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Global solutions and general decay for the dispersive wave equation with memory and source terms

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Abstract: This paper concerns with the global solutions and general decay to an initial-boundary value problem of the dispersive wave equation with memory and source terms.

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1. Introduction

This paper deals with the initial boundary value problem of the dispersive wave equation with memory and source terms

$$u_{tt} - \Delta u + \alpha \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau + u_t = |u|^{p-1}u, \quad x \in \Omega,\ t > 0, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^d$ $(d \geq 1)$ with a smooth boundary $\partial\Omega$, α is a positive constant and $g(t)$ is a positive function that represents the kernel of the memory term, which will be specified in Section 2. Here, we understand $\Delta^2 u$ to be the dispersive term. In the absence of the viscoelastic term and the dispersive term (that is, if $g = \alpha = 0$), the model (1) reduces to the weakly damped wave equation

$$u_{tt} - \Delta u + u_t = |u|^{p-1}u, \quad x \in \Omega,\ t > 0. \tag{2}$$

The interaction between the weak damping term and the source term are considered by many authors. We refer the reader to, Haraux and Zuazua [1], Ikehata [2] and Levine [3,4]. If $\alpha = 0$ and g is not trivial on $\mathbb{R}$, but replacing the fourth order memory term in (1) by a weaker memory of the form $\int_0^t g(t-\tau)\Delta u(\tau)d\tau$, then (1) can be rewritten as follows

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + u_t = |u|^{p-1}u, \quad x \in \Omega,\ t > 0, \tag{3}$$

The Equation (3) has been considered by Wang *et al* [5]. Under some appropriate assumptions on g, by introducing potential wells they obtained the existence of global solution and the explicit exponential energy decay estimates. Our main goal in the present paper is to discuss the global solutions and general decay to the following weakly damped wave equation with dispersive term, the fourth order memory term and the nonlinear source term

$$u_{tt} - \Delta u + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau + u_t = |u|^{p-1}u \quad \text{in } \Omega \times \mathbb{R}^+, \tag{4}$$

with simply supported boundary condition

$$u = 0, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \tag{5}$$

and initial conditions

$$u(\cdot,0) = u_0 \quad \text{and} \quad u_t(\cdot,0) = u_1 \quad \text{in } \Omega, \tag{6}$$

where Ω is a bounded domain of $\mathbb{R}^d$ with a smooth boundary $\partial\Omega$ and $p > 1$. Here, ν is the unit outward normal to $\partial\Omega$, and $g(t)$ is a positive function that represents the kernel of the memory term, which will be specified in Section 2. We prove that Problem (4)-(6) has a global weak solution assuming small initial data. In addition, we show the general decay of solutions. The global solutions are constructed by means of the Galerkin approximations and the general decay is obtained by employing the technique used in [6].

2. Preliminaries

Before proceeding to our analysis, we use the following abbreviations $\|\cdot\|_q = \|\cdot\|_{L^q(\Omega)}$ $(1 \le q \le +\infty)$ denotes usual L^q norm, $(\cdot,\cdot)$ denotes the L^2-inner product, and consider the Sobolev spaces $H_0^1(\Omega)$ and $H_0^2(\Omega)$ with their usual scalar products and norms. We also use the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for $2 < q < \frac{2d}{d-2}$ if $d \ge 3$ or $2 < q < \infty$ if $d = 1, 2$. In this case, the embedding constant is denoted by C_*, that is $\|u\|_q \le C_* \|\nabla u\|_2$. We define the polynomial Q by $Q(z) = \frac{1}{2}z^2 - \frac{C_*^{p+1}}{p+1} z^{p+1}$, which is increasing in $[0, z_0]$, where $z_0 = C_*^{\frac{p+1}{1-p}}$ is its unique local maximum. Next, we give the assumptions for Problem (4)-(6).

(G1) The relaxation function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded C^1 function such that $g(0) > 0$, $\quad 0 < \eta = 1 - \int_0^\infty g(\tau)d\tau \le 1 - \int_0^t g(\tau)d\tau = \eta(t)$.

(G2) There exist positive constants ξ_1 and ξ_2 such that $-\xi_1 g(t) \le g'(t) \le -\xi_2 g(t) \quad \forall t \ge 0$.

(G3) We also assume that $1 < p \le \frac{d+2}{d-2}$ if $d \ge 3$ and $p > 1$ if $d = 1, 2$. where λ_1 is the first eigenvalue of the following problem

$$\Delta^2 u = \lambda_1 u \quad \text{in } \Omega, \quad u = \frac{\partial u}{\partial \nu} = 0 \quad \text{in } \partial\Omega. \tag{7}$$

Remark 1. [7] Assuming λ_1 is the first eigenvalue of the problem (7), we have

$$\|\Delta u\|_2^2 \ge \lambda_1 \|\nabla u\|_2^2. \tag{8}$$

Now, we define the following energy function associated with a solution u of the Problem (4)-(6)

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|\Delta u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \Delta u)(t) - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \tag{9}$$

for $u \in H_0^2(\Omega)$, and

$$E(0) = \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\Delta u_0\|_2^2 + \frac{1}{2}\|\nabla u_0\|_2^2 - \frac{1}{p+1}\|u_0\|_{p+1}^{p+1} \tag{10}$$

is the initial total energy. To facilitate further on our analysis, we use the following notation

$$(g \circ \Delta u)(t) = \int_0^t g(t-\tau)\|\Delta u(\tau) - \Delta u(t)\|_2^2 d\tau.$$

Now, we are in a position to state our main results.

3. Main results

Theorem 1. *Assume that* $(G1) - (G3)$ *hold,* $u_0 \in H_0^2(\Omega)$, $u_1 \in L^2(\Omega)$. *Further assume that* $\|\nabla u_0\|_2 < z_0$ *and* $E(0) < Q(z_0)$, *then the Problem (4)-(6) possesses a global weak solution satisfying;* $u \in L^\infty(0,\infty; H_0^2(\Omega))$, $\quad u_t \in L^\infty(0,\infty; L^2(\Omega))$ *for* $0 \le t < \infty$, *and the energy identity*

$$E(t) + \int_0^t \|u_t(\tau)\|_2^2 d\tau - \frac{1}{2}\int_0^t (g' \circ \Delta u)(\tau)d\tau + \frac{1}{2}\int_0^t g(\tau)\|\Delta u(\tau)\|_2^2 d\tau = E(0), \tag{11}$$

holds for $0 \le t < \infty$. *Moreover, for* $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ *a increasing* C^2 *function satisfying*

$$\zeta(0) = 0, \quad \zeta_t(0) > 0, \quad \lim_{t \to +\infty} \zeta(t) = +\infty, \quad \zeta_{tt}(t) < 0 \quad \forall t \ge 0, \tag{12}$$

and, if $\|g\|_{L^1(0,\infty)}$ is sufficiently small, we have for $\kappa > 0$; $E(t) \le E(0)e^{-\kappa\zeta(t)}, \quad \forall t \ge 0$.

Remark 2. From (11) and $(G2)$, we can easily obtain

$$\frac{d}{dt}E(t) = -\|u_t(t)\|_2^2 + \frac{1}{2}(g' \circ \Delta u)(t) - \frac{1}{2}g(t)\|\Delta u(t)\|_2^2 \le -\|u_t(t)\|_2^2 - \frac{1}{2}\xi_2(g \circ \Delta u)(t) - \frac{1}{2}g(t)\|\Delta u(t)\|_2^2 \le 0. \tag{13}$$

Remark 3. For $\zeta(t) = t + \frac{t}{t+1}$, we can get the exponential decay rate $E(t) \le E(0)e^{-\kappa t}, \quad \forall t \ge 0$. For $\zeta(t) = ln(1+t)$, we can get polynomial decay rate $E(t) \le E(0)(1+t)^{-\kappa}, \quad \forall t \ge 0$.

4. Proof of main results

In this section, we shall divide the proof into two steps. In Step 1, we prove the global existence of weak solutions by using Galerkin's approximations. In Step 2, we establish the general decay of energy employing the method used in [6].

Step 1 Global existence of weak solutions

Let $\{\omega_j\}_{j=1}^{\infty}$ be an orthogonal basis of $H_0^2(\Omega)$ with ω_j being the eigenfunction of the problem $-\Delta\omega_j = \lambda_j\omega_j, \quad x \in \Omega, \quad \omega_j = 0, \quad x \in \partial\Omega$. Let $V^n = \text{Span}\{\omega_1, \omega_2, \cdots, \omega_n\}$. By the standard method of ODE, we know that $u^n(t) = \sum_{j=1}^n b_j^n(t)\omega_j(x)$ of the Cauchy problem as follows

$$\begin{aligned} &\int_\Omega u_{tt}^n \omega dx + \int_\Omega \nabla u^n \cdot \nabla\omega dx + \int_\Omega \Delta u^n \cdot \Delta\omega dx - \int_0^t g(t-\tau)\int_\Omega \Delta u^n(\tau)\cdot\Delta\omega dx d\tau \\ &+ \int_\Omega u_t^n \omega dx - \int_\Omega |u^n|^{p-1}u^n\omega dx = 0, \end{aligned} \tag{14}$$

$$u^n(0) = u_0^n \to u_0, \text{ in } H_0^2(\Omega), \quad u_t^n(0) = u_1^n \to u_1 \text{ in } L^2(\Omega). \tag{15}$$

By the standard theory of ODE system, we prove the existence of solutions of Problem (14)-(15) on some interval $[0, t_n)$, $0 < t_n < T$ for arbitrary $T > 0$, then, this solution can be extended to the whole interval $[0, T]$ using the first estimate given below. Taking $\omega = u_t^n(t)$ in (14), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_t^n\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u^n\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\Delta u^n\|_2^2 - \frac{1}{p+1}\frac{d}{dt}\|u^n\|_{p+1}^{p+1} + \|u_t^n\|_2^2 \\ &- \int_0^t g(t-\tau)\int_\Omega \Delta u^n(\tau)\cdot\Delta u_t^n(t) dx d\tau = 0. \end{aligned} \tag{16}$$

For the last term on the left hand side of (16) we have

$$\begin{aligned} &-\int_0^t g(t-\tau)\int_\Omega \Delta u^n(\tau)\cdot\Delta u_t^n(t)dxd\tau = \frac{1}{2}\frac{d}{dt}(g\circ\Delta u^n)(t) - \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(\tau)d\tau\right)\|\Delta u^n(t)\|_2^2 \\ &-\frac{1}{2}(g'\circ\Delta u^n)(t) + \frac{1}{2}g(t)\|\Delta u^n(t)\|_2^2. \end{aligned} \tag{17}$$

Inserting (17) into (16) and integrating over $[0, t] \subset [0, T]$, we obtain

$$\begin{aligned} &\frac{1}{2}\|u_t^n\|_2^2 + \frac{\eta(t)}{2}\|\Delta u^n(t)\|_2^2 + \frac{1}{2}\|\nabla u^n\|_2^2 - \frac{1}{p+1}\|u^n\|_{p+1}^{p+1} + \int_0^t \|u_t^n(\tau)\|_2^2 d\tau + \frac{1}{2}(g\circ\Delta u^n)(t) \\ &-\frac{1}{2}\int_0^t (g'\circ\Delta u^n)(\tau)d\tau + \frac{1}{2}\int_0^t g(\tau)\|\Delta u^n(\tau)\|_2^2 d\tau = E^n(0). \end{aligned} \tag{18}$$

Now from assumption $(G3)$ and the Sobolev embedding, we have that

$$\|u^n\|_{p+1}^{p+1} \le C_*^{p+1}\|\nabla u^n\|_2^{p+1}, \tag{19}$$

and then we have

$$\frac{1}{2}\|u_t^n\|_2^2+\frac{\eta(t)}{2}\|\Delta u^n(t)\|_2^2+\mathcal{Q}(\|\nabla u^n\|_2^2)+\int_0^t\|u_t^n(\tau)\|_2^2d\tau+\frac{1}{2}(g\circ\Delta u^n)(t)$$
$$-\frac{1}{2}\int_0^t(g'\circ\Delta u^n)(\tau)d\tau+\frac{1}{2}\int_0^t g(\tau)\|\Delta u^n(\tau)\|_2^2d\tau\leq E^n(0). \tag{20}$$

By using the fact that $-\int_0^t(g'\circ\Delta u^n)(\tau)d\tau+\int_0^t g(\tau)\|\Delta u^n(\tau)\|_2^2d\tau\geq 0$, estimate (20) yields

$$\frac{1}{2}\|u_t^n\|_2^2+\frac{\eta(t)}{2}\|\Delta u^n(t)\|_2^2+\frac{1}{2}(g\circ\Delta u^n)(t)+\mathcal{Q}(\|\nabla u^n\|_2^2)+\int_0^t\|u_t^n(\tau)\|_2^2d\tau\leq E^n(0). \tag{21}$$

From $E(0)<\mathcal{Q}(z_0)$ and (15), it follows that

$$E^n(0)<\mathcal{Q}(z_0), \tag{22}$$

for sufficiently large n. We claim that there exists an integer N such that

$$\|\nabla u^n(t)\|_2^2<z_0\quad\forall t\in[0,t_n)\quad n>N. \tag{23}$$

Suppose the claim is proved. Then $\mathcal{Q}(\|\nabla u^n\|_2^2)\geq 0$ and from (21) and (22),

$$\frac{1}{2}\|u_t^n\|_2^2+\frac{\eta(t)}{2}\|\Delta u^n(t)\|_2^2+\frac{1}{2}(g\circ\Delta u^n)(t)+\int_0^t\|u_t^n(\tau)\|_2^2d\tau\leq E^n(0)<\mathcal{Q}(z_0) \tag{24}$$

for sufficiently large n and $0\leq t<\infty$.

Proof of the claim. Suppose that (23) false. Then for each $n>N$, there exists $t\in[0,t_n)$ such that $\|\nabla u^n(t)\|_2\geq z_0$. We note that from $\|\nabla u_0\|_2<z_0$ and (15) there exists N_0 such that $\|\nabla u^n(0)\|_2<z_0\quad\forall n>N_0$. Then by continuity there exits a first $t_n^*\in[0,t_n)$ such that

$$\|\nabla u^n(t_n^*)\|_2=z_0, \tag{25}$$

from where $\mathcal{Q}(\|\nabla u^n(t)\|_2)\geq 0\quad\forall t\in[0,t_n^*]$. Now from $E(0)<\mathcal{Q}(z_0)$ and (24), there exists $N>N_0$ and $\gamma\in(0,z_0)$ such that $0\leq\frac{1}{2}\|u_t^n(t)\|_2^2+\frac{\eta(t)}{2}\|\Delta u^n(t)\|_2^2+\frac{1}{2}(g\circ\Delta u^n)(t)+\mathcal{Q}(\|\nabla u^n(t)\|_2^2)\leq\mathcal{Q}(\gamma)\quad\forall t\in[0,t_n^*]\quad\forall n>N$. Then the monotonicity of $\mathcal{Q}$ in $[0,z_0]$ implies that $0\leq\|\nabla u^n(t)\|_2^2\leq\gamma<z_0\quad\forall t\in[0,t_n^*]$, and in particular, $\|\nabla u^n(t)\|_2^2<z_0$, which is a contradiction to (24). From (24), we have

$$\|\Delta u^n\|_2^2<\frac{2\mathcal{Q}(z_0)}{\eta},\quad 0\leq t<\infty, \tag{26}$$

$$\|u_t^n\|_2^2<2\mathcal{Q}(z_0),\quad 0\leq t<\infty, \tag{27}$$

$$\int_0^t\|u_t^n(\tau)\|_2^2d\tau<\mathcal{Q}(z_0),\quad 0\leq t<\infty. \tag{28}$$

Using Sobolev inequality, (8) and (26), it follows that

$$\|u^n\|_{p+1}^2\leq C_*^2\|\nabla u^n\|_2^2\leq C_*^2\lambda_1^{-1}\|\Delta u^n\|_2^2<\frac{2C_*^2\lambda_1^{-1}\mathcal{Q}(z_0)}{\eta},\quad 0\leq t<\infty. \tag{29}$$

Furthermore, by (29), we get

$$|(|u^n|^{p-1}u^n,u^n)|\leq\|u^n\|_{p+1}^{p+1}<C_*^{p+1}\left(\frac{2C_*^2\lambda_1^{-1}\mathcal{Q}(z_0)}{\eta}\right)^{\frac{p+1}{2}},\quad 0\leq t<\infty. \tag{30}$$

The estimates (26)-(30) permit us to obtain a subsequences of $\{u_n\}$ which from now on will be also denoted by $\{u_n\}$ and functions u,χ such that

$$u_n \to u \text{ weak star in } L^\infty(0,\infty;H_0^2(\Omega)), \quad n\to+\infty, \tag{31}$$

$$u_t^n \to u_t \text{ weak star in } L^\infty(0,\infty;L^2(\Omega)), \quad n\to+\infty, \tag{32}$$

$$|u^n|^{p-1}u^n \to \chi \text{ weak star in } L^\infty(0,\infty;L^{\frac{p+1}{p}}(\Omega)), \quad n\to+\infty. \tag{33}$$

Besides, from Lions-Aubin Lemma we also have

$$u^n \to u \text{ strongly in } L^2(0,\infty;L^2(\Omega)), \quad n\to+\infty, \tag{34}$$

and consequently, making use of the Lemma 1.3 in [8], we deduce

$$|u^n|^{p-1}u^n \to \chi = |u|^{p-1}u \text{ weak star in } L^\infty(0,\infty;L^{\frac{p+1}{p}}(\Omega)), \quad n\to+\infty. \tag{35}$$

Thus, we obtain that u is a global weak of problem (4)-(6). Next, we shall prove that u satisfies (11). From the discussion above, we obtain for each fixed $t>0$ that

$$\lim_{n\to+\infty}(g\circ\Delta u^n)(t) = (g\circ\Delta u)(t), \quad \lim_{n\to+\infty}\|u^n\|_{p+1}^{p+1} = \|u\|_{p+1}^{p+1}. \tag{36}$$

We obtain for each fixed $t>0$ that

$$\begin{aligned} |(g\circ\Delta u)(t)-(g\circ\Delta u^n)(t)| &= \left|\int_0^t g(t-\tau)\|\Delta u(\tau)-\Delta u(t)\|_2^2 d\tau - \int_0^t g(t-\tau)\|\Delta u^n(\tau)-\Delta u^n(t)\|_2^2 d\tau\right| \\ &\le \int_0^t g(t-\tau)\|\Delta u(\tau)-\Delta u^n(\tau)\|_2\|\Delta u(\tau)+\Delta u^n(\tau)\|_2 d\tau \\ &\quad + \int_0^t g(t-\tau)\|\Delta u(\tau)-\Delta u^n(\tau)\|_2 d\tau\|\Delta u(t)+\Delta u^n(t)\|_2 \\ &\quad + \int_0^t g(t-\tau)\|\Delta u(\tau)+\Delta u^n(\tau)\|_2 d\tau\|\Delta u(t)-\Delta u^n(t)\|_2 \\ &\quad + \int_0^t g(\tau)d\tau\|\Delta u(t)+\Delta u^n(t)\|_2\|\Delta u(t)-\Delta u^n(t)\|_2 \\ &\le C\int_0^t g(t-\tau)\|\Delta u(\tau)-\Delta u^n(\tau)\|_2 d\tau + C\int_0^t g(\tau)d\tau\|\Delta u(t)-\Delta u^n(t)\|_2 \to 0, \end{aligned} \tag{37}$$

as $n\to+\infty$, and

$$\begin{aligned} \|u^n\|_{p+1}^{p+1} - \|u\|_{p+1}^{p+1} &\le (p+1)\left|\int_\Omega |u+\theta_n u^n|^{p-1}(u+\theta_n u^n)(u^n-u)dx\right| \\ &\le (p+1)\|u+\theta_n u^n\|_{p+1}^p\|u^n-u\|_{p+1} \le C\|u^n-u\|_{p+1} \to 0, \end{aligned} \tag{38}$$

as $n\to+\infty$, where $0<\theta_n<1$. Hence, we have

$$\lim_{n\to+\infty}(g\circ\Delta u^n)(t) = (g\circ\Delta u)(t), \quad \lim_{n\to+\infty}\|u^n\|_{p+1}^{p+1} = \|u\|_{p+1}^{p+1}. \tag{39}$$

From (15), it follows that $E^n(0)\to E(0)$ as $n\to+\infty$. Finally, taking $n\to+\infty$ in (18), we deduce that the energy identity (11) holds for $0\le t<\infty$. □

Step 2 General decay of the energy

Firstly, we state several Lemmas to prove the decay rate estimate of the energy.

Lemma 1. *Let $u\in L^\infty(0,\infty;H_0^2(\Omega))$ be the solution of (4)-(6) and $E(0)<\mathcal{Q}(z_0)$, $\|\nabla u_0\|_2<z_0$, then we have*

$$0\le E(t)\le \frac{1}{2}\|u_t\|_2^2 + C_1\|\Delta u\|_2^2 + \frac{1}{2}(g\circ\Delta u)(t), \tag{40}$$

where $C_1 = \frac{1}{2} + (2\lambda_1)^{-1}$.

Proof. From $E(0) < \mathcal{Q}(z_0)$ and $\|\nabla u_0\|_2 < z_0$, we can obtain $\mathcal{Q}(\|\nabla u(t)\|_2) \geq 0$ for $0 \leq t < \infty$. Thus we have

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|\Delta u\|_2^2 + \frac{1}{2}(g \circ \Delta u)(t) + \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &\geq \frac{1}{2}\|u_t\|_2^2 + \frac{\eta}{2}\|\Delta u\|_2^2 + \frac{1}{2}(g \circ \Delta u)(t) + \mathcal{Q}(\|\nabla u(t)\|_2) \geq 0, \end{aligned} \tag{41}$$

and

$$E(t) \leq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{2}(g \circ \Delta u)(t) + \frac{1}{2}\|\nabla u\|_2^2 \leq \frac{1}{2}\|u_t\|_2^2 + C_1\|\Delta u\|_2^2 + \frac{1}{2}(g \circ \Delta u)(t). \tag{42}$$

□

Lemma 2. *The energy $E(t)$ satisfies*

$$\frac{dE(t)}{dt} \leq -\|u_t(t)\|_2^2 - \frac{1}{2}\xi_2(g \circ \Delta u)(t) - \frac{1}{2}\left[g(0) - \xi_1\|g\|_{L^1(0,\infty)}\right]\|\Delta u(t)\|_2^2 \quad \forall t \geq 0. \tag{43}$$

Proof. From (13), we have

$$\frac{dE(t)}{dt} \leq -\|u_t(t)\|_2^2 - \frac{\xi_2}{2}(g \circ \Delta u)(t) - \frac{1}{2}g(t)\|\Delta u(t)\|_2^2. \tag{44}$$

From assumptions $(G2)$ and since $\int_0^t g'(\tau)d\tau = g(t) - g(0)$, we obtain

$$\begin{aligned} -\frac{1}{2}g(t)\|\Delta u(t)\|_2^2 &= -\frac{1}{2}g(0)\|\Delta u(t)\|_2^2 - \frac{1}{2}\left(\int_0^t g'(\tau)d\tau\right)\|\Delta u(t)\|_2^2 \\ &\leq -\frac{1}{2}g(0)\|\Delta u(t)\|_2^2 + \frac{\xi_1}{2}\|g\|_{L^1(0,\infty)}\|\Delta u(t)\|_2^2 \\ &= -\frac{1}{2}\left[g(0) - \xi_1\|g\|_{L^1(0,\infty)}\right]\|\Delta u(t)\|_2^2. \end{aligned} \tag{45}$$

Then, Combining (45) and (44) our conclusion holds. Multiplying (43) by $e^{\kappa\zeta(t)}$ $(\kappa > 0)$ and using (40), we have

$$\begin{aligned} \frac{d}{dt}\left(e^{\kappa\zeta(t)}E(t)\right) &\leq -\|u_t(t)\|_2^2 e^{\kappa\zeta(t)}E(t) - \frac{1}{2}\xi_2(g \circ \Delta u)(t)e^{\kappa\zeta(t)}E(t) \\ &\quad -\frac{1}{2}\left[g(0) - \xi_1\|g\|_{L^1(0,\infty)}\right]\|\Delta u(t)\|_2^2 e^{\kappa\zeta(t)}E(t) + \kappa\zeta_t(t)e^{\kappa\zeta(t)}E(t) \\ &\leq -\frac{1}{2}\left[2 - \kappa\zeta_t(t)\right]\|u_t(t)\|_2^2 e^{\kappa\zeta(t)}E(t) - \frac{1}{2}\left[\xi_2 - \kappa\zeta_t(t)\right](g \circ \Delta u)(t)e^{\kappa\zeta(t)}E(t) \\ &\quad -\frac{1}{2}\left[g(0) - \xi_1\|g\|_{L^1} - 2C_1\kappa\zeta_t(t)\right]\|\Delta u(t)\|_2^2 e^{\kappa\zeta(t)}E(t). \end{aligned} \tag{46}$$

Using the fact that ζ_t is decreasing by (12), we arrive at

$$\begin{aligned} \frac{d}{dt}\left(e^{\kappa\zeta(t)}E(t)\right) &\leq -\frac{1}{2}\left[2 - \kappa\zeta_t(0)\right]\|u_t(t)\|_2^2 e^{\kappa\zeta(t)}E(t) - \frac{1}{2}\left[\xi_2 - \kappa\zeta_t(0)\right](g \circ \Delta u)(t)e^{\kappa\zeta(t)}E(t) \\ &\quad -\frac{1}{2}\left[g(0) - \xi_1\|g\|_{L^1(0,\infty)} - 2C_1\kappa\zeta_t(0)\right]\|\Delta u(t)\|_2^2 e^{\kappa\zeta(t)}E(t). \end{aligned} \tag{47}$$

Choosing $\|g\|_{L^1(0,\infty)}$ sufficiently small so that $g(0) - \xi_1\|g\|_{L^1(0,\infty)} = B > 0$ and defining $\kappa_0 = \min\left\{\frac{2}{\zeta_t(0)}, \frac{\xi_2}{\zeta_t(0)}, \frac{B}{2C_1\zeta_t(0)}\right\}$, we conclude by taking $\kappa \in (0, \kappa_0]$ in (47) that $\frac{d}{dt}\left(e^{\kappa\zeta(t)}E(t)\right) \leq 0$, $t > 0$. Integrating the above inequality over $(0,t)$, it follows that $E(t) \leq E(0)e^{-\kappa\zeta(t)}$, $t > 0$. □

Conflicts of Interest: "The author declares no conflict of interest."

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