau(t)), \varphi_n(t-\sigma(t))) - a(t) F(t, \varphi_n(t-\sigma(t)))]. \end{aligned}$$

Consequently, by invoking (30), we obtain

$$\begin{aligned} \left| \frac{d}{dt}(A_1\varphi_n)(t) \right| &\leq \|a\| L(1-k_2) + \left| \frac{1}{1-\mu} - \frac{\mu}{1-\mu} \right| \frac{L(1-k_2)}{MT} \\ &\leq \|a\| L(1-k_2) + \frac{L(1-k_2)}{MT} \leq \xi, \end{aligned}$$

for some positive constant ξ. Hence the sequence $(A_1\varphi_n)$ is equicontinuous. The Ascoli-Arzela theorem implies that a subsequence $(A_1\varphi_{n_k})$ of $(A_1\varphi_n)$ converges uniformly to a continuous T–periodic function. Thus A_1 is continuous and $A_1(\Omega_2)$ is contained in a compact subset of X. □

Lemma 6. *Suppose that (27) holds. If A_2 is given by (26) with*

$$\alpha < 1. \tag{31}$$

Then $A_2 : \Omega_2 \to X$ is a contraction.

Proof. Let A_2 be defined by (26). Obviously, $A_2\varphi$ is continuous and it is easy to show that $(A_2\varphi)(t+T) = (A_2\varphi)(t)$. So, for any $\phi, \varphi \in \Omega_2$ we have

$$\begin{aligned} |(A_2\varphi)(t) - (A_2\phi)(t)| &\leq |F(t, \phi(t-\sigma(t))) - F(t, \varphi(t-\sigma(t)))| \\ &\leq \alpha \|\phi - \varphi\|. \end{aligned}$$

This yields $\|A_2\varphi - A_2\phi\| \leq \alpha \|\phi - \varphi\|$. Thus $A_2 : \Omega_2 \to X$ is a contraction by (31). □

Lemma 7. *Under the hypotheses of Lemmas 5 and 6. Then, equation (2) has at least one positive T–periodic solution u in the subset Ω_2 of X.*

Proof. By Lemma 5, the operator $A_1 : \Omega_2 \to X$ is compact and continuous. Also, from Lemma 6, the operator $A_2 : \Omega_2 \to X$ is a contraction. Moreover, let $\phi, \varphi \in \Omega_2$ then

$$\begin{aligned}(A_1\varphi)(t) + (A_2\phi)(t) &= \int_{t-T}^{t} \Phi(t,s)\left[c(s)\,G(s,x(s-\tau(s)),\varphi(s-\sigma(s))) - a(s)\,F(s,\varphi(s-\sigma(s)))\right] ds \\ &\quad + F(t,\phi(t-\sigma(t))) \\ &\le L(1-k_2) + k_2\|\phi\| \le L(1-k_2) + k_2 L = L\end{aligned}$$

On the other hand,

$$\begin{aligned}(A_1\varphi)(t) + (A_2\phi)(t) &= \int_{t-T}^{t} \Phi(t,s)\left[c(s)\,G(s,x(s-\tau(s)),\varphi(s-\sigma(s))) - a(s)\,F(s,\varphi(s-\sigma(s)))\right] ds \\ &\quad + F(t,\phi(t-\sigma(t))) \\ &\ge l(1-k_1) + k_1 l = l.\end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii theorem are satisfied. Thus there exists a fixed point $u \in \Omega_2$ such that $u = A_1u + A_2u$. By (21), (22) and (23) we claim that this fixed point is a solution of (2) for all continuous T–periodic functions $\|x\| \le J$. □

In the case $F(t,u) \le 0$, we substitute conditions (28)-(30) with the following conditions respectively. We assume that there exist a negative constant k_3 and a non-positive constant k_4 such that

$$k_3 u \le F(t,u) \le k_4 u \text{ for all } t \in [0,T], u \in \Omega \tag{32}$$

$$-k_3 < 1 \tag{33}$$

and for all $u \in \Omega_2$ and $\|x\| \le J$

$$\frac{l - k_3 L}{Tm} \le c(t)\,G(t,x,u) - a(t)\,F(t,u) \le \frac{L - k_4 l}{MT} \text{ for all } t \in [0,T], \tag{34}$$

Note that the proof of Lemmas 7 and 8 differ only by conditions (16)–(20) and (32)–(33). So the treatment is the same as in the first case. So, we have the following lemma which can be proved by a similar argumentation.

Lemma 8. *Assume that (16)–(20) and (32)–(33) hold. Then equation (2) has a positive T–periodic solution $u \in \Omega_2$.*

Theorem 9. *Assume that either all hypotheses of Lemmas 2 and 7 or, 2 and 8 or, 3 and 7 or, 3 and 8 hold true. Then, system (1)–(2) has at least one T–periodic solution $(x(t), u(t)) \in \Omega_1 \times \Omega_2$.*

Example 1. Suppose that $\dfrac{1}{(6+4(\pi+1))} > \beta > 0$ and let K be a positive constant. Consider the following the neutral differential system equation

$$\frac{dx(t)}{dt} = \beta x'(t-\tau(t)) + f(x(t-\tau(t))) + Ke^{-u^2(t-\sigma(t))} + p(t) \tag{35}$$

$$\frac{du(t)}{dt} = -a(t)\,u(t) + \frac{d}{dt}F(t,u(t-\sigma(t))) + c(t)\,G(t,x(t-\tau(t)),u(t-\sigma(t))), \tag{36}$$

where

$$\tau(t) = \sigma(t) = 0.4\cos^2(t),$$

and the functions a, p, c, f, F and G are defined as follows $a(t) = \frac{1}{2}\sin^2(t)$, $p(t) = \frac{1+\cos(2t)}{4}$, $c(t) = \frac{1}{2}\sin^2(t) + \frac{1}{5}$, $F(t, u(t-\sigma(t))) = 0.4\sin(u(t-\sigma(t)))$,

$$G(t, x(t-\tau(t)), u(t-\sigma(t))) = \frac{1}{5}\frac{\cos(u(t-\sigma(t))\,x(t-\tau(t))) + 2}{u^2(t-\sigma(t)) + 1} + \sin u(t-\sigma(t)),$$

and

$$f(x) = \begin{cases} \dfrac{1}{3+2(\pi+1)}x + K + \dfrac{1}{2} + \arctan x, & x > 1, \\ \dfrac{1}{3+2(\pi+1)} + K + \dfrac{1}{2} + \dfrac{\pi}{4}, & |x| \le 1, \\ \dfrac{1}{3+2(\pi+1)}x - K - \dfrac{1}{2} + \arctan x, & x < -1. \end{cases}$$

Then system (35)-(36) has a π-periodic solution.

To show this, we first remark that $H = \dfrac{1}{3+2(\pi+1)}$. A simple calculation yields

$$R_0 = \sup_{t\in[-\tau(0),T-\tau(T)]} \frac{1}{[1-\tau'(r(t))]^2} \le \frac{1}{[1-0.4]^2} = 2.7778 < 2\sqrt{2} \tag{37}$$

$$R = \sup_{t\in[0,\pi]} \left|\frac{1}{1-\tau'(t)}\right| \le \frac{1}{1-0.4} = 1.6667 \le 2. \tag{38}$$

In fact $\beta < \dfrac{1}{6+4(\pi+1)}$ and from (37) we obtain

$$\beta 4\sqrt{2} + \frac{2(\pi+1)}{3+2(\pi+1)} < \frac{4\sqrt{2}}{6+4(\pi+1)} + \frac{2(\pi+1)}{3+2(\pi+1)} = \frac{2\sqrt{2}+2(\pi+1)}{3+2(\pi+1)} < 1.$$

Thus

$$1 > 2\beta R_0 + 2H(\pi+1).$$

Moreover, for all $\|x\| \ge 1$ we have

$$|f(x)| \ge \frac{1}{3+2(\pi+1)}x + K + \frac{1}{2} > K + \|p\| + 2\beta\|x\| \ge K + \|p\| + R\beta\|x\|.$$

Consequently, for a any positive number K, we can choose $J > 0$ so that

$$J := \frac{(\pi+1)\left[3K + \dfrac{2}{3+2(\pi+1)} + \dfrac{\pi+1}{2} + 1\right]}{1 - \left(\beta 4\sqrt{2} + \dfrac{2(\pi+1)}{3+2(\pi+1)}\right)}.$$

On the other hand since the function $\sin(u)$ is a strictly increasing on $[0, \pi/2] \supset [0.1, 1.5]$ we have

$$|F(t,u) - F(t,y)| \le \alpha\|u-y\| \text{ for all } t \in [0,\pi],\, u, y \in \Omega \text{ with } \alpha = 0.4$$

and $0.1u \le F(t,u) \le 0.4u$ with $0.4 = k_2$,$k_1 = 0.1$. Also

$$2.1 \ge m = \frac{\mu}{1-\mu} \ge 2 \text{ and } 3.1 \ge M = \frac{1}{1-\mu} \ge 3$$

where μ is given by $\mu = \exp\left(-\int_0^\pi a(t)\,dt\right) = \exp\left(-\int_0^\pi \frac{1}{2}\sin^2(t)\,dt\right)$.

Let $\Omega_2 = [0.1, 1.5]$. We have

$$k_1 u \le F(t,u) \le k_2 u \text{ for all } t \in [0,\pi],\ u \in \Omega_2,\ k_2 < 1,$$

and for all $u \in \Omega_2$, $\|x\| \leq J$,

$$c(t)\,G(t,x,u) - a(t)\,F(t,u) = 0.4\left(\frac{1}{2}\sin^2(t) + \frac{1}{5}\right)\left(\frac{1}{5}\frac{\cos\left(u\left(t-\sigma(t)\right)x\left(t-\sigma(t)\right)\right)+2}{u^2\left(t-\sigma(t)\right)+1}\right) + 0.4\frac{1}{5}\sin u\left(t-\sigma(t)\right).$$

Thus, we deduce that

$$0.039987 \leq c(t)\,G(t,x,u) - a(t)\,F(t,u) \leq 0.062.$$

Furthermore,

$$\begin{aligned} \frac{l(1-k_1)}{m\pi} &\leq \frac{0.1(1-0.1)}{2\pi} = 0.014324 \\ 0.092413 &= \frac{1.5(1-0.4)}{3.1\pi} \leq \frac{L(1-k_2)}{M\pi}. \end{aligned}$$

These calculations, prove that

$$\frac{l(1-k_1)}{m\pi} \leq c(t)\,G(t,x,u) - a(t)\,F(t,u) \leq \frac{L(1-k_2)}{M\pi}.$$

Thus, under these hypotheses on the system (35)-(36), all the conditions of Theorem 9 are satisfied. Hence, the system (35)-(36) has at least one positive π–periodic solution.

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