Open Journal of Mathematical Analysis

PSR Press

Article

Exponential growth of solution with L_p-norm for class of non-linear viscoelastic wave equation with distributed delay term for large initial data

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Received: 23 March 2020; Accepted: 13 May 2020; Published: 22 June 2020.

Abstract: In this work, we are concerned with a problem for a viscoelastic wave equation with strong damping, nonlinear source and distributed delay terms. We show the exponential growth of solution with L_p-norm, *i.e.*, $\lim_{t\to\infty} \|u\|_p^p \to \infty$.

Keywords: Strong damping, viscoelasticity, nonlinear source, exponential growth, distributed delay.

MSC: 35L05, 35L20, 58G16, 93D20.

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1. Introduction

The well known *"Growth"* phenomenon is one of the most important phenomena of asymptotic behavior, where many researches omit from its study especially when it comes from the evolution problems. It gives us very important information to know the behavior of equation when time arrives at infinity, it differs from global existence and blow up in both mathematically and in applications point of view. Although the interest of the scientific community for the study of delayed problems is fairly recent, multiple techniques have already been explored in depth.

In this direction, we are concerned with the delayed damped system

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \int_0^t \varpi(t-q)\Delta u(q)dq \\ +\mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(q)| u_t(x, t-q)dq = b|u|^{p-2}.u, \ x \in \Omega, t > 0, \\ u(x,t) = 0, x \in \partial\Omega, \\ u_t(x,-t) = f_0(x,t), \ (x,t) \in \Omega \times (0, \tau_2), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \end{cases} \tag{1}$$

where ω, b, μ_1 are positive constants, $p > 2$ and τ_1, τ_2 are the time delay with $0 \leq \tau_1 < \tau_2$ and μ_2 is bounded function and ϖ is a differentiable function.

It is well known that viscous materials are the opposite of elastic materials which have the capacity to store and dissipate mechanical energy. As the mechanical properties of these viscous substances are of great importance when they appear in many applications of other applied sciences. Many searchers have paid attention to this problem.

In the absence of the strong damping Δu_t, that is for $w = 0$ and in absence of the distributed delay term. Our problem (1) has been investigated by many authors and results on the local/global existence, and stability have been established. See for example [1–4]. In [5], the authors looked into the following system

$$u_{tt} - \Delta u + \int_0^t \varpi(t-s)\Delta u(s)ds + a(x)u_t + |u|^\gamma u = 0, \tag{2}$$

where decay result of an exponential rate was showed.

In [6], Song and Xue considered with the following viscoelastic equation with strong damping:

$$\begin{cases} u_{tt} - \Delta u + \int_0^\infty g(t-s)\Delta u(s)ds - \Delta u_t = |u|^{p-2}.u, \ x \in \Omega, t > 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x). \end{cases} \tag{3}$$

The authors showed, under suitable conditions on g, that there were solutions of (3) with arbitrarily high initial energy that blow up in a finite time. For the same Problem (3), in [7], Song and Zhong showed that there were solutions of (3) with positive initial energy that blew up in finite time. In [8], Zennir considered with the following viscoelastic equation with strong damping:

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \int_0^t g(t-s)\Delta u(s)ds \\ +a|u_t|^{m-2}.u_t = |u|^{p-2}.u, \ x \in \Omega, t > 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \ x \in \Omega \\ u(x,t) = 0, \ x \in \partial\Omega. \end{cases} \tag{4}$$

They proved the exponential growth result under suitable assumptions.

In [9] the authors considered the following problem for a nonlinear viscoelastic wave equation with strong damping, nonlinear damping and source terms

$$\begin{cases} u_{tt} - \Delta u + \int_0^\infty g(s)\Delta u(t-s)ds - \varepsilon_1 \Delta u_t + \varepsilon_2 u_t|u_t|^{m-2} = \varepsilon_3 u|u|^{p-2}, x \in \Omega, t > 0, \\ u(x,t) = 0, \ x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \ x \in \Omega. \end{cases} \tag{5}$$

They proved a blow up result if $p > m$ and established the global existence.

In this article, we investigated Problem (1), in which all the damping mechanism have been considered in the same time, these assumptions make our problem different form those studied in the literature, specially the Exponential Growth of solutions. We will prove that if the initial energy $E(0)$ of our solutions is negative (this means that our initial data are large enough), then our local solutions in bounded and

$$\|u\|_p^p \to \infty, \tag{6}$$

as t tends to ∞, used idea in [10–13].

Our aim in the present work is to extend the existing exponential growth results to strong damping for a viscoelastic problem with distributed delay under the following assumptions:

(A1) $\varpi \in (\mathbb{R}_+, \mathbb{R}_+)$ is decreasing function so that

$$\varpi(t) \geq 0 \quad , 1 - \int_0^\infty \varpi(q)\, dq = l > 0. \tag{7}$$

(A2) There exists a constant $\xi > 0$ such that

$$\varpi'(t) \leq -\xi \varpi(t) \quad , t \geq 0. \tag{8}$$

(A3) $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is bounded function so that

$$\left(\frac{2\delta - 1}{2}\right) \int_{\tau_1}^{\tau_2} |\mu_2(q)| dq \leq \mu_1 \quad , \delta > \frac{1}{2}. \tag{9}$$

2. Main results

First, as in [14], we introduce the new variable

$$y(x, \rho, q, t) = u_t(x, t - q\rho),$$

then we obtain

$$\begin{cases} qy_t(x,\rho,q,t) + y_\rho(x,\rho,q,t) = 0 \\ y(x,0,q,t) = u_t(x,t). \end{cases} \tag{10}$$

Let us denote by

$$(\varpi o u) = \int_\Omega \int_0^t \varpi(t-q)|u(t)-u(q)|^2 dq. \tag{11}$$

Therefore, Problem (1) takes the form

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \int_0^t \varpi(t-q)\Delta u(q)dq + \mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(q)| y(x,1,q,t)dq = b|u|^{p-2}u, \; x \in \Omega, t > 0, \\ qy_t(x,\rho,q,t) + y_\rho(x,\rho,q,t) = 0, \end{cases} \tag{12}$$

with initial and boundary conditions

$$\begin{cases} u(x,t) = 0, \; x \in \partial\Omega, \\ y(x,\rho,q,0) = f_0\left(x, q\rho\right), \\ u\left(x,0\right) = u_0\left(x\right), u_t\left(x,0\right) = u_1\left(x\right), \end{cases} \tag{13}$$

where

$$(x,\rho,q,t) \in \Omega \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty).$$

We state without proof the local existence theorem that can be established by combining arguments of [15].

Theorem 1. *Assume (7), (8) and (9) holds. Let*

$$\begin{cases} 2 < p < \dfrac{2n-2}{n-2}, & n \geq 3; \\ p \geq 2, & n = 1,2. \end{cases} \tag{14}$$

Then for any initial data

$$(u_0, u_1, f_0) \in \mathcal{H} \;\; / \;\; \mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega \times (0,1) \times (\tau_1,\tau_2)),$$

with compact support, Problem (13) has a unique solution

$$u \in C([0,T];\mathcal{H}),$$

for some $T > 0$.

In the next theorem we give the global existence result, its proof based on the potential well depth method in which the concept of so-called stable set appears, where we show that if we restrict our initial data in the stable set, then our local solution obtained is global in time. One can make use of arguments in [16].

Theorem 2. *Suppose that (7), (8), (9) and (14) hold. If $u_0 \in W$, $u_1 \in H_0^1(\Omega)$ and*

$$\frac{bC_*^p}{l}\left(\frac{2p}{(p-2)l}E(0)\right)^{\frac{p-2}{2}} < 1, \tag{15}$$

where C_* is the best Poincare's constant. Then the local solution $u(t,x)$ is global in time.

Lemma 1. *Assume (7), (8), (9) and (14) hold, let $u(t)$ be a solution of (12), then $\mathcal{E}(t)$ is non-increasing, that is*

$$\mathcal{E}(t)=\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\Big(1-\int_0^t \varpi(q)dq\Big)\|\nabla u\|_2^2+\frac{1}{2}(\varpi o\nabla u)+\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} q|\mu_2(q)|y^2(x,\rho,q,t)dqd\rho dx-\frac{b}{p}\|u\|_p^p. \tag{16}$$

satisfies

$$\mathcal{E}(t)\le -c_1\Big(\|u_t\|_2^2+\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(q)|y^2(x,1,q,t)dqdx\Big). \tag{17}$$

Proof. By multiplying the Equation $(12)_1$ by u_t and integrating over Ω, we get

$$\begin{aligned}&\frac{d}{dt}\Big\{\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\Big(1-\int_0^t \varpi(q)dq\Big)\|\nabla u\|_2^2+\frac{1}{2}(\varpi o\nabla u)-\frac{b}{p}\|u\|_p^p\Big\}\\ &=-\mu_1\|u_t\|_2^2-\int_\Omega u_t\int_{\tau_1}^{\tau_2}|\mu_2(q)|y(x,1,q,t)dqdx+\frac{1}{2}(\varpi' o\nabla u)-\frac{1}{2}\varpi(t)\|\nabla u\|_2^2-\omega\|\nabla u_t\|_2^2,\end{aligned} \tag{18}$$

and, we have

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} q|\mu_2(q)|y^2(x,\rho,q,t)dqd\rho dx=-\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}2|\mu_2(q)|yy_\rho dqd\rho dx\\ &=\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(q)|y^2(x,0,q,t)dqdx-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(q)|y^2(x,1,q,t)dqdx\\ &=\frac{1}{2}\Big(\int_{\tau_1}^{\tau_2}|\mu_2(q)dq\Big)\|u_t\|_2^2-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(q)|y^2(x,1,q,t)dqdx.\end{aligned} \tag{19}$$

Then, we get

$$\begin{aligned}&\frac{d}{dt}\mathcal{E}(t)=-\mu_1\|u_t\|_2^2-\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(q)|u_ty(x,1,q,t)dqdx+\frac{1}{2}(\varpi' o\nabla u)\\ &-\frac{1}{2}\varpi(t)\|\nabla u\|_2^2-\omega\|\nabla u_t\|_2^2+\frac{1}{2}\Big(\int_{\tau_1}^{\tau_2}|\mu_2(q)dq\Big)\|u_t\|_2^2-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(q)|y^2(x,1,q,t)dqdx.\end{aligned} \tag{20}$$

By (18) and (19), we get (16). Further using Young's inequality, (7), (8) and (9) in (20), we obtain (17). □

Now we are ready to state and prove our main result. For this purpose, we define

$$\begin{aligned}H(t)&=-\mathcal{E}(t) \qquad (21)\\ &=\frac{b}{p}\|u\|_p^p-\frac{1}{2}\|u_t\|_2^2-\frac{1}{2}(1-\int_0^t\varpi(q)dq)\|\nabla u\|_2^2-\frac{1}{2}(\varpi o\nabla u)-\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}q|\mu_2(q)|y^2(x,\rho,q,t)dqd\rho dx.\end{aligned}$$

Theorem 3. *Suppose that (7)-(9) and (14). Assume further that $E(0)<0$ holds. Then the unique local solution of problem (12) grows exponentially.*

Proof. From (16), we have

$$\mathcal{E}(t)\le\mathcal{E}(0)\le 0. \tag{22}$$

Hence

$$\begin{aligned}H'(t)=-\mathcal{E}'(t)&\ge c_1\Big(\|u_t\|_2^2+\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(q)|y^2(x,1,q,t)dqdx\Big)\\ &\ge c_1\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(q)|y^2(x,1,q,t)dqdx\ge 0,\end{aligned} \tag{23}$$

and

$$0\le H(0)\le H(t)\le\frac{b}{p}\|u\|_p^p. \tag{24}$$

We set

$$\mathcal{K}(t)=H(t)+\varepsilon\int_\Omega uu_t dx+\frac{\varepsilon\mu_1}{2}\int_\Omega u^2dx+\frac{\varepsilon\omega}{2}\int_\Omega(\nabla u)^2dx, \tag{25}$$

where $\varepsilon>0$ to be specified later.

Multiplying $(12)_1$ by u and taking derivative of (25), we obtain

$$\begin{aligned}\mathcal{K}'(t) &= H'(t)+\varepsilon\|u_t\|_2^2+\varepsilon\int_\Omega\nabla u\int_0^t\varpi(t-q)\nabla u(q)dqdx\\ &-\varepsilon\|\nabla u\|_2^2+\varepsilon b\int_\Omega|u|^pdx-\varepsilon\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(q)|uy(x,1,q,t)dqdx.\end{aligned} \tag{26}$$

Using

$$\varepsilon\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(q)|uy(x,1,q,t)dqdx \le \varepsilon\Big\{\delta_1\Big(\int_{\tau_1}^{\tau_2}|\mu_2(q)|dq\Big)\|u\|_2^2+\frac{1}{4\delta_1}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(q)|y^2(x,1,q,t)dqdx\Big\}, \tag{27}$$

and

$$\begin{aligned}\varepsilon\int_0^t\varpi(t-q)dq\int_\Omega\nabla u\nabla u(q)dxdq &= \varepsilon\int_0^t\varpi(t-q)dq\int_\Omega\nabla u(\nabla u(q)-\nabla u(t))dxdq+\varepsilon\int_0^t\varpi(q)dq\|\nabla u\|_2^2\\ &\ge \frac{\varepsilon}{2}\int_0^t\varpi(q)dq\|\nabla u\|_2^2-\frac{\varepsilon}{2}(\varpi o\nabla u).\end{aligned} \tag{28}$$

We obtain, from (26),

$$\begin{aligned}\mathcal{K}'(t) &\ge H'(t)+\varepsilon\|u_t\|_2^2-\varepsilon\Big(1-\frac{1}{2}\int_0^t\varpi(q)dq\Big)\|\nabla u\|_2^2+\varepsilon b\|u\|_p^p\\ &-\varepsilon\delta_1\Big(\int_{\tau_1}^{\tau_2}|\mu_2(q)|dq\Big)\|u\|_2^2-\frac{\varepsilon}{4\delta_1}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(q)|y^2(x,1,q,t)dqdx+\frac{\varepsilon}{2}(\varpi o\nabla u).\end{aligned} \tag{29}$$

Therefore, using (23) and by setting δ_1 so that, $\frac{1}{4\delta_1 c_1}=\kappa$, substituting in (29), we get

$$\begin{aligned}\mathcal{K}'(t) &\ge [1-\varepsilon\kappa]H'(t)+\varepsilon\|u_t\|_2^2-\varepsilon\Big[\Big(1-\frac{1}{2}\int_0^t\varpi(q)dq\Big)\Big]\|\nabla u\|_2^2+\varepsilon b\|u\|_p^p\\ &-\frac{\varepsilon}{4c_1\kappa}\Big(\int_{\tau_1}^{\tau_2}|\mu_2(q)|dq\Big)\|u\|_2^2+\frac{\varepsilon}{2}(\varpi o\nabla u).\end{aligned} \tag{30}$$

For $0<a<1$, from (21)

$$\begin{aligned}\varepsilon b\|u\|_p^p &= \varepsilon p(1-a)H(t)+\frac{\varepsilon p(1-a)}{2}\|u_t\|_2^2+\varepsilon ba\|u\|_p^p\\ &+\frac{\varepsilon p(1-a)}{2}\Big(1-\int_0^t\varpi(q)dq\Big)\|\nabla u\|_2^2+\frac{\varepsilon}{2}p(1-a)(\varpi o\nabla u)\\ &+\frac{\varepsilon p(1-a)}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}q|\mu_2(q)|y^2(x,\rho,q,t)dqd\rho dx.\end{aligned} \tag{31}$$

Substituting in (30), we get

$$\begin{aligned}\mathcal{K}'(t) &\ge [1-\varepsilon\kappa]H'(t)+\varepsilon\Big[\frac{p(1-a)}{2}+1\Big]\|u_t\|_2^2\\ &+\varepsilon\Big[\Big(\frac{p(1-a)}{2}\Big)\Big(1-\int_0^t\varpi(q)dq\Big)-\Big(1-\frac{1}{2}\int_0^t\varpi(q)dq\Big)\Big]\|\nabla u\|_2^2\\ &-\frac{\varepsilon}{4c_1\kappa}\Big(\int_{\tau_1}^{\tau_2}|\mu_2(q)|dq\Big)\|u\|_2^2+\varepsilon p(1-a)H(t)+\varepsilon ba\|u\|_p^p\\ &+\frac{\varepsilon p(1-a)}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}q|\mu_2(q)|y^2(x,\rho,q,t)dqd\rho dx+\frac{\varepsilon}{2}(p(1-a)+1)(\varpi o\nabla u).\end{aligned} \tag{32}$$

Using Poincare's inequality, we obtain

$$\begin{aligned} \mathcal{K}'(t) &\geq [1-\varepsilon\kappa]H'(t)+\varepsilon\Big[\frac{p(1-a)}{2}+1\Big]\|u_t\|_2^2+\frac{\varepsilon}{2}(p(1-a)+1)(\varpi o\nabla u) \\ &+\varepsilon\Big\{\Big(\frac{p(1-a)}{2}-1\Big)-\int_0^t\varpi(q)dq(\frac{p(1-a)-1}{2}) \\ &-\frac{c}{4c_1\kappa}\Big(\int_{\tau_1}^{\tau_2}|\mu_2(q)|dq\Big)\Big\}\|\nabla u\|_2^2+\varepsilon ab\|u\|_p^p+\varepsilon p(1-a)H(t) \\ &+\frac{\varepsilon p(1-a)}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}q|\mu_2(q)|y^2(x,\rho,q,t)dqd\rho dx. \end{aligned} \tag{33}$$

At this point, we choose $a>0$ so small that

$$\alpha_1=\frac{p(1-a)}{2}-1>0,$$

and assume

$$\int_0^\infty\varpi(q)dq<\frac{\frac{p(1-a)}{2}-1}{\Big(\frac{p(1-a)}{2}-\frac{1}{2}\Big)}=\frac{2\alpha_1}{2\alpha_1+1}, \tag{34}$$

then we choose κ so large that

$$\alpha_2=\Big(\frac{p(1-a)}{2}-1\Big)-\int_0^t\varpi(q)dq\Big(\frac{p(1-a)-1}{2}\Big)-\frac{c}{4c_1\kappa}\Big(\int_{\tau_1}^{\tau_2}|\mu_2(q)|dq\Big)>0.$$

Once κ and a are fixed, we pick ε so small enough so that

$$\alpha_4=1-\varepsilon\kappa>0,$$

and

$$\mathcal{K}(t)\leq\frac{b}{p}\|u\|_p^p. \tag{35}$$

Thus, for some $\beta>0$, estimate (33) becomes

$$\mathcal{K}'(t) \geq \beta\Big\{H(t)+\|u_t\|_2^2+\|\nabla u\|_2^2+(\varpi o\nabla u)+\|u\|_p^p+\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}q|\mu_2(q)|y^2(x,\rho,q,t)dqd\rho dx\Big\}, \tag{36}$$

and

$$\mathcal{K}(t)\geq\mathcal{K}(0)>0,\ t>0. \tag{37}$$

Next, using Young's and Poincare's inequalities, from (25) we have

$$\begin{aligned} \mathcal{K}(t) &= \Big(H+\varepsilon\int_\Omega uu_tdx+\frac{\varepsilon\mu_1}{2}\int_\Omega u^2dx+\frac{\varepsilon\omega}{2}\int_\Omega\nabla u^2dx\Big) \\ &\leq c[H(t)+|\int_\Omega uu_tdx|+\|u\|_2^2+\|\nabla u\|_2^2]\leq c[H(t)+\|\nabla u\|_2^2+\|u_t\|_2^2]. \end{aligned} \tag{38}$$

For some $c>0$. Since, $H(t)>0$, we have from (12)

$$\begin{aligned} &-\frac{1}{2}\|u_t\|_2^2-\frac{1}{2}\Big(1-\int_0^t\varpi(q)dq\Big)\|\nabla u\|_2^2-\frac{1}{2}(\varpi o\nabla u) \\ &-\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}q|\mu_2(q)|y^2(x,\rho,q,t)dqd\rho dx+\frac{b}{p}\|u\|_p^p>0, \end{aligned} \tag{39}$$

then

$$\frac{1}{2}\Big(1-\int_0^t\varpi(q)dq\Big)\|\nabla u\|_2^2 < \frac{b}{p}\|u\|_p^p<\frac{b}{p}\|u\|_p^p+(\varpi o\nabla u)+\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}q|\mu_2(q)|y^2(x,\rho,q,t)dqd\rho dx.$$

In the other hand, using (7), to get

$$\frac{1}{2}(1-l)\|\nabla u\|_2^2 < \frac{b}{p}\|u\|_p^p < \frac{b}{p}\|u\|_p^p + (\varpi o \nabla u) + \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q|\mu_2(q)|y^2(x,\rho,q,t)dqd\rho dx. \tag{40}$$

Consequently,

$$\|\nabla u\|_2^2 < \frac{2b}{p}\|u\|_p^p + 2(\varpi o \nabla u) + l\|\nabla u\|_2^2 + 2\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q|\mu_2(q)|y^2(x,\rho,q,t)dqd\rho dx. \tag{41}$$

Inserting (41) into (38), to see that there exists a positive constant k_1 such that

$$\begin{aligned}\mathcal{K}(t) \leq\ & k_1[H(t) + \|\nabla u\|_2^2 + \|u_t\|_2^2 + \frac{b}{p}\|u\|_p^p + (\varpi o \nabla u)(t) \\ & + \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q|\mu_2(q)|y^2(x,\rho,q,t)dqd\rho dx], \forall t > 0.\end{aligned} \tag{42}$$

From inequalities (36) and (42) we obtain the differential inequality

$$\mathcal{K}'(t) \geq \lambda \mathcal{K}(t), \tag{43}$$

where $\lambda > 0$, depending only on β and k_1.

A simple integration of (43), we obtain

$$\mathcal{K}(t) \geq \mathcal{K}(0)e^{(\lambda t)}, \forall t > 0. \tag{44}$$

From (24) and (35), we have

$$\mathcal{K}(t) \leq H(t) \leq \frac{b}{p}\|u\|_p^p. \tag{45}$$

By (44) and (45), we have

$$\|u\|_p^p \geq Ce^{(\lambda t)}, \forall t > 0.$$

Therefore, we conclude that the solution in the L_p-norm growths exponentially. This completes the proof. □

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: "The authors declare no conflict of interest."

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