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Article

On certain subclasses of p-valent functions with negative coefficients defined by a generalized differential operator

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Abstract: In this article, we introduce new subclasses of normalized analytic functions in the unit disk U, defined by a generalized Raducanu-Orhan differential Operator. Various results are driven including coefficient inequalities, growth and distortion theorem, closure property, δ-neighborhoods, extreme points, radii of close-to-convexity, starlikeness and convexity for these subclasses.

Keywords: Multivalent functions, Raducanu-Orhan differential operator, extreme points, coefficient inequality, closure properties.

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1. Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$.

For a function $f \in \mathcal{A}$, Raducanu and Orhan [1] introduced the following operator:

$$D^0_{\alpha\nu} f(z) = f(z)$$

$$D^1_{\alpha\nu} f(z) = \alpha\nu z^2 f''(z) + (\alpha - \nu) z f'(z) + (1 - \alpha + \nu) f(z)$$

$$D^n_{\alpha\nu} f(z) = D_{\alpha\nu}(D^{n-1}_{\alpha\nu} f(z)), (0 \le \nu \le \alpha \le 1, n \in N). \tag{2}$$

If f is given by (1), then from the definition of the operator $D^n_{\alpha\nu} f$, the Equation (2) can be rewritten as:

$$D^n_{\alpha\nu} f(z) = z + \sum_{k=2}^{\infty} \left[1 + (\alpha\nu k + \alpha - \nu)(k-1)\right]^n a_k z^k, \tag{3}$$

where $(n \in N_0 = N \cup \{0\})$.

Remark 1.
1. When $\alpha = 1, \nu = 0$, we get the Sălăgean differential operator introduced by Sălăgean in [2].
2. When $\nu = 0$, we obtain differential operator defined by Al-Oboudi in [3].

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \qquad (p = 1, 2, 3, \ldots) \tag{4}$$

which are analytic and p-valent in the open unit disk $U = \{z : |z| < 1\}$. We can write the following equalities for the functions $f \in \mathcal{A}_p$:

$$D_{\alpha\nu}^{0,p} f(z) = f(z)$$

$$D_{\alpha\nu}^{1,p} f(z) = \frac{\alpha\nu}{p} z^2 f''(z) + \frac{1}{p}[(1-p)\alpha\nu + \alpha - \nu] z f'(z) + (1-\alpha+\nu) f(z) \tag{5}$$

$$D_{\alpha\nu}^{n,p} f(z) = D_{\alpha\nu}(D_{\alpha\nu}^{n-1} f(z)), \qquad (n \in N = 1,2,3,...) \tag{6}$$

If f is given by Equation (4), then from Equation (5) and Equation (6), we see that

$$D_{\alpha\nu}^{n,p} f(z) = z^p + \sum_{k=p+1}^{\infty} \left[1 + (\alpha\nu k + \alpha - \nu)\left(\frac{k}{p} - 1\right)\right]^n a_k z^k, \quad (n \in N_0 = N \cup \{0\}, p \in N = 1,2,3,\ldots). \tag{7}$$

Remark 2.

1. If $\nu = 0$, $D_{\alpha\nu}^{n,p} f = D_{\alpha,p}^{n} f$ defined by Bulut in [4]
2. If $p = 1$, $D_{\alpha\nu}^{n,p} f = D_{\alpha\nu}^{n} f$ introduced by Raducanu and Orhan in [1]
3. If $p = 1, \alpha = 1, \nu = 0$, $D_{\alpha\nu}^{n,p} f = D^n f$ defined by Sălăgean in [2]
4. If $p = 1, \nu = 0$, $D_{\alpha\nu}^{n,p} f = D_{\alpha}^{n} f$ defined by Al-Oboudi in [3].

Let $\mathcal{T}_p$ denote the subclass of $\mathcal{A}_p$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, \qquad (a_k \geq 0, p = 1,2,3,...). \tag{8}$$

If f is given by Equation (8), then from Equation (5) and Equation (6), we get

$$D_{\alpha\nu}^{n,p} f(z) = z^p - \sum_{k=p+1}^{\infty} \left[1 + (\alpha\nu k + \alpha - \nu)\left(\frac{k}{p} - 1\right)\right]^n a_k z^k, \qquad (n \in N_0) \tag{9}$$

Definition 1. A function $f \in \mathcal{T}_p$ is in the class, $S_p^n(\vartheta, \beta, \gamma, \varphi)$ if and only if

$$\left| \frac{(D_{\alpha\nu}^{n,p} f(z))' - p z^{p-1}}{\vartheta (D_{\alpha\nu}^{n,p} f(z))' + (\beta - \gamma)} \right| < \varphi, (z \in U, n \in N_0) \tag{10}$$

for $0 \leq \nu \leq \alpha \leq 1, 0 \leq \vartheta < 1, 0 \leq \gamma < 1, 0 < \beta \leq 1, 0 < \varphi < 1, p \in N, D_{\alpha\nu}^{n,p} f(z)$ as in (9).

In this paper, basic properties of the class $S_p^n(\vartheta, \beta, \gamma, \varphi)$ are studied such as: coefficient inequalities, growth and distortion theorem, closure property, δ-neighborhoods, extreme points, radii of close-to-convexity, starlikeness and convexity for these subclasses.

Remark 3. If $\nu = 0$, $\vartheta = \alpha$, $\varphi = \mu$, the class $S_p^n(\vartheta, \beta, \gamma, \varphi)$ reduces to the class $R_p^n(\alpha, \beta, \gamma, \mu)$ investigated by Bulut [4]

Definition 2. A function $f \in \mathcal{T}_p$ is in the class $S_p^{n,(\delta_0)}(\vartheta, \beta, \gamma, \varphi)$, if there exists a function $g(z) \in S_p^n(\vartheta, \beta, \gamma, \varphi)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \delta_0 ...(z \in U, 0 \leq \delta_0 < 1)$$

for $0 \leq \vartheta < 1, 0 \leq \gamma < 1, 0 < \beta \leq 1, 0 < \varphi < 1$.

Definition 3. For a function $f \in \mathcal{T}_p$, $\delta \geq 0$, δ-neighborhood of f is defined as:

$$N_\delta^p(f, g) = \left\{ g : g = z^p - \sum_{k=p+1}^{\infty} b_k z^k \in \mathcal{T}_p \text{and} \sum_{k=p+1}^{\infty} k|a_k - b_k| \leq \delta \right\}, \tag{11}$$

in particular, for a function $h \in \mathcal{T}_p$, given by $h(z) = z^p$ $(p \in N)$, we immediately have

$$N_\delta^p(h,g) = \left\{ g : g = z^p - \sum_{k=p+1}^{\infty} b_k z^k \in \mathcal{T}_p, \text{and} \sum_{k=p+1}^{\infty} k|b_k| \leq \delta \right\}. \tag{12}$$

The concept of neighborhoods was first introduced by Goodman [5] and generalized by Ruschewey [6] and Altintas [7] (see also [8,9]).

2. Coefficient inequalities

Theorem 4. *A function $f \in \mathcal{T}_p$ is in the class $S_p^n(\vartheta,\beta,\gamma,\varphi)$ if and only if*

$$\sum_{k=p+1}^{\infty} k\left[1 + (\alpha\nu k + \alpha - \nu)\left(\frac{k}{p} - 1\right)\right]^n (1+\varphi\vartheta)a_k \leq \varphi(\vartheta p + \beta - \gamma), \tag{13}$$

for $0 \leq \nu \leq \alpha \leq 1, 0 \leq \vartheta < 1, 0 \leq \gamma < 1, 0 < \beta \leq 1, 0 < \varphi < 1$, $n \in N_0$, $p \in N$. Furthermore, the result is sharp for the function given as

$$f(z) = z^p - \frac{\varphi(\vartheta p + \beta - \gamma)}{k\left[1 + (\alpha\nu k + \alpha - \nu)\left(\frac{k}{p} - 1\right)\right]^n (1+\varphi\vartheta)} a_k, (k \geq p+1).$$

Proof. Suppose that $f \in S_p^n(\vartheta,\beta,\gamma,\varphi)$, then from inequality (10), we have

$$\begin{aligned}
\left|\frac{(D_{\alpha\nu}^{n,p} f(z))' - pz^{p-1}}{\vartheta(D_{\alpha\nu}^{n,p} f(z))' + (\beta-\gamma)}\right| &= \left|\frac{pz^{p-1} - \sum_{k=p+1}^{\infty} k\left[1 + (\alpha\nu k + \alpha - \nu)\left(\frac{k}{p} - 1\right)\right]^n a_k z^{k-1} - pz^{p-1}}{\vartheta(pz^{p-1} - \sum_{k=p+1}^{\infty} k\left[1 + (\alpha\nu k + \alpha - \nu)\left(\frac{k}{p} - 1\right)\right]^n a_k z^{k-1}) + (\beta-\gamma)}\right| \\
&= \left|\frac{\sum_{k=p+1}^{\infty} k\left[1 + (\alpha\nu k + \alpha - \nu)\left(\frac{k}{p} - 1\right)\right]^n a_k z^{k-1}}{\vartheta(pz^{p-1} - \sum_{k=p+1}^{\infty} k\left[1 + (\alpha\nu k + \alpha - \nu)\left(\frac{k}{p} - 1\right)\right]^n a_k z^{k-1}) + (\beta-\gamma)}\right| \\
&< \varphi, (z \in U, n \in N_0)
\end{aligned}$$

it is well known that $\Re z \leq |z|$, therefore, we obtain

$$\Re\left\{\frac{\sum_{K=p+1}^{\infty} k\left[1 + (\alpha\nu k + \alpha - \nu)\left(\frac{k}{p} - 1\right)\right]^n a_k z^{k-1}}{\vartheta(pz^{p-1} - \sum_{k=p+1}^{\infty} k\left[1 + (\alpha\nu k + \alpha - \nu)\left(\frac{k}{p} - 1\right)\right]^n a_k z^{k-1}) + (\beta-\gamma)}\right\} < \varphi.$$

If we choose z real and let $z \to 1^-$, then we get

$$\sum_{K=p+1}^{\infty} k\left[1 + (\alpha\nu k + \alpha - \nu)\left(\frac{k}{p} - 1\right)\right]^n a_k \leq \varphi\{\vartheta(p - \sum_{k=p+1}^{\infty} k\left[1 + (\alpha\nu k + \alpha - \nu)\left(\frac{k}{p} - 1\right)\right]^n a_k) + (\beta-\gamma)\}$$

which is precisely the assertion (13).

On contrary, suppose that the inequality (13) hold true and let $z \in \delta U = \{z \in C : |z| = 1\}$. Then, from (10), we have

$$\begin{aligned}
&\left|(D_{\alpha\nu}^{n,p} f(z))' - pz^{p-1}\right| - \varphi\left|\vartheta(D_{\alpha\nu}^{n,p} f(z))' + (\beta-\gamma)\right| \leq \sum_{k=p+1}^{\infty} k\left[1 + (\alpha\nu k + \alpha - \nu)\left(\frac{k}{p} - 1\right)\right]^n a_k |z|^{k-1} \\
&-\varphi(\vartheta p + \beta - \gamma) + \varphi\vartheta \sum_{k=p+1}^{\infty} k\left[1 + (\alpha\nu k + \alpha - \nu)\left(\frac{k}{p} - 1\right)\right]^n a_k |z|^{k-1} \\
&= \sum_{k=p+1}^{\infty} k\left[1 + (\alpha\nu k + \alpha - \nu)\left(\frac{k}{p} - 1\right)\right]^n a_k |z|^{k-1} (1+\varphi\vartheta)a_k - \varphi(\vartheta p + \beta - \gamma) \leq 0.
\end{aligned}$$

By maximum modulus theorem, we have $f \in S_p^n(\vartheta,\beta,\gamma,\varphi)$. $\square$

Corollary 5. *If $f \in S_p^n(\vartheta,\beta,\gamma,\varphi)$, then* $a_{p+1} \leq \frac{\varphi(\vartheta p+\beta-\gamma)}{(p+1)\left[1+(\alpha\nu(p+1)+\alpha-\nu)(\frac{1}{p})\right]^n(1+\varphi\vartheta)}$.

3. Growth and distortion theorem

Theorem 6. *For each $f(z) \in S_p^n(\vartheta,\beta,\gamma,\varphi)$, we have*
$|z|^p - \frac{\varphi(\vartheta p+\beta-\gamma)}{\left[1+(\alpha\nu(p+1)+\alpha-\nu)(\frac{1}{p})\right]^n(1+\varphi\vartheta)(p+1)}|z|^{p+1} \leq |f(z)| \leq |z|^p + \frac{\varphi(\vartheta p+\beta-\gamma)}{\left[1+(\alpha\nu(p+1)+\alpha-\nu)(\frac{1}{p})\right]^n(1+\varphi\vartheta)(p+1)}|z|^{p+1}$.

Proof. Let $f(z) \in S_p^n(\vartheta,\beta,\gamma,\varphi), z \in U$, the bound on $f(z)$ is given by

$$|f(z)| \leq |z|^p + |z|^{p+1}\sum_{k=p+1}^{\infty} a_k, z \in U, \tag{14}$$

from Theorem 4, we have

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{\varphi(\vartheta p+\beta-\gamma)}{(p+1)\left[1+(\alpha\nu(p+1)+\alpha-\nu)(\frac{1}{p})\right]^n(1+\varphi\vartheta)}, \tag{15}$$

by using (15) in (14), we obtain

$$|f(z)| \leq |z|^p + \frac{\varphi(\vartheta p+\beta-\gamma)}{(p+1)\left[1+(\alpha\nu(p+1)+\alpha-\nu)(\frac{1}{p})\right]^n(1+\varphi\vartheta)}|z|^{p+1}, \tag{16}$$

again using (15), we have

$$|f(z)| \geq |z|^p - \frac{\varphi(\vartheta p+\beta-\gamma)}{(p+1)\left[1+(\alpha\nu(p+1)+\alpha-\nu)(\frac{1}{p})\right]^n(1+\varphi\vartheta)}|z|^{p+1}. \tag{17}$$

Consequently, combining (16) and (17) we obtain the desired result. $\square$

Theorem 7. *For each $f(z) \in S_p^n(\vartheta,\beta,\gamma,\varphi)$, we have*
$p|z|^{p-1} - \frac{\varphi(\vartheta p+\beta-\gamma)}{\left[1+(\alpha\nu(p+1)+\alpha-\nu)(\frac{1}{p})\right]^n(1+\varphi\vartheta)}|z|^p \leq |f'(z)| \leq p|z|^{p-1} + \frac{\varphi(\vartheta p+\beta-\gamma)}{\left[1+(\alpha\nu(p+1)+\alpha-\nu)(\frac{1}{p})\right]^n(1+\varphi\vartheta)}|z|^p$.

Proof. Let $f(z) \in S_p^n(\vartheta,\beta,\gamma,\varphi), z \in U$, the bound on the derivative of $f(z)$ is given by

$$\left|f'(z)\right| \leq p|z|^{p-1} + (p+1)|z|^p\sum_{k=p+1}^{\infty} a_k, z \in U,$$

and, in the same way as above, we get our desired result. $\square$

4. Closure properties

Theorem 8. *Let the functions*

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, \qquad (a_k \geq 0)$$

$$g(z) = z^p - \sum_{k=p+1}^{\infty} b_k z^k, \qquad (b_k \geq 0),$$

be in the class $S_p^n(\vartheta,\beta,\gamma,\varphi)$. *Then for* $0\leq\lambda\leq 1$, *the function h is defined as*

$$h(z)=(1-\lambda)f(z)+\lambda g(z)=z^p-\sum_{k=p+1}^{\infty}c_kz^k,$$

where $c_k:=(1-\lambda)a_k+\lambda b_k\geq 0$, *is also in* $S_p^n(\vartheta,\beta,\gamma,\varphi)$.

Proof. Suppose that each of the functions f and g is in the class $S_p^n(\vartheta,\beta,\gamma,\varphi)$. Then making use of inequality (13), we have

$$\begin{aligned}&\sum_{k=p+1}^{\infty}k\left[1+(\alpha\nu k+\alpha-\nu)\left(\frac{k}{p}-1\right)\right]^n(1+\varphi\vartheta)c_k\\&=(1-\lambda)\sum_{k=p+1}^{\infty}k\left[1+(\alpha\nu k+\alpha-\nu)\left(\frac{k}{p}-1\right)\right]^n(1+\varphi\vartheta)a_k\\&+\lambda\sum_{k=p+1}^{\infty}k\left[1+(\alpha\nu k+\alpha-\nu)\left(\frac{k}{p}-1\right)\right]^n(1+\varphi\vartheta)b_k\\&\leq(1-\lambda)\varphi(\vartheta p+\beta-\gamma)+\lambda\varphi(\vartheta p+\beta-\gamma)\\&=\varphi(\vartheta p+\beta-\gamma),\end{aligned}$$

which completes the proof. □

5. δ-Neighborhoods

Theorem 9. *If*

$$\delta:=\frac{\varphi(\vartheta p+\beta-\gamma)}{\left[1+(\alpha\nu(p+1)+\alpha-\nu)(\frac{1}{p})\right]^n(1+\varphi\vartheta)},\tag{18}$$

then $S_p^n(\vartheta,\beta,\gamma,\varphi)\subset N_\delta^p(h,g)$.

Proof. For a function $f(z)\in S_p^n(\vartheta,\beta,\gamma,\varphi)$ of the form (8), Theorem 4 immediately yields

$$(p+1)\left[1+(\alpha\nu(p+1)+\alpha-\nu)(\frac{1}{p})\right]^n(1+\varphi\vartheta)\sum_{k=p+1}^{\infty}a_k\leq\varphi(\vartheta p+\beta-\gamma),$$

therefore,

$$\sum_{k=p+1}^{\infty}a_k\leq\frac{\varphi(\vartheta p+\beta-\gamma)}{(p+1)\left[1+(\alpha\nu(p+1)+\alpha-\nu)(\frac{1}{p})\right]^n(1+\varphi\vartheta)}.\tag{19}$$

On the other hand, we also find from (13) that

$$\sum_{k=p+1}^{\infty}ka_k\leq\frac{\varphi(\vartheta p+\beta-\gamma)}{\left[1+(\alpha\nu(p+1)+\alpha-\nu)(\frac{1}{p})\right]^n(1+\varphi\vartheta)},\tag{20}$$

that is

$$\sum_{k=p+1}^{\infty}ka_k\leq\frac{\varphi(\vartheta p+\beta-\gamma)}{\left[1+(\alpha\nu(p+1)+\alpha-\nu)(\frac{1}{p})\right]^n(1+\varphi\vartheta)}:=\delta,\tag{21}$$

which completes the proof. □

Theorem 10. *If $g(z) \in S_p^n(\vartheta, \beta, \gamma, \varphi)$ and*

$$\delta_0 = 1 - \frac{\delta}{p+1} \frac{(p+1)\left[1 + (\alpha\nu(p+1) + \alpha - \nu)(\frac{1}{p})\right]^n (1+\varphi\vartheta)}{(p+1)\left[1 + (\alpha\nu(p+1) + \alpha - \nu)(\frac{1}{p})\right]^n (1+\varphi\vartheta) - \varphi(\vartheta p + \beta - \gamma)}, \tag{22}$$

then $N_\delta^p(f,g) \subset S_p^{n,(\delta_0)}(\vartheta, \beta, \gamma, \varphi)$.

Proof. Suppose that $f \in N_\delta^p(f,g)$, then by Definition 3, we have

$$\sum_{k=p+1}^{\infty} k|a_k - b_k| \leq \delta,$$

which readily implies the coefficient inequality given by

$$\sum_{k=p+1}^{\infty} |a_k - b_k| \leq \frac{\delta}{p+1}(p \in N).$$

Next, since $g \in S_p^n(\vartheta, \beta, \gamma, \varphi)$, we have from inequality (13) that

$$\sum_{k=p+1}^{\infty} b_k \leq \frac{\varphi(\vartheta p + \beta - \gamma)}{(p+1)\left[1 + (\alpha\nu(p+1) + \alpha - \nu)(\frac{1}{p})\right]^n (1+\varphi\vartheta)},$$

so from the definition of the class, we have

$$\begin{aligned} \left|\frac{f(z)}{g(z)} - 1\right| &< \frac{\sum_{k=p+1}^{\infty} |a_k - b_k|}{1 - \sum_{k=p+1}^{\infty} b_k} \\ &\leq \frac{\delta}{p+1} \frac{(p+1)\left[1 + (\alpha\nu(p+1) + \alpha - \nu)(\frac{1}{p})\right]^n (1+\varphi\vartheta)}{(p+1)\left[1 + (\alpha\nu(p+1) + \alpha - \nu)(\frac{1}{p})\right]^n (1+\varphi\vartheta) - \varphi(\vartheta p + \beta - \gamma)} \\ &= 1 - \delta_0, \end{aligned}$$

provided that δ_0 is given precisely by (22). Thus, by the definition, $f \in S_p^{n,\delta_0}(\vartheta, \beta, \gamma, \varphi)$ for δ_0 given by (22), this completes our proof. □

6. Extreme points

Theorem 11. *If $f_p(z) = z^p, f_k(z) = z^p - \frac{\varphi(\vartheta p + \beta - \gamma)}{k\left[1 + (\alpha\nu k + \alpha - \nu)\left(\frac{k}{p} - 1\right)\right]^n (1+\varphi\vartheta)} z^k (k \geq p+1)$ then, $f \in S_p^n(\vartheta, \beta, \gamma, \varphi)$ if and only if it can be expressed in the form $f(z) = \lambda_p f_p(z) + \sum_{k=p+1}^{\infty} \lambda_k f_k(z)$, where $\lambda_k \geq 0$ and λ_p=$1 - \sum_{k=p+1}^{\infty} \lambda_k$.*

Proof. Assume that $f(z) = \lambda_p f_p(z) + \sum_{k=p+1}^{\infty} \lambda_k f_k(z)$, then

$$\begin{aligned} f(z) &= (1 - \sum_{k=p+1}^{\infty} \lambda_k) z^p + \sum_{k=p+1}^{\infty} \lambda_k \left\{ z^p - \frac{\varphi(\vartheta p + \beta - \gamma)}{k\left[1 + (\alpha\nu k + \alpha - \nu)\left(\frac{k}{p} - 1\right)\right]^n (1+\varphi\vartheta)} z^k \right\} \\ &= z^p - \sum_{k=p+1}^{\infty} \lambda_k \left\{ \frac{\varphi(\vartheta p + \beta - \gamma)}{k\left[1 + (\alpha\nu k + \alpha - \nu)\left(\frac{k}{p} - 1\right)\right]^n (1+\varphi\vartheta)} z^k \right\}. \end{aligned}$$

Thus,

$$\sum_{k=p+1}^{\infty} k\left[1+(\alpha\nu k+\alpha-\nu)\left(\frac{k}{p}-1\right)\right]^n (1+\varphi\vartheta)\lambda_k \frac{\varphi(\vartheta p+\beta-\gamma)}{k\left[1+(\alpha\nu k+\alpha-\nu)\left(\frac{k}{p}-1\right)\right]^n(1+\varphi\vartheta)}$$
$$=\varphi(\vartheta p+\beta-\gamma)\sum_{k=p+1}^{\infty}\lambda_k=\varphi(\vartheta p+\beta-\gamma)(1-\lambda_p)\leq\varphi(\vartheta p+\beta-\gamma),$$

which shows that f satisfies condition (13) and therefore,$f \in S_p^n(\vartheta,\beta,\gamma,\varphi)$. Conversely, suppose that $f \in S_p^n(\vartheta,\beta,\gamma,\varphi)$, since

$$a_k \leq \frac{\varphi(\vartheta p+\beta-\gamma)}{k\left[1+(\alpha\nu k+\alpha-\nu)\left(\frac{k}{p}-1\right)\right]^n(1+\varphi\vartheta)}, (k\geq p+1),$$

we may set

$$\lambda_k=\frac{k\left[1+(\alpha\nu k+\alpha-\nu)\left(\frac{k}{p}-1\right)\right]^n(1+\varphi\vartheta)}{\varphi(\vartheta p+\beta-\gamma)}a_k, \text{ and } \lambda_p=1-\sum_{k=p+1}^{\infty}\lambda_k,$$

then we obtain from

$$\begin{aligned}
f(z) &= z^p-\sum_{k=p+1}^{\infty}a_k z^k\\
&= (\lambda_p+\sum_{k=p+1}^{\infty}\lambda_k)z^p-\sum_{k=p+1}^{\infty}\lambda_k\frac{\varphi(\vartheta p+\beta-\gamma)}{k\left[1+(\alpha\nu k+\alpha-\nu)\left(\frac{k}{p}-1\right)\right]^n(1+\varphi\vartheta)}z^k\\
&= \lambda_p z^p+\sum_{k=p+1}^{\infty}\lambda_k(z^p-\frac{\varphi(\vartheta p+\beta-\gamma)}{k\left[1+(\alpha\nu k+\alpha-\nu)\left(\frac{k}{p}-1\right)\right]^n(1+\varphi\vartheta)}z^k)\\
&= \lambda_p z^p+\sum_{k=p+1}^{\infty}\lambda_k f_k(z),
\end{aligned}$$

which completes the proof. □

Corollary 12. *The extreme points of $S_p^n(\vartheta,\beta,\gamma,\varphi)$ are given by*

$$f_p(z)=z^p, f_k(z)=z^p-\frac{\varphi(\vartheta p+\beta-\gamma)}{k\left[1+(\alpha\nu k+\alpha-\nu)\left(\frac{k}{p}-1\right)\right]^n(1+\varphi\vartheta)}z^k(k\geq p+1)$$

7. Radii of close-to-convexity,starlikeness and convexity

A function $f\in\mathcal{T}_p$ is said to be p-valently close-to-convex of order ρ if it satisfies

$$\Re\left\{f'(z)\right\}>\rho$$

for some $\rho(0\leq\rho<p)$ and for all $z\in U$.

Also, a function $f\in\mathcal{T}_p$ is said to be p-valently starlike of order ρ if it satisfies

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\}>\rho,$$

for some $\rho(0\leq\rho<p)$ and for all $z\in U$.

Further, a function $f\in\mathcal{T}_p$ is said to be p-valently convex of order ρ if it satisfies

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\rho,$$

for some $\rho(0\leq\rho<p)$ and for all $z\in U$.

Theorem 13. *If $f \in S_p^n(\vartheta,\beta,\gamma,\varphi)$ then f is p-valently close-to-convex of order ρ in $|z| < r_1(\vartheta,\beta,\gamma,\varphi,\rho)$, where*

$$r_1(\vartheta,\beta,\gamma,\varphi,\rho) = \inf_k \left\{ \frac{\left[1+(\alpha\nu k+\alpha-\nu)\left(\frac{k}{p}-1\right)\right]^n (1+\varphi\vartheta)a_k(p-\rho)}{\varphi(\vartheta p+\beta-\gamma)} \right\}^{\frac{1}{k-p}} \quad k \geq p+1.$$

Proof. It is sufficient to show that $\left|\frac{f'(z)}{z^{p-1}} - p\right| < p-\rho$. Since $\left|\frac{pz^{p-1}-\sum_{k=p+1}^{\infty} ka_K z^{k-1}}{z^{p-1}} - p\right| < p-\rho$, which implies that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq \sum_{k=p+1}^{\infty} ka_k |z|^{k-p} < p-\rho,$$

implies

$$\frac{\sum_{k=p+1}^{\infty} ka_k |z|^{k-p}}{p-\rho} < 1, \tag{23}$$

and by applying the result of Theorem 4, we get

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{\varphi(\vartheta p+\beta-\gamma)}{k\left[1+(\alpha\nu k+\alpha-\nu)\left(\frac{k}{p}-1\right)\right]^n (1+\varphi\vartheta)a_k}.$$

Hence,(23) is true if

$$\frac{k|z|^{k-p}}{p-\rho} \leq \frac{k\left[1+(\alpha\nu k+\alpha-\nu)\left(\frac{k}{p}-1\right)\right]^n (1+\varphi\vartheta)}{\varphi(\vartheta p+\beta-\gamma)}, \tag{24}$$

solving (24) for z we obtain

$$|z| \leq \left\{ \frac{\left[1+(\alpha\nu k+\alpha-\nu)\left(\frac{k}{p}-1\right)\right]^n (1+\varphi\vartheta)(p-\rho)}{\varphi(\vartheta p+\beta-\gamma)} \right\}^{\frac{1}{k-p}}$$

which completes the proof. □

Theorem 14. *If $f \in S_p^n(\vartheta,\beta,\gamma,\varphi)$ then f is p-valently starlike of order ρ in $|z| < r_2(\vartheta,\beta,\gamma,\varphi,\rho)$, where*

$$r_2(\vartheta,\beta,\gamma,\varphi,\rho) = \inf_k \left\{ \frac{k\left[1+(\alpha\nu k+\alpha-\nu)\left(\frac{k}{p}-1\right)\right]^n (1+\varphi\vartheta)(p-\rho)}{\varphi(\vartheta p+\beta-\gamma)(k-\rho)} \right\}^{\frac{1}{k-p}} \quad k \geq p+1.$$

Proof. In order to prove, it suffices to show that $\left|\frac{zf'(z)}{f(z)} - p\right| < p-\rho$.

$$\begin{aligned} \left|\frac{zf'(z)}{f(z)} - p\right| &= \left|\frac{zf'(z)-pf(z)}{f(z)}\right| \\ &= \left|\frac{z(pz^{p-1}-\sum_{k=p+1}^{\infty} ka_k z^{k-1}) - p(z^p - \sum_{k=p+1}^{\infty} a_k z^k)}{z^p - \sum_{k=p+1}^{\infty} a_k z^k}\right| \\ &= \left|\frac{-\sum_{k=p+1}^{\infty}(k-p)a_k z^{k-p}}{1-\sum_{k=p+1}^{\infty} a_k z^{k-p}}\right| \leq \frac{\sum_{k=p+1}^{\infty}(k-p)a_k |z|^{k-p}}{1-\sum_{k=p+1}^{\infty} a_k |z|^{k-p}} < p-\rho, \end{aligned} \tag{25}$$

and by using inequality (13), we get

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{\varphi(\vartheta p+\beta-\gamma)}{k\left[1+(\alpha\nu k+\alpha-\nu)\left(\frac{k}{p}-1\right)\right]^n(1+\varphi\vartheta)a_k},$$

so, (25) holds true if

$$\frac{(k-\rho)\,|z|^{k-\rho}}{p-\rho} \leq \frac{k\left[1+(\alpha\nu k+\alpha-\nu)\left(\frac{k}{p}-1\right)\right]^n(1+\varphi\vartheta)}{\varphi(\vartheta p+\beta-\gamma)},$$

and then f is starlike of order ρ. □

Theorem 15. *If $f \in S_p^n(\vartheta,\beta,\gamma,\varphi)$, then f is p-valently convex of order ρ in $|z| < r_3(\vartheta,\beta,\gamma,\varphi,\rho)$, where*

$$r_3(\vartheta,\beta,\gamma,\varphi,\rho) = \inf_k \left\{ \frac{\left[1+(\alpha\nu k+\alpha-\nu)\left(\frac{k}{p}-1\right)\right]^n(1+\varphi\vartheta)p(p-\rho)}{\varphi(\vartheta p+\beta-\gamma)(k-\rho)} \right\}^{\frac{1}{k-p}} \quad k \geq p+1.$$

Proof. To prove this, it suffices to show that $\left|1+\frac{zf''(z)}{f'(z)}-p\right| < p-\rho$.
Since

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| = \left|\frac{f'(z)+zf''(z)-pf'(z)}{f'(z)}\right|$$
$$= \left|\frac{pz^{p-1}-\sum_{k=p+1}^{\infty}ka_kz^{k-1}+z(p(p-1)z^{p-2}-\sum_{k=p+1}^{\infty}k(k-1)a_kz^{k-2})-p(pz^{p-1}-\sum_{k=p+1}^{\infty}ka_kz^{k-1})}{pz^{p-1}-\sum_{k=p+1}^{\infty}ka_kz^{k-1}}\right| \tag{26}$$

it implies that

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| = \left|\frac{-\sum_{k=p+1}^{\infty}k(k-p)a_kz^{k-p}}{p-\sum_{k=p+1}^{\infty}ka_kz^{k-p}}\right| \leq \frac{\sum_{k=p+1}^{\infty}k(k-p)a_k\,|z|^{k-p}}{p-\sum_{k=p+1}^{\infty}ka_k\,|z|^{k-p}} < p-\rho$$

and by applying the result in Theorem 4, we get

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{\varphi(\vartheta p+\beta-\gamma)}{k\left[1+(\alpha\nu k+\alpha-\nu)\left(\frac{k}{p}-1\right)\right]^n(1+\varphi\vartheta)a_k}$$

so, (26) holds true if

$$\frac{k(k-\rho)\,|z|^{k-p}}{p(p-\rho)} \leq \frac{k\left[1+(\alpha\nu k+\alpha-\nu)\left(\frac{k}{p}-1\right)\right]^n(1+\varphi\vartheta)}{\varphi(\vartheta p+\beta-\gamma)}$$

and then f is convex of order ρ. □

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