New inequalities based on harmonic log-convex functions

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Abstract: Harmonic convexity is very important new class of non-convex functions, it gained prominence in the Theory of Inequalities and Applications as well as in the rest of Mathematics’s branches. The harmonic convexity of a function is the basis for many inequalities in mathematics. Furthermore, harmonic convexity provides an analytic tool to estimate several known definite integrals like \( \int_a^b e^{x^2} \, dx \), \( \int_a^b \sin x \, dx \) and \( \int_a^b \cos x \, dx \) \( \forall n \in \mathbb{N} \), where \( a, b \in (0, \infty) \). In this article, some un-weighted inequalities of Hermite-Hadamard type for harmonic log-convex functions defined on real intervals are given.

Keywords: Harmonic convex functions, Hermite-Hadamard type inequalities, integral inequalities, harmonic log-convex functions.

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1. Introduction

During the investigation of convexity, many researchers founded new classes of functions which are not convex in general. Some of them are the so called harmonic convex functions [1], harmonic \((a, m)\)-convex functions [2], harmonic \((s, m)\)-convex functions [3,4] and harmonic \((p, (s, m))\)-convex functions [5]. For a quick glance on importance of these classes and applications, see [1–19] and references therein.

Definition 1. A function \( f : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) is said to be harmonic convex function on \( I \) if

\[
f \left( \frac{xy}{tx + (1-t)y} \right) \leq tf(y) + (1-t)f(x) \tag{1}
\]

holds for all \( x, y \in I \) and \( t \in [0,1] \). If the inequality is reversed, then \( f \) is said to be harmonic concave.

In [5,20], Baloch et al. and Noor et al. also gave the definition of harmonic log-convex functions as follow:

Definition 2. A function \( f : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow (0, \infty) \) is said to be harmonic log-convex function on \( I \) if

\[
f \left( \frac{xy}{tx + (1-t)y} \right) \leq [f(x)]^{1-t}[f(y)]^t \tag{2}
\]

holds for all \( x, y \in I \) and \( t \in [0,1] \). If the inequality is reversed, then \( f \) is said to be harmonic log-concave.

In [20], Noor et al. proved the following result for harmonic log-convex functions:

Theorem 3. Let \( I \subseteq \mathbb{R} \setminus \{0\} \) be an interval. If \( f : I \rightarrow (0, \infty) \) is harmonic convex function, then

\[
f \left( \frac{2ab}{a+b} \right) \leq \exp \left[ ab \int_a^b \log \left( \frac{f(x)}{x^2} \right) \, dx \right] \leq \sqrt{f(a)f(b)} \tag{3}
\]
for all \(a, b \in \mathbb{I} \) and \(a < b\).

Here, motivated by the above result we study the class of harmonic log-convex functions and present some new inequalities for this class of functions.

2. Main Results

The following result holds.

**Theorem 4.** Let \(f : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow (0, \infty)\) be harmonic log-convex function. Then, for every \(t \in [0, 1]\), we have

\[
\int_a^b f(x) \, dx \geq \int_a^b [f(x)]^{1-t} \left[ \frac{a^2 b^2}{(a + b) x - ab} f\left( \frac{ab x}{(a + b) x - ab} \right) \right]^t \, dx
\]

\[
\geq \begin{cases} 
(1 - 2t)a^2 b^2 \int_{\frac{ab}{a+b} x}^{\frac{ab}{a+b} x-t} \frac{|(a+b)tu-ab|^{2(t-1)}}{|ab-(1-t)(a+b)u|^{2t}} f(u) \, du & \text{if } t \neq \frac{1}{2}; \\
\frac{2ab}{a+b} \ln(\frac{b}{a}) f\left( \frac{2ab}{a+b} \right) & \text{if } t = \frac{1}{2}.
\end{cases}
\]

(4)

**Proof.** The cases \(t = 0, \frac{1}{2}, 1\) are obvious. Assume that \(t \in (0, 1) \setminus \{\frac{1}{2}\}\). By the harmonic log-convexity of \(f\) we have

\[
[f(x)]^{1-t} \left[ f\left( \frac{ab x}{(a + b) x - ab} \right) \right]^t \geq \frac{ab x}{tx + (1-t) \frac{ab}{a+b} x - (2t-1)ab} f\left( \frac{ab x}{(a + b) x - (2t-1)ab} \right)
\]

(5)

for any \(x \in [a, b]\). This allows that

\[
[f(x)]^{1-t} \left[ \frac{a^2 b^2}{(a + b) x - ab} f\left( \frac{ab x}{(a + b) x - ab} \right) \right]^t \geq \frac{a^2 t b^{2t}}{(a + b) x - ab} f\left( \frac{ab x}{(a + b) x - (2t-1)ab} \right).
\]

Integrating the inequality (6) over \(x\) on \([a, b]\), we have

\[
\int_a^b [f(x)]^{1-t} \left[ \frac{a^2 b^2}{(a + b) x - ab} f\left( \frac{ab x}{(a + b) x - ab} \right) \right]^t \, dx \geq \int_a^b \frac{a^2 t b^{2t}}{(a + b) x - ab} f\left( \frac{ab x}{(a + b) x - (2t-1)ab} \right) \, dx.
\]

Since \(t \neq \frac{1}{2}\), then \(u = \frac{ab x}{(a + b) x - (2t-1)ab} = \frac{ab x}{(a + b) x - ab} \) is the change of variable with \(dx = \frac{(1-2t)ab^2}{(a + b) x - ab} \, du\). For \(x = a\), we get \(u = \frac{a}{a+b(1-t)}\) and for \(x = b\), we get \(u = \frac{b}{(1-t)a+b}\). Therefore,

\[
\int_a^b \frac{a^2 t b^{2t}}{(a + b) x - ab} f\left( \frac{ab x}{(a + b) x - (2t-1)ab} \right) \, dx = (1-2t)a^2 b^2 \int_{\frac{ab}{a+b} x}^{\frac{ab}{a+b} x-t} \frac{|(a+b)tu-ab|^{2(t-1)}}{|ab-(1-t)(a+b)u|^{2t}} f(u) \, du,
\]

and hence the second inequality (4) is proved. By the Hölder integral inequality for \(p = \frac{1}{1-t}, q = \frac{1}{t}\), we have

\[
\int_a^b \frac{a^2 b^2}{(a + b) x - ab} f\left( \frac{ab x}{(a + b) x - ab} \right) \, dx
\]

\[
\leq \left( \int_a^b \frac{1}{[f(x)]^{1-t}} \, dx \right)^{1-t} \left( \int_a^b \left[ \frac{a^2 b^2}{(a + b) x - ab} f\left( \frac{ab x}{(a + b) x - ab} \right) \right]^t \, dx \right)^{1\over t}
\]

\[
= \left( \int_a^b f(x) \, dx \right)^{1-t} \left( \int_a^b \frac{a^2 b^2}{(a + b) x - ab} f\left( \frac{ab x}{(a + b) x - ab} \right) \, dx \right)^t
\]

\[
= \left( \int_a^b f(x) \, dx \right)^{1-t} \left( \int_a^b f(x) \, dx \right)^t = \frac{1}{ab} \int_a^b f(x) \, dx.
\]

This proves the first part of inequality (4). 

\(\square\)
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References


