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Article

Global well-posedness and analyticity for generalized porous medium equation in critical Fourier-Besov-Morrey spaces

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Abstract: In this paper, we study the generalized porous medium equations with Laplacian and abstract pressure term. By using the Fourier localization argument and the Littlewood-Paley theory, we get global well-posedness results of this equation for small initial data u_0 belonging to the critical Fourier-Besov-Morrey spaces. In addition, we also give the Gevrey class regularity of the solution.

Keywords: Porous medium equation, well-posedness, analyticity, Fourier-Besov-Morrey space.

MSC: 35K55, 74G25,76S05.

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1. Introduction

We investigate the generalized porous medium equation in the whole space $\mathbb{R}^3$,

$$\begin{cases} u_t + \mu\Lambda^\alpha u = \nabla\cdot(u\nabla Pu); & (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \\ u(0,x) = u_0 \;\; x\in\mathbb{R}^3, \end{cases} \tag{1}$$

where $u = u(t,x)$ is a real-valued function, which denotes a density or concentration. The dissipative coefficient $\mu > 0$ corresponds to the viscous case, while $\mu = 0$ corresponds to the inviscid case. The fractional Laplacian operator Λ^α is defined by Fourier transform as $\widehat{\Lambda^\alpha u} = |\xi|^\alpha \hat{u}$, and P is an abstract operator.

The equation (1) was introduced in the first by Zhou *et al.* [1]. In fact, Equation (1) is obtained by adding the fractional dissipative term $\mu\Lambda^\alpha u$ to the continuity equation (PME) $u_t + \nabla\cdot(uV) = 0$ given by Caffarelli and Vázquez [2], where the velocity V derives from a potential, $V = -\nabla p$ and the velocity potential or pressure p is related to u by an abstract operator $p = Pu$ [3].

For $\mu = 0$ and $Pu = (-\Delta)^{-s}u = \Lambda^{-2s}u$, $0 < s < 1$; X. Zhou *et al.* [4] were interested in finding the strong solutions of the equation (1) which becomes the fractional porous medium equation in the Besov spaces $B^\alpha_{p,\infty}$ and they obtained the local solution for any initial data in $B^\alpha_{1,\infty}$. Moreover, in the critical case $s = 1$, the Equation (1) leads to a mean field equation [4,5]. Let's take this opportunity to briefly quote some works on the well-posedness and regularity on those equations such as [4,6] and the references therein.

On the other hand, an another similar model occurs in the aggregation equation, and plays a fundamental role in applied sciences such as physics, biology, chemistry, population dynamics. It describes a collective motion and aggregation phenomena in biology and in mechanics of continuous media [7,8]. In the aggregation equation, the abstract form pressure term Pu can also be represented by convolution with a kernel K as $Pu = K * u$. The typical kernels are the Newton potential $|x|^\gamma$ [9,10], and the exponent potential $-e^{-|x|}$ [11,12]. For more results on this equation, we refer to [13,14] and the references therein.

Recently, Zhou *et al.* [1] obtained the local well-posedness in Besov spaces for large initial data, and proved that the solution becomes global if the initial data is small, also, they studied a blowup criterion for the solution.

In addition, we can represent the Equation (1) with the same initial data by

$$u_t + \mu\Lambda^{\alpha} u + v \cdot \nabla u = -u(\nabla \cdot v);\\ v = -\nabla P u\,. \tag{2}$$

As a consequence, this equation must be compared to the geostrophic model. So, the convective velocity is not absolutely divergence-free for the generalized porous medium equation. Additionally, if we assume that v is divergence-free vector function ($\nabla \cdot v = 0$), the form (2) can contain the quasi-geostrophic (Q-G) equation [15,16].

Inspired by the works [1,17]; the aim of this paper is to prove the well-posedness results of Equation (1) and to give the Gevrey class regularity of the solution in homogeneous Fourier Besov-Morrey spaces under the condition that the abstract operator P is commutative with the operator $e^{-\mu\sqrt{t}|D|^{\frac{\alpha}{2}}}$ and

$$\|\varphi_j \widehat{\nabla P u}\|_{M_p^{\lambda}} \le C 2^{j\sigma} \|\varphi_j \hat{u}\|_{M_p^{\lambda}}\,. \tag{3}$$

Clearly, for the fractional porous medium equation, i.e. $Pu = \Lambda^{-2s} u$, we get $\sigma = 1 - 2s$. If $Pu = K * u$ in the aggregation equation, Wu and Zhang [18] proved a similar result under the condition $\nabla K \in W^{1,1}$, $\alpha \in (0,1)$. Corresponding to their case we give a same result for $\sigma = 0$ when $\nabla K \in L^1$, and also a similar result for $\sigma = 1$ when $K \in L^1$.

Throughout this paper, we use $\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q}$ to denote the homogenous Fourier Besov-Morrey spaces, C will denote constants which can be different at different places, $\mathrm{U} \lesssim \mathrm{V}$ means that there exists a constant $C > 0$ such that $\mathrm{U} \le C\mathrm{V}$, and p' is the conjugate of p satisfying $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 \le p \le \infty$.

2. Preliminaries and main results

We start with a dyadic decomposition of $\mathbb{R}^n$. Suppose $\chi \in C_0^{\infty}(\mathbb{R}^n)$, $\varphi \in C_0^{\infty}(\mathbb{R}^n \setminus \{0\})$ satisfying

$$\operatorname{supp}\chi \subset \left\{\xi \in \mathbb{R}^n : |\xi| \le \frac{4}{3}\right\},\\ \operatorname{supp}\varphi \subset \left\{\xi \in \mathbb{R}^n : \frac{3}{4} \le |\xi| \le \frac{8}{3}\right\},\\ \chi(\xi) + \sum_{j\ge 0}\varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^n,\\ \sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^n\backslash\{0\},$$

and denote $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ and $\mathcal{P}$ the set of all polynomials.

First, we recall the definition of Morrey spaces which are a complement of L^p spaces.

Definition 1 ([19]). For $1 \le p < \infty$, $0 \le \lambda < n$, the Morrey spaces $M_p^{\lambda} = M_p^{\lambda}(\mathbb{R}^n)$ is defined as the set of functions $f \in L^p_{loc}(\mathbb{R}^n)$ such that

$$\|f\|_{M_p^{\lambda}} = \sup_{x_0\in\mathbb{R}^n}\sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x_0,r))} < \infty, \tag{4}$$

where $B(x_0, r)$ denotes the ball in $\mathbb{R}^n$ with center x_0 and radius r.

It is easy to see that the injection $M_{p_1}^{\lambda} \hookrightarrow M_{p_2}^{\mu}$ provided $\frac{n-\mu}{p_2} \ge \frac{n-\lambda}{p_1}$ and $p_2 \le p_1$, and $M_p^0 = L^p$.

If $1 \le p_1, p_2, p_3 < \infty$ and $0 \le \lambda_1,\ \lambda_2,\ \lambda_3 < n$ with $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{\lambda_3}{p_3} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}$, then we have the Hölder type inequality

$$\|fg\|_{M_{p_3}^{\lambda_3}} \le \|f\|_{M_{p_1}^{\lambda_1}} \|g\|_{M_{p_2}^{\lambda_2}}\,.$$

Also, for $1 \le p < \infty$ and $0 \le \lambda < n$,

$$\|\varphi * g\|_{M_p^{\lambda}} \le \|\varphi\|_{L^1} \|g\|_{M_p^{\lambda}}, \tag{5}$$

for all $\varphi \in L^1$ and $g \in \mathrm{M}_p^\lambda$.

Definition 2. (homogeneous Fourier-Besov-Morrey spaces) Let $s \in \mathbb{R}$, $0 \leq \lambda < n$, $1 \leq p < +\infty$ and $1 \leq q \leq +\infty$. The space $\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}(\mathbb{R}^n)$ denotes the set of all $u \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$ such that

$$\|u\|_{\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}(\mathbb{R}^n)} = \Big\{ \sum_{j\in\mathbb{Z}} 2^{jqs} \|\varphi_j \widehat{u}\|^q_{\mathrm{M}_p^\lambda} \Big\}^{1/q} < +\infty, \tag{6}$$

with suitable modification made when $q = \infty$.

Note that the space $\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}(\mathbb{R}^n)$ equipped with the norm (6) is a Banach space. Since $\mathrm{M}_p^0 = L^p$, we have $\mathcal{F}\dot{\mathcal{N}}^s_{p,0,q} = F\dot{B}^s_{p,q}$, $\mathcal{F}\dot{\mathcal{N}}^s_{1,0,q} = F\dot{B}^s_{1,q} = \dot{\mathcal{B}}^s_q$ and $\mathcal{F}\dot{\mathcal{N}}^{-1}_{1,0,1} = \chi^{-1}$ where $\dot{\mathcal{B}}^s_q$ is the Fourier-Herz space and χ^{-1} is the Lei-Lin space [20].

Now, we recall the definition of the mixed space-time spaces.

Definition 3. Let $s \in \mathbb{R}$, $1 \leq p < \infty$, $1 \leq q, \rho \leq \infty$, $0 \leq \lambda < n$, and $I = [0, T)$, $T \in (0, \infty]$. The space-time norm is defined on $u(t, x)$ by

$$\|u(t,x)\|_{\mathcal{L}^\rho(I;\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})} = \Big\{ \sum_{j\in\mathbb{Z}} 2^{jqs} \|\varphi_j \widehat{u}\|^q_{L^\rho(I,\mathrm{M}_p^\lambda)} \Big\}^{1/q},$$

and denote by $\mathcal{L}^\rho(I; \mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})$ the set of distributions in $S'(\mathbb{R} \times \mathbb{R}^n)/\mathcal{P}$ with finite $\|.\|_{\mathcal{L}^\rho(I;\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})}$ norm.

According to Minkowski inequality, we have

$$L^\rho(I; \mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}) \hookrightarrow \mathcal{L}^\rho(I; \mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}), \quad \text{if } \rho \leq q,$$
$$\mathcal{L}^\rho(I; \mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}) \hookrightarrow L^\rho(I; \mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}), \quad \text{if } \rho \geq q,$$

where $\|u(t,x)\|_{L^\rho(I;\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})} := \Big(\int_I \|u(\tau,\cdot)\|^\rho_{\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}} d\tau \Big)^{1/\rho}$.

Our first main result is the following theorem.

Theorem 4. *Assume that the abstract operator P satisfies the condition (3). If $0 \leq \lambda < 3$, $1 \leq q \leq \infty$, $1 \leq p < \infty$ and $\max\{1 + \sigma, 0\} < \alpha < 2 + \frac{3}{p'} + \frac{\lambda}{p} + \sigma$ then there exists a constant C_0 such that for any $u_0 \in \mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}$ satisfies $\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}} < C_0\mu$, the equation(1) admits a unique global solution u,*

$$\|u\|_{\mathcal{L}^\infty\left([0,\infty);\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}\right)} + \mu\|u\|_{\mathcal{L}^1\left([0,\infty);\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}\right)} \leq 2C\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}}$$

where C is a positive constant.

Now, we give some remarks about this result.

Remark 1. The result stated in Theorem 4 is based on the works [3]. In particular, this result remains true if we replace the Fourier-Besov-Morrey space $\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}$ by other functional spaces such as Fourier-Herz space $\dot{\mathcal{B}}^s_q$, Fourier-Besov space $\mathrm{F}\dot{\mathrm{B}}^s_{p,q}$ and Lei-Lin space χ^{-1}.

The analyticity of the solution is also an important subject developed by several researchers, particularly with regard to the Navier-Stokes equations, see [17] and its references. In this paper, we will prove the Gevrey class regularity for (1) in the Fourier-Besov-Morrey space. Inspired by this, we have obtained the following specific results.

Theorem 5. *Let $0 \le \lambda < 3$, $1 \le q \le \infty$, $1 \le p < \infty$ and $\max\{1+\sigma, 0\} < \alpha < \min\{2, 2+\frac{3}{p'}+\frac{\lambda}{p}+\sigma\}$. There exists a constant C_0 such that, if $u_0 \in \mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}$ satisfies $\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}} < C_0\mu$, then the Cauchy problem (1) admits a unique analytic solution u, in the sense that*

$$\|e^{\mu\sqrt{t}|D|^{\frac{\alpha}{2}}}u\|_{\mathcal{L}^\infty\left([0,\infty);\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}\right)} + \mu\|e^{\mu\sqrt{t}|D|^{\frac{\alpha}{2}}}u\|_{\mathcal{L}^1\left([0,\infty);\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}\right)} \le 2C\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}}.$$

We finish this section with a Bernstein type lemma in Fourier variables in Morrey spaces.

Lemma 6 ([21]). *Let $1 \le q \le p < \infty$, $0 \le \lambda_1, \lambda_2 < n$, $\frac{n-\lambda_1}{p} \le \frac{n-\lambda_2}{q}$, and let γ be a multiindex. If $supp(\widehat{f}) \subset \{|\xi| \le A2^j\}$ then there is a constant $C > 0$ independent of f and j such that*

$$\|(i\xi)^\gamma \widehat{f}\|_{M^{\lambda_2}_q} \le C2^{j|\gamma|+j\left(\frac{n-\lambda_2}{q}-\frac{n-\lambda_1}{p}\right)}\|\widehat{f}\|_{M^{\lambda_1}_p}. \tag{7}$$

3. The well-posedness

First, we consider the linear nonhomogeneous dissipative equation

$$\begin{cases} u_t + \mu\Lambda^\alpha u = f(t,x) & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3 \\ u(0,x) = u_0(x) & x \in \mathbb{R}^3, \end{cases} \tag{8}$$

for which we recall the following result.

Lemma 7 ([22]). *Let $I = [0,T)$, $0 < T \le \infty$, $s \in \mathbb{R}$, $0 \le \lambda < 3$, $1 \le p < \infty$, and $1 \le q, \rho \le \infty$. Assume that $u_0 \in \mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}$ and $f \in \mathcal{L}^\rho\left(I;\mathcal{F}\dot{\mathcal{N}}^{s-\alpha+\frac{\alpha}{\rho}}_{p,\lambda,q}\right)$. Then the Cauchy problem (8) has a unique solution $u(t,x)$ such that for all $\rho_1 \in [\rho,+\infty]$*

$$\mu^{\frac{1}{\rho_1}}\|u\|_{\mathcal{L}^{\rho_1}\left(I;\mathcal{F}\dot{\mathcal{N}}^{s+\frac{\alpha}{\rho_1}}_{p,\lambda,q}\right)} \le \left(\frac{4}{3}\right)^\alpha\left(\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}} + \mu^{\frac{1}{\rho}-1}\|f\|_{\mathcal{L}^\rho(I;\mathcal{F}\dot{\mathcal{N}}^{s+\frac{\alpha}{\rho}-\alpha}_{p,\lambda,q})}\right)$$

and

$$\|u\|_{\mathcal{L}^\infty(I;\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})} + \mu\|u\|_{\mathcal{L}^1\left(I;\mathcal{F}\dot{\mathcal{N}}^{s+\alpha}_{p,\lambda,q}\right)} \le (1+\left(\frac{4}{3}\right)^\alpha)\left(\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}} + \|f\|_{\mathcal{L}^1(I;\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})}\right).$$

If in addition q is finite, then u belongs to $\mathcal{C}(I;\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})$.

Proposition 8. *Let $1 \le p < \infty$, $1 \le \rho, q \le \infty$, $1+\sigma < \alpha < \frac{2+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}{2-\frac{1}{\rho}}$, $0 \le \lambda < 3$, $I = [0,T)$, $T \in (0,\infty]$, and set*

$$X = \mathcal{L}^\infty\left(I;\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}\right) \cap \mathcal{L}^\rho\left(I;\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho}+\sigma}_{p,\lambda,q}\right),$$

with the norm

$$\|u\|_X = \|u\|_{\mathcal{L}^\infty\left(I;\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}\right)} + \mu\|u\|_{\mathcal{L}^\rho\left(I;\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho}+\sigma}_{p,\lambda,q}\right)}.$$

There exists a constant $C = C(p,q) > 0$ depending on p, q such that

$$\|u\partial_i Pv\|_{\mathcal{L}^\rho\left(I;\mathcal{F}\dot{\mathcal{N}}^{-2(\alpha-1)+\frac{3}{p'}+\frac{\alpha}{\rho}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}\right)} \le C\mu^{-1}\|u\|_X\|v\|_X. \tag{9}$$

Proof. Let us introduce some notations about the standard localization operators. We set

$$u_j = \dot{\Delta}_j u = \left(\mathscr{F}^{-1}\varphi_j\right) * u, \quad \dot{S}_j u = \sum_{k\le j-1} \dot{\Delta}_k u, \quad \widetilde{\dot{\Delta}}_j u = \sum_{|k-j|\le 1} \dot{\Delta}_k u, \quad \forall j \in \mathbb{Z}.$$

Using the decomposition of Bony's paraproducts for the fixed j, we have

$$\begin{aligned}\dot{\Delta}_j(u\partial_i Pv) &= \sum_{|k-j|\le 4} \dot{\Delta}_j(\dot{S}_{k-1}u\dot{\Delta}_k(\partial_i Pv)) + \sum_{|k-j|\le 4} \dot{\Delta}_j(\dot{S}_{k-1}(\partial_i Pv)\dot{\Delta}_k u) + \sum_{k\ge j-3} \dot{\Delta}_j(\dot{\Delta}_k u\widetilde{\dot{\Delta}}_k(\partial_i Pv)) \\ &= I_j + II_j + III_j.\end{aligned}$$

To prove this proposition, we can write

$$\begin{aligned}\|u\partial_i Pv\|_{\mathcal{L}^\rho\left(I;\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{-2(\alpha-1)+\frac{3}{p'}+\frac{\alpha}{\rho}+\frac{\lambda}{p}+\sigma}\right)} \quad &\lesssim \quad \Big\{\sum_{j\in\mathbb{Z}} 2^{j(-2(\alpha-1)+\frac{3}{p'}+\frac{\alpha}{\rho}+\frac{\lambda}{p}+\sigma)q}\|\widehat{I_j}\|^q_{L^\rho(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q} \\ &+ \quad \Big\{\sum_{j\in\mathbb{Z}} 2^{j(-2(\alpha-1)+\frac{3}{p'}+\frac{\alpha}{\rho}+\frac{\lambda}{p}+\sigma)q}\|\widehat{II_j}\|^q_{L^\rho(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q} \\ &+ \quad \Big\{\sum_{j\in\mathbb{Z}} 2^{j(-2(\alpha-1)+\frac{3}{p'}+\frac{\alpha}{\rho}+\frac{\lambda}{p}+\sigma)q}\|\widehat{III_j}\|^q_{L^\rho(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q}. \qquad (10)\end{aligned}$$

We treat the above three terms differently. First, using Young's inequality (5) in Morrey spaces, and Lemma 6 with $|\gamma| = 0$, we get

$$\begin{aligned}\|\widehat{I_j}\|_{L^\rho(I,\mathrm{M}_p^\lambda)} &\le \sum_{|k-j|\le 4} \|\dot{S}_{k-1}\widehat{u\dot{\Delta}_k(\partial_i Pv)}\|_{L^\rho(I,\mathrm{M}_p^\lambda)} \\ &\le \sum_{|k-j|\le 4} \|\varphi_k\mathcal{F}(\partial_i Pv)\|_{L^\rho(I,\mathrm{M}_p^\lambda)} \sum_{l\le k-2} \|\varphi_l\hat{u}\|_{L^\infty(I,L^1)} \\ &\le \sum_{|k-j|\le 4} \|\varphi_k\mathcal{F}(\partial_i Pv)\|_{L^\rho(I,\mathrm{M}_p^\lambda)} \sum_{l\le k-2} 2^{l(\frac{3}{p'}+\frac{\lambda}{p})}\|\widehat{u}_l\|_{L^\infty(I,\mathrm{M}_p^\lambda)} \\ &\lesssim \sum_{|k-j|\le 4} 2^{k\sigma}\|\widehat{v}_k\|_{L^\rho(I,\mathrm{M}_p^\lambda)}\Big(\sum_{l\le k-2} 2^{l(\alpha-1-\sigma)q'}\Big)^{\frac{1}{q'}}\|u\|_{\mathcal{L}^\infty\left(I;\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}\right)} \\ &\lesssim \sum_{|k-j|\le 4} 2^{k(\alpha-1)}\|\widehat{v}_k\|_{L^\rho(I,\mathrm{M}_p^\lambda)}\|u\|_{\mathcal{L}^\infty\left(I;\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}\right)}.\end{aligned}$$

Multiplying by $2^{j(-2(\alpha-1)+\frac{3}{p'}+\frac{\alpha}{\rho}+\frac{\lambda}{p}+\sigma)}$, and taking l^q-norm of both sides in the above estimate, we obtain

$$\begin{aligned}&\Big\{\sum_{j\in\mathbb{Z}} 2^{j(-2(\alpha-1)+\frac{3}{p'}+\frac{\alpha}{\rho}+\frac{\lambda}{p}+\sigma)q}\|\widehat{I_j}\|^q_{L^\rho(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q} \\ &\lesssim \Big\{\sum_{j\in\mathbb{Z}}\Big(\sum_{|k-j|\le 4} 2^{k(1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho}+\sigma)}2^{(j-k)(-2(\alpha-1)+\frac{3}{p'}+\frac{\alpha}{\rho}+\frac{\lambda}{p}+\sigma)}\|\widehat{v}_k\|_{L^\rho(I,\mathrm{M}_p^\lambda)}\Big)^q\Big\}^{1/q} \times \|u\|_{\mathcal{L}^\infty\left(I;\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}\right)} \\ &\lesssim \|u\|_{\mathcal{L}^\infty\left(I;\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}\right)}\|v\|_{\mathcal{L}^\rho\left(I;\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho}+\sigma}\right)}. \qquad (11)\end{aligned}$$

Likewise, we prove that

$$\Big\{\sum_{j\in\mathbb{Z}} 2^{j(-2(\alpha-1)+\frac{3}{p'}+\frac{\alpha}{\rho}+\frac{\lambda}{p}+\sigma)q}\|\widehat{II_j}\|^q_{L^\rho(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q} \lesssim \|v\|_{\mathcal{L}^\infty\left(I;\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}\right)}\|u\|_{\mathcal{L}^\rho\left(I;\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho}+\sigma}\right)}. \qquad (12)$$

To evaluate III_j, we apply the Young inequality (5) in Morrey spaces and Lemma 6 with $|\gamma| = 0$, we obtain

$$\begin{aligned}
&2^{j(-2(\alpha-1)+\frac{3}{p'}+\frac{\alpha}{\rho}+\frac{\lambda}{p}+\sigma)}\|\widehat{III_j}\|_{L^\rho(I,M_p^\lambda)}\\
&\le 2^{j(-2(\alpha-1)+\frac{3}{p'}+\frac{\alpha}{\rho}+\frac{\lambda}{p}+\sigma)}\sum_{k\ge j-3}\sum_{|l-k|\le 1}\left\|\mathcal{F}(\dot\Delta_k u\dot\Delta_l(\partial_i Pv))\right\|_{L^\rho(I,M_p^\lambda)}\\
&\le 2^{j(-2(\alpha-1)+\frac{3}{p'}+\frac{\alpha}{\rho}+\frac{\lambda}{p}+\sigma)}\sum_{k\ge j-3}\sum_{|l-k|\le 1}\left\|\widehat{u}_k\right\|_{L^\rho(I,M_p^\lambda)}\left\|\varphi_l\mathcal{F}(\partial_i Pv)\right\|_{L^\infty(I,L^1)}\\
&\le 2^{j(-2(\alpha-1)+\frac{3}{p'}+\frac{\alpha}{\rho}+\frac{\lambda}{p}+\sigma)}\sum_{k\ge j-3}\sum_{|l-k|\le 1}2^{l(\frac{3}{p'}+\frac{\lambda}{p})}\left\|\widehat{u}_k\right\|_{L^\rho(I,M_p^\lambda)}2^{l\sigma}\left\|\widehat{v}_l\right\|_{L^\infty(I,M_p^\lambda)}\\
&\le \sum_{k\ge j-3}\sum_{l=-1}^{1}2^{(-2(\alpha-1)+\frac{3}{p'}+\frac{\alpha}{\rho}+\frac{\lambda}{p}+\sigma)(j-k)}2^{(\alpha-1)l}\left(2^{(-(\alpha-1)+\frac{3}{p'}+\frac{\alpha}{\rho}+\frac{\lambda}{p}+\sigma)k}\left\|\widehat{u}_k\right\|_{L^\rho(I,M_p^\lambda)}\right)\\
&\times\left(2^{(l+k)(-(\alpha-1)+\frac{3}{p'}+\frac{\lambda}{p}+\sigma)}\left\|\widehat{v}_{l+k}\right\|_{L^\infty(I,M_p^\lambda)}\right).
\end{aligned}$$

Taking the l^q-norm on both sides in the above estimate and using Hölder's inequalities for series with $-2(\alpha-1)+\frac{\alpha}{\rho}+\frac{3}{p'}+\frac{\lambda}{p}+\sigma>0$, we get

$$\begin{aligned}
&\Big(\sum_{j\in\mathbb{Z}}2^{j(-2(\alpha-1)+\frac{3}{p'}+\frac{\alpha}{\rho}+\frac{\lambda}{p}+\sigma)q}\|\widehat{III_j}\|^q_{L^\rho(I,M_p^\lambda)}\Big)^{\frac{1}{q}}\\
&\le \Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{m\le 3}\sum_{l=-1}^{1}2^{(-2(\alpha-1)+\frac{3}{p'}+\frac{\alpha}{\rho}+\frac{\lambda}{p}+\sigma)m}2^{(\alpha-1)l}2^{(-(\alpha-1)+\frac{3}{p'}+\frac{\alpha}{\rho}+\frac{\lambda}{p}+\sigma)(j-m)}\\
&\quad\times\left\|\widehat{u}_{j-m}\right\|_{L^\rho(I,M_p^\lambda)}2^{(-(\alpha-1)+\frac{3}{p'}+\frac{\lambda}{p}+\sigma)(j-m+l)}\left\|\widehat{v}_{j-m+l}\right\|_{L^\infty(I,M_p^\lambda)}\Big)^q\Big)^{\frac{1}{q}}\\
&\le \sum_{l=-1}^{1}\sum_{m\le 3}2^{(-2(\alpha-1)+\frac{3}{p'}+\frac{\alpha}{\rho}+\frac{\lambda}{p}+\sigma)m}2^{(\alpha-1)l}\|u\|_{\mathcal{L}^\rho\left(I;\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-\alpha+\frac{3}{p'}+\frac{\alpha}{\rho}+\frac{\lambda}{p}+\sigma}\right)}\\
&\quad\times\|v\|_{\mathcal{L}^\infty\left(I;\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,\infty}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}\right)}.
\end{aligned}$$

Since $l^q\hookrightarrow l^\infty$, we obtain

$$\Big(\sum_{j\in\mathbb{Z}}2^{j(-2(\alpha-1)+\frac{3}{p'}+\frac{\alpha}{\rho}+\frac{\lambda}{p}+\sigma)q}\|\widehat{III_j}\|^q_{L^\rho(I,M_p^\lambda)}\Big)^{\frac{1}{q}}\lesssim\|u\|_{\mathcal{L}^\rho\left(I;\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-\alpha+\frac{3}{p'}+\frac{\alpha}{\rho}+\frac{\lambda}{p}+\sigma}\right)}\|v\|_{\mathcal{L}^\infty\left(I;\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}\right)}. \tag{13}$$

Estimates (10), (11), (12) and (13) yield (9). □

Lemma 9. *Let X be a Banach space with norm $\|.\|_X$ and $B: X\times X\longmapsto X$ be a bounded bilinear operator satisfying*

$$\|B(u,v)\|_X\le\eta\|u\|_X\|v\|_X$$

for all $u,v\in X$ and a constant $\eta>0$. Then, if $0<\varepsilon<\frac{1}{4\eta}$ and if $y\in X$ such that $\|y\|_X\le\varepsilon$, the equation $x:=y+B(x,x)$ has a solution $\overline{x}$ in X such that $\|\overline{x}\|_X\le 2\varepsilon$. This solution is the only one in the ball $\overline{B}(0,2\varepsilon)$. Moreover, the solution depends continuously on y in the sense: if $\|y'\|_X\le\varepsilon$, $x'=y'+B(x',x')$, and $\|x'\|_X\le 2\varepsilon$, then

$$\|\overline{x}-x'\|_X\le\frac{1}{1-4\varepsilon\eta}\|y-y'\|_X.$$

Proof of theorem 4

Proof. To ensure the existence of global solutions with small initial data, we will use Lemma 9.

In the following, we consider the Banach space

$$X = \mathcal{L}^\infty\left([0,+\infty);\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}\right)\cap\mathcal{L}^1\left([0,+\infty);\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}\right).$$

First, we start with the integral equation

$$\begin{aligned} u &= e^{-\mu t\Lambda^\alpha}u_0 + \int_0^t e^{-\mu(t-\tau)\Lambda^\alpha}\nabla\cdot(u(\tau)\nabla Pu(\tau))d\tau \\ &= e^{-\mu t\Lambda^\alpha}u_0 + B(u,u). \end{aligned} \tag{14}$$

We notice that $B(u,v)$ can be thought as the solution to the heat Equation (8) with $u_0 = 0$ and force $f = \nabla\cdot(u(\tau)\nabla Pv(\tau))$. According to Lemma 7 with $s = 1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma$ and Proposition 8 with $\rho = 1$, we obtain

$$\begin{aligned} \|B(u,v)\|_X &\le \left(1+\left(\frac{4}{3}\right)^\alpha\right)\|\nabla\cdot(u\nabla Pv)\|_{\mathcal{L}^1\left([0,+\infty);\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}\right)} \\ &\le \left(1+\left(\frac{4}{3}\right)^\alpha\right)C\mu^{-1}\|u\|_X\|v\|_X. \end{aligned}$$

By Lemma 9, we know that if $\|e^{-\mu t\Lambda^\alpha}u_0\|_X < R$ with $R = \frac{\mu}{4(1+(\frac{4}{3})^\alpha)C}$ then the equation (14) has a unique solution in $B(0,2R) := \{x\in X : \|x\|_X \le 2R\}$. To prove $\|e^{-\mu t\Lambda^\alpha}u_0\|_X < R$, notice that $e^{-\mu t\Lambda^\alpha}u_0$ is the solution to the dissipative equation with $u_0 = u_0$ and $f = 0$. So, Lemma 7 yields

$$\|e^{-\mu t\Lambda^\alpha}u_0\|_X \le \left(1+\left(\frac{4}{3}\right)^\alpha\right)\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}}. \tag{15}$$

If $\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}} \le C_0\mu$ with $C_0 = \frac{1}{4(1+(\frac{4}{3})^\alpha)^2C}$, then (14) has a unique global solution $u\in X$ satisfying

$$\|u\|_X \le 2\left(1+\left(\frac{4}{3}\right)^\alpha\right)\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}}.$$

□

Proof of theorem 5

Proof. To prove Theorem 5, we note $a(t,x) := e^{\mu\sqrt{t}|D|^{\frac{\alpha}{2}}}u(t,x)$. Using the integral Equation (14), we obtain

$$\begin{aligned} a(t,x) &= e^{\mu(\sqrt{t}|D|^{\frac{\alpha}{2}}-\frac{1}{2}t\Lambda^\alpha)}e^{-\frac{1}{2}\mu t\Lambda^\alpha}u_0 \\ &\quad + \int_0^t e^{\mu[(\sqrt{t}-\sqrt{\tau})|D|^{\frac{\alpha}{2}}-\frac{1}{2}(t-\tau)\Lambda^\alpha]}e^{-\frac{1}{2}\mu(t-\tau)\Lambda^\alpha}e^{\mu\sqrt{\tau}|D|^{\frac{\alpha}{2}}}\nabla\cdot(u\nabla(Pu))d\tau \\ &:= Lu_0 + \widetilde{B}(u,u). \end{aligned}$$

In order to obtain the Gevrey class regularity of the solution, we use Lemma 9. Firstly, we start by estimating the term $Lu_0 = e^{-\frac{1}{2}\mu(\sqrt{t}|D|^{\frac{\alpha}{2}}-1)^2+\frac{\mu}{2}}e^{-\frac{1}{2}\mu t\Lambda^\alpha}u_0$.

Using the Fourier transform, multiplying by φ_j and taking the M_p^λ-norm we obtain

$$\|\varphi_j\widehat{Lu_0}\|_{\mathrm{M}_p^\lambda} \le Ce^{-\frac{1}{2}\mu t2^{j\alpha}(3/4)^\alpha}\|\varphi_j\widehat{u_0}\|_{\mathrm{M}_p^\lambda}.$$

Multiplying by $2^{j(1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma)}$ and taking l^q-norm we get

$$\|Lu_0\|_{\mathcal{L}^\infty\left([0,+\infty);\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}\right)} \leq C\,\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}} .$$

Similarly

$$2^{j(1+\frac{3}{p'}+\frac{\lambda}{p}+\sigma)}\left\|\varphi_j\widehat{Lu_0}\right\|_{L^1\left([0,+\infty);\mathrm{M}_p^\lambda\right)} \leq \left(\int_0^\infty e^{-\frac{1}{2}\mu t 2^{j\alpha}(3/4)^\alpha}2^{j\alpha}dt\right)2^{j(1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma)}\left\|\varphi_j\widehat{u_0}\right\|_{\mathrm{M}_p^\lambda} .$$

We conclude by taking l^q-norm that

$$\mu\,\|Lu_0\|_{\mathcal{L}^1\left([0,+\infty);\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}\right)} \leq C\,\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}} .$$

Finally,

$$\|Lu_0\|_X \leq C\,\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}} .$$

On the other hand, we notice that $\widetilde{B}(u,v)$ as $\widetilde{B}\left(e^{-\mu\sqrt{\tau}|D|^{\frac{\alpha}{2}}}a, e^{-\mu\sqrt{\tau}|D|^{\frac{\alpha}{2}}}b\right)$ with $b := e^{\mu\sqrt{\tau}|D|^{\frac{\alpha}{2}}}v$. Since $e^{\mu[(\sqrt{t}-\sqrt{\tau})|\xi|^{\frac{\alpha}{2}}-\frac{1}{2}(t-\tau)|\xi|^\alpha]}$ is uniformly bounded on $t\in(0,\infty)$ and $\tau\in[0,t]$, it sufficient to consider the estimate of $\|e^{\mu\sqrt{\tau}|D|^{\frac{\alpha}{2}}}u\partial_i(Pv)\|_{\mathcal{L}^1\left(I;\mathcal{F}\dot{\mathcal{N}}^{2-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}\right)}$ for which we prove the flowing lemma.

Lemma 10. *Let* $1\leq p<\infty$, $1\leq q\leq\infty$, $0\leq\lambda<3$, $1+\sigma<\alpha<\min\{2,2+\frac{3}{p'}+\frac{\lambda}{p}+\sigma\}$, $I=[0,T)$, $T\in(0,\infty]$, *and set*

$$X = \mathcal{L}^\infty\left(I;\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}\right)\cap\mathcal{L}^1\left(I;\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}\right) .$$

There exists a constant $C=C(p,q)>0$ *depending on* p,q *such that*

$$\|e^{\mu\sqrt{\tau}|D|^{\frac{\alpha}{2}}}u\partial_i(Pv)\|_{\mathcal{L}^1\left(I;\mathcal{F}\dot{\mathcal{N}}^{2-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}\right)} \leq C\mu^{-1}\|a\|_X\|b\|_X .$$

Proof. Based on the same procedure in the proof of Proposition 8, we evaluate the estimate of $\|e^{\mu\sqrt{\tau}|D|^{\frac{\alpha}{2}}}u\partial_i(Pv)\|_{\mathcal{L}^1\left(I;\mathcal{F}\dot{\mathcal{N}}^{2-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}\right)}$, in fact, we have for fixed j

$$\begin{aligned}
\dot{\Delta}_j e^{\mu\sqrt{\tau}|D|^{\frac{\alpha}{2}}}(u\partial_i(Pv)) &= \sum_{|k-j|\leq 4}\dot{\Delta}_j e^{\mu\sqrt{\tau}|D|^{\frac{\alpha}{2}}}\left(\dot{S}_{k-1}u\dot{\Delta}_k\partial_i(Pv)\right)\\
&+\sum_{|k-j|\leq 4}\dot{\Delta}_j e^{\mu\sqrt{\tau}|D|^{\frac{\alpha}{2}}}\left(\dot{S}_{k-1}\partial_i(Pv)\dot{\Delta}_k u\right)\\
&+\sum_{k\geq j-3}\dot{\Delta}_j e^{\mu\sqrt{\tau}|D|^{\frac{\alpha}{2}}}\left(\dot{\Delta}_k u\widetilde{\Delta}_k\partial_i(Pv)\right)\\
&:= S_{1,j}+S_{2,j}+S_{3,j} .
\end{aligned}$$

Since $e^{\mu\sqrt{\tau}\left(|\xi|^{\frac{\alpha}{2}}-|\xi-\eta|^{\frac{\alpha}{2}}-|\eta|^{\frac{\alpha}{2}}\right)}$ is uniformly bounded on τ when $\alpha\in[0,2]$, we obtain

$$\begin{aligned}
\|\widehat{S_{1,j}}\|_{\mathrm{M}_p^\lambda} &= \|\sum_{|k-j|\le 4}\varphi_j e^{\mu\sqrt{\tau}|\xi|^{\frac{\alpha}{2}}}\mathscr{F}\left(\dot{S}_{k-1}u\dot{\Delta}_k\partial_i(Pv)\right)\|_{\mathrm{M}_p^\lambda}\\
&= \|\sum_{|k-j|\le 4}\varphi_j e^{\mu\sqrt{\tau}|\xi|^{\frac{\alpha}{2}}}\left[\left(\sum_{l\le k-2}e^{-\mu\sqrt{\tau}|\xi|^{\frac{\alpha}{2}}}\widehat{a}_l\right)*\left(e^{-\mu\sqrt{\tau}|\xi|^{\frac{\alpha}{2}}}\mathscr{F}(\dot{\Delta}_k\partial_i(Pb))\right)\right]\|_{\mathrm{M}_p^\lambda}\\
&= \|\sum_{|k-j|\le 4}\varphi_j\int_{\mathbb{R}^3}e^{\mu\sqrt{\tau}\left(|\xi|^{\frac{\alpha}{2}}-|\xi-\eta|^{\frac{\alpha}{2}}-|\eta|^{\frac{\alpha}{2}}\right)}\left(\sum_{l\le k-2}\widehat{a}_l\right)(\xi-\eta)\mathscr{F}(\dot{\Delta}_k\partial_i(Pb))(\eta)d\eta\|_{\mathrm{M}_p^\lambda}\\
&\le C\|\sum_{|k-j|\le 4}\mathscr{F}\left(\dot{S}_{k-1}a\dot{\Delta}_k\partial_i(Pb)\right)\|_{\mathrm{M}_p^\lambda}.
\end{aligned}$$

The same calculus as in Proposition 8 gives

$$\Big\{\sum_{j\in\mathbb{Z}}2^{j(2-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma)q}\|\widehat{S_{1,j}}\|^q_{L^1(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q}\lesssim\|a\|_{\mathcal{L}^\infty\left(I;\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}\right)}\|b\|_{\mathcal{L}^1\left(I;\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}\right)}.$$

Similarly, we show that

$$\Big\{\sum_{j\in\mathbb{Z}}2^{j(2-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma)q}\|\widehat{S_{2,j}}\|^q_{L^1(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q}\lesssim\|b\|_{\mathcal{L}^\infty\left(I;\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}\right)}\|a\|_{\mathcal{L}^1\left(I;\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}\right)}.$$

Similarly,

$$\left\|\widehat{S_{3,j}}\right\|_{\mathrm{M}_p^\lambda}\le\sum_{k\ge j-3}\sum_{|l-k|\le 1}\left\|\mathcal{F}\left(\dot{\Delta}_k a\dot{\Delta}_l\left(\partial_i(Pb)\right)\right)\right\|_{M_p^\lambda}.$$

Using again the same procedure described in the proof of Proposition 8 we obtain

$$\Big\{\sum_{j\in\mathbb{Z}}2^{j(2-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma)q}\|\widehat{S_{3,j}}\|^q_{L^1(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q}\lesssim\|a\|_{\mathcal{L}^\infty\left(I;\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}\right)}\|b\|_{\mathcal{L}^1\left(I;\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}\right)}.$$

Finally,

$$\left\|e^{\mu\sqrt{\tau}|D|^{\frac{\alpha}{2}}}u\partial_i(Pv)\right\|_{\mathcal{L}^1\left(I;\mathcal{F}\dot{\mathcal{N}}^{2-\alpha+\frac{3}{p'}+\frac{\lambda}{p}+\sigma}_{p,\lambda,q}\right)}\le C\mu^{-1}\|a\|_X\|b\|_X.$$

□

To finish the proof of Theorem 5, it is easy to obtain the requested result by repeating the same step in the proof of Theorem 4 and Proposition 8. □

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