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Exponential decay of solutions with L^p -norm for a class to semilinear wave equation with damping and source terms

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Abstract: In this paper, we consider an initial value problem related to a class of hyperbolic equation in a bounded domain is studied. We prove local existence and uniqueness of the solution by using the Faedo–Galerkin method and that the local solution is global in time. We also prove that the solutions with some conditions exponentially decay. The key tool in the proof is an idea of Haraux and Zuazua with is based on the construction of a suitable Lyapunov function.

Keywords: Wave equation, source termes, Faedo-Galerkin method, global existence, exponential decay.

MSC: 35B40, 35L90.

1. Introduction

Consider the following problem:

$$u_{tt} - div\left(\frac{|\nabla u|^{2m-2}\nabla u}{\sqrt{1+|\nabla u|^{2m}}}\right) - \omega\Delta u_t + \mu u_t = u\,|\,u\,|^{p-2},\ x\in\Omega,\ t\geq 0, \tag{1}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \tag{2}$$

$$u(x,t) = 0,\ x\in\partial\Omega,\ t\geq 0, \tag{3}$$

where Ω is a bounded regular domain in $\mathbb{R}^n$, $n\geq 1$ with a smooth boundary $\partial\Omega$. ω, μ and m, p are real numbers.

The nonlinear wave equations

$$u_{tt} - \Delta u - \omega\Delta u_t + \mu u_t = u\,|\,u\,|^{p-2},\ x\in\Omega,\ t\geq 0, \tag{4}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ t\geq 0, \tag{5}$$

$$u(x,t) = 0, x\in\partial\Omega, t\geq 0, \tag{6}$$

has been investigated by many authors [1–10]. In the absence of the nonlinear source term, it is well know that the presence of one damping term ensures global existence and decay of solutions for arbitrary initial condition [5,6]. For $\omega=\mu=0$ the nolinear term $u\,|\,u\,|^{p-2}$ causes finite time blow up of solutions with negative energy [2]. The interaction between the damping and the source terms was first considered by Levine [11]. He showed that solutions with negative initial energy blows up in finite time. When $\omega=0$ and the linear term u_t is replaced by $|\,u_t\,|^{r-2}\,u_t$, Georgiev and Todorowa [12] extended Levin's result to the case where $r>2$. In their work, the authors introduced a method different from the one know as the concavity method. The termined suitable relations between r and p, for whith there is global existence or alternatively fnite time blow-up.

For the initial boundary value problem of a quasilinear equation

$$u_{tt} - div\left(|\,\nabla u\,|^{m-2}\nabla u\right) + au_t\,|\,u_t\,|^{p-2} - \Delta u_t = bu\,|\,u\,|^{r-2},$$

$$x \in \Omega,\ t \geq 0, \tag{7}$$

$$u(x,0) = u_0(x) \in W_0^{1,m}(\Omega),\ u_t(x,0) = u_1(x) \in L^2(\Omega),\ t \geq 0, \tag{8}$$

$$u(x,t) = 0,\ x \in \partial\Omega,\ t \geq 0. \tag{9}$$

Yang and Chen [13,14] studied the problem (7)-(9) and obtained global existence results under the growth assumptions on the nonlinear terms and the initial value. This global existence results have been improved by Liu and Zhao [15] by using a new method. In [13], the author considered a similar problem to (7)-(9) and proved a blow-up result under the condition $p > max(r,m)$ and the energy is sufficiently negative. Messaoudi and Said-Houari [16] improved the results in [15] and showed that blow-up takes place for negative initial data only regardless of the size of Ω Messaoudi in [17] showed that for $m = 2$, the decay is exponential. In absence of strong damping $-\Delta u_t$ equation (7) becames

$$u_{tt} - div\left(|\nabla u|^{m-2}\nabla u\right) + au_t\,|u_t|^{p-2} = bu\,|u|^{r-2},\ x \in \Omega,\ t \geq 0. \tag{10}$$

For $b = 0$, it is well known that the damping term assures global existence and decay of the solution energy for arbitrary initial value [18]. For $a = 0$, the source term causes finite time blow-up of solutions with negative initial energy if $r > m$ (see [2]). When the quasilinear operator $-div\left(|\nabla u|^{m-2}\nabla u\right)$ is replaced by $\Delta^2 u$, Wu and Tsai [19] showed that the solution is global in time under some conditions without the relation between p and r. They also proved that the local solution blows up infinite time if $r > p$ and the initial energy is nonnegative, and gave the decay estimates of the energy function and the lifespan of solutions. In this paper, we show that the local solutions of the problem (1)- (3) can be extented in infinite time to global solutions with the some conditions on initial data in the stable set for which the solutions decay expontially with L_p norm. The key tool in the proof is an idea of Haraux and Zuazua [6] and [9] with is based on the construction of a suitable Lyapunov function.

2. Assumptions and preliminaries

In this section, we present some material needed in the proof in our result.

Lemma 1. *(Young's inequality) Let $a,\ b \geq 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ for $1 < p,q < +\infty$, then one has the inequality $ab \leq \delta a^p + C(\delta)\,b^q$, where $\delta > 0$ is an arbitrary constant, and $C(\delta)$ is a positive constant depending on δ.*

Lemma 2. *Let s be a number with $2 \leq s < +\infty$ if $n \leq r$ and $2 \leq s \leq \frac{nr}{n-r}$ if $n > r$. Then there is a constant C depending on Ω and s such that $\|u\|_s \leq C\,\|\nabla u\|_r,\ u \in W_0^{1,r}(\Omega)$.*

We denote the total energy related to the problem (1)-(3) by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{m}\int_\Omega \sqrt{1+|\nabla u|^{2m}}dx - \frac{1}{p}\|u\|_p^p. \tag{11}$$

We also introduce the following functionals:

$$I(t) = \int_\Omega \sqrt{1+|\nabla u|^{2m}}dx - \|u\|_p^p, \tag{12}$$

$$J(t) = \frac{1}{m}\int_\Omega \sqrt{1+|\nabla u|^{2m}}dx - \frac{1}{p}\|u\|_p^p. \tag{13}$$

As in [20], we can now define the so called " Nehari manifold" as follows:

$$\mathcal{N} = \left\{u \in W_0^{1,m}(\Omega)\setminus\{0\}\,;\ I(t) = 0\right\}.$$

$\mathcal{N}$ separates the two unbounded sets:

$$\mathcal{N}^{+}=\left\{u \in W_0^{1,m}(\Omega);\ I(t)>0\right\} \cup \{0\}$$

and

$$\mathcal{N}^{-}=\left\{u \in W_0^{1,m}(\Omega);\ I(t)<0\right\}.$$

Assumptions:
$(A1)$: Assume that $I(0)>0$, and $0<E(0)$ such that

$$B=c^{p}\left(\frac{mp}{p-m}E(0)\right)^{\frac{p-m}{m}}<1. \tag{14}$$

where c *is the Poincaré constant*.
$(A2)$: p satisfies

$$2<m<p\leq \frac{nm}{n-m}, n\geq m;\ 2<m<p\leq +\infty, n<m.$$

For simplicity, we define the weak solutions of (1)-(3) over the interval $[0,T)$, but it is to be understood throughout that T is either infinity or the limit of the existence interval.

Definition 1. We say that $u(x,t)$ is a weak solution of the problem (1)-(3) on the interval $\Omega\times[0,T)$, if $u\in L^{\infty}\left([0,T);W_0^{1,m}(\Omega)\right), u_t\in L^{\infty}\left([0,\ T);\ L^2(\Omega)\right)\cap L^2\left([0,\ T);\ H_0^1(\Omega)\right)$ satisfy the following conditions:
(i)

$$\left(u''(t),\phi\right)+\left(\frac{|\nabla u|^{2m-2}\nabla u}{\sqrt{1+|\nabla u|^{2m}}},\nabla\phi\right)+\omega\left(\nabla u',\nabla\phi\right)+\mu\left(u',\phi\right)=\left(u|u|^{p-2},\phi\right), \tag{15}$$

for any function $\phi\in W_0^{1,m}(\Omega)$ and a.e. $t\in[0,T)$.
(ii)

$$u(x,0)=u_0(x)\in L^2(\Omega),\ u_t(x,0)=u_1(x)\in L^1(\Omega). \tag{16}$$

Theorem 1. *(Local existence) Suppose that* $u_0\in L^2(\Omega)$, $u_1\in L^1(\Omega)$ *and* $E(0)>0$, *then there exists* $T>0$ *such that problem* (1)- (3) *has a unique solution* u *satisfying* $u\in L^{\infty}\left([0,T];\ W_0^{1,m}(\Omega)\right), u_t\in L^{\infty}\left([0,\ T);\ L^2(\Omega)\right)\cap L^2\left([0,\ T);\ H_0^1(\Omega)\right)$.

3. Global existence and exponential decay of solutions

In this section we are going to obtain the existence of local solutions to the problem (1)-(3) and exponential decay of solution. We will use the Faedo- Galerkin's method approximation.

Let $\{w_l\}_{l=1}^{\infty}$ be a basis of $W_0^{1,m}(\Omega)$ wich constructs a complete orthonormal system in $L^2(\Omega)$. Denote by $V_k=span\{w_1,w_2,...,w_k\}$ the subspace generated by the first k vectors of the basis $\{w_l\}_{l=1}^{\infty}$. By the normalization, we have $\|w_l\|=1$. for any given integer k, we consider the approximation solution

$$u_k(t)=\sum_{l=1}^{k}u_{lk}(t)v_l,$$

where u_k is the solutions to the following Cauchy problem

$$\left(u_k''(t),v_l\right)+\left(\frac{|\nabla u_k|^{2m-2}\nabla u_k}{\sqrt{1+|\nabla u_k|^{2m}}},\nabla v_l\right)+\omega\left(\nabla u_k',\nabla v_l\right)+\mu\left(u_k',v_l\right)=\left(u_k|u_k|^{p-2},v_l\right), \tag{17}$$

where $l=1,...,k,$ with initial conditions $u_k(0)=u_{0k}$ and $u_k'(0)=u_{1k}$, $u_k(0)$ and $u_k'(0)$ are chosen in V_k such that

$$\sum_{l=1}^{k}(u_0,v_l)v_l=u_{0k}\longrightarrow u_0\ in\ L^2(\Omega);\sum_{l=1}^{k}(u_1,v_l)v_l=u_{1k}\longrightarrow u_1\ in\ L^1(\Omega). \tag{18}$$

Well known results on the solvability of nonlinear ODE provide the existence of a solution to problem (17)-(18) on interval $[0,\tau)$ for some $\tau>0$ and we can extend this solution to the whole interval $[0,T]$ for any given $T>0$ by making use of the a priori estimates below. Multiplying equation (17) by $u'_{lk}(t)$ and sum for $l=1,...,k$, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\left\|u'_k\right\|_2^2+\frac{1}{m}\int_\Omega\sqrt{1+|\nabla u_k|^{2m}}dx-\frac{1}{p}\left\|u_k\right\|_p^p\right)=-\left(\omega\int_\Omega|\nabla u_k|_2^2dx+\mu\int_\Omega|u'_k|_2^2dx\right). \tag{19}$$

Integrating (19) over $(0,t)$, we obtain the estimate

$$\frac{1}{2}\left\|u'_k\right\|_2^2+\frac{1}{m}\int_\Omega\sqrt{1+|\nabla u_k|^{2m}}dx-\frac{1}{p}\left\|u_k\right\|_p^p+\omega\int_0^t\int_\Omega|\nabla u_k|_2^2dx+\mu\int_0^t\int_\Omega|u'_k|_2^2dx\leq E(0). \tag{20}$$

Since $I(0)>0$, then there exists $\tau<T$ by continuity such that $I(t)\geq 0$,. We get from (12) and (13) that

$$J(u_k(t))=\frac{p-m}{mp}\int_\Omega\sqrt{1+|\nabla u_k|^{2m}}dx+\frac{1}{p}I(u_k(t)) \tag{21}$$

$$J(u_k(t))\geq\frac{p-m}{mp}\int_\Omega\sqrt{1+|\nabla u_k|^{2m}}dx,\ \ \forall t\in[0,\ \tau]. \tag{22}$$

Hence we have

$$\int_\Omega\sqrt{1+|\nabla u_k|^{2m}}dx\leq\frac{mp}{p-m}J(u_k(t)). \tag{23}$$

From (11) and (13), we obvioulsy have $\forall t\in[0,\ \tau]$, $J(u_k(t))\leq E(u_k(t))$. Thus we obtain

$$\int_\Omega\sqrt{1+|\nabla u_k|^{2m}}dx\leq\frac{mp}{p-m}E(u_k(t)). \tag{24}$$

Since E is a decreasing function of t, we have

$$\int_\Omega\sqrt{1+|\nabla u_k|^{2m}}dx\leq\frac{mp}{p-m}E(0),\ \ \forall t\in[0,\ \tau] \tag{25}$$

By using Lemma 2, we easily have

$$\begin{aligned}\|u_k\|_p^p &\leq c^p\|\nabla u_k\|_m^p=c^p\left(\int_\Omega|\nabla u_k|^m dx\right)^{\frac{p}{m}}\leq c^p\left(\int_\Omega\sqrt{1+|\nabla u_k|^{2m}}dx\right)^{\frac{p}{m}}\\ &\leq c^p\left(\int_\Omega\sqrt{1+|\nabla u_k|^{2m}}dx\right)^{\frac{p-m}{m}}\int_\Omega\sqrt{1+|\nabla u_k|^{2m}}dx\end{aligned}$$

Using the inequality (25), we deduce

$$\|u_k\|_p^p\leq c^p\left(\frac{mp}{p-m}E(0)\right)^{\frac{p-m}{m}}\int_\Omega\sqrt{1+|\nabla u_k|^{2m}}dx.$$

Now exploiting the inequality (14), we obtain

$$\|u_k\|_p^p\leq\int_\Omega\sqrt{1+|\nabla u_k|^{2m}}dx. \tag{26}$$

Hence $\int\limits_{\Omega}\sqrt{1+|\nabla u_k|^{2m}}dx - \ \|u_k\|_p^p > 0,\ \forall t \in [0,\ \tau]$, this shows that $I(u_k(t)) > 0$, by repeating this procedure, τ is extended to T.

Since $\int\limits_{\Omega}\sqrt{1+|\nabla u_k|^{2m}}dx > \|\nabla u_k\|_m^m$, it follows from (20) and (26) that

$$\frac{1}{2}\|u_k'\|_2^2 + \frac{p-m}{pm}\|\nabla u_k\|_m^m + +\omega\int_0^t\int\limits_{\Omega}|\nabla u_k|_2^2 dx + \mu\int_0^t\int\limits_{\Omega}|u_k'|_2^2 dx \le E(0). \tag{27}$$

From (27), we have

$$\begin{cases} \{u_k\} \text{ is uniformly bounded in } L^\infty\left([0,T];W_0^{1,m}(\Omega)\right), \\ \{u_k\} \rightharpoonup u \text{ is uniformly bounded in } L^2\left([0,T];H_0^1(\Omega)\right), \\ \{u_k'\} \text{ is uniformly bounded in } L^\infty\left([0,T];L^2(\Omega)\right), \\ \{u_k'\} \text{ is uniformly bounded in } L^2\left([0,T];L^2(\Omega)\right). \end{cases} \tag{28}$$

Furthermore, we have from Lemma 2 and (28) that

$$\{|u_k|^p u_k\} \ \text{ is uniformly bounded in } L^\infty\left([0,T];L^2(\Omega)\right). \tag{29}$$

By (28) and (29), we infer that there exists a subsequence of u_k (denote still by the same symbol) and a function u such that

$$\begin{cases} u_k \rightharpoonup u \text{ weakly star in } L^\infty\left([0,T];W_0^{1,m}(\Omega)\right), \\ u_k \rightharpoonup u \text{ weakly star in } L^2\left([0,T];H_0^1(\Omega)\right), \\ u_k' \rightharpoonup u' \ \text{ weakly star in } L^\infty\left([0,T];L^2(\Omega)\right), \\ u_k' \rightharpoonup u' \ \text{ weakly star in } L^2\left([0,T];L^2(\Omega)\right), \\ |u_k|^{p-2}u_k \rightharpoonup \mathcal{X} \ \text{ weakly star in } L^\infty\left([0,T];L^2(\Omega)\right). \end{cases} \tag{30}$$

By the Aubin-Lions compactness Lemma [7], we conclude from (30) that

$$\begin{cases} u_k \rightharpoonup u \ \text{ strongly in } C\left([0,T];L^2(\Omega)\right), \\ u_k' \rightharpoonup u' \ \text{ strongly in } C\left([0,T];L^2(\Omega)\right), \end{cases}$$

and

$$u_k \rightharpoonup u \text{ almost everywhere in } [0,T]\times\Omega. \tag{31}$$

It follows from Lemma 1.3 in [21] and (31)

$$|u_k|^{p-2}u_k \rightharpoonup |u|^{p-2}u \text{ weakly star in } L^\infty\left([0,T];L^2(\Omega)\right). \tag{32}$$

By the last formula (32) and (30), we obtain $\mathcal{X} = |u|^{p-2}u$ On the other hand, taking $\phi = 1$,(17) become

$$\left(u_k''(t),1\right) + \mu\left(u_k',1\right) = \left(u_k|u_k|^{p-2},1\right). \tag{33}$$

We have

$$|\left(u_k''(t),1\right) + \mu\left(u_k',1\right)| \ge \|u_k''\| - \mu\|u_k'\|.$$

Since, the measure of Ω is finite, by the embedding theorem, (30) and (33), we obtain

$$\|u_k''\| \le C,$$

then

$$\{u_k''\} \text{ is uniformly bounded in } L^\infty\left([0,T];L^1(\Omega)\right).$$

Similarly, we have

$$u_k'' \rightharpoonup u'' \text{ weakly star in } L^\infty\left([0,T]; L^1(\Omega)\right), \tag{34}$$

Setting up $k \longrightarrow \infty$ and passing to the limit in (17), we obtain

$$\left(u''(t), v_l\right) + \left(\frac{|\nabla u|^{2m-2}\nabla u}{\sqrt{1+|\nabla u_k|^{2m}}}, \nabla v_l\right) + \omega\left(\nabla u', \nabla v_l\right) + \mu\left(u', v_l\right) = \left(u|u|^{p-2}, v_l\right),$$

$l = 1, ..., k$. Since $\{v_l\}_{l=1}^{\infty}$ is a base of $W_0^{1,m}(\Omega)$, we deduce that u satisfies (1).

From (30), (34) and Lemma 3.1.7 in [22], with $B = L^2(\Omega)$ and $B = L^1(\Omega)$, respectively, we infer that

$$\begin{cases} u_k(0) \rightharpoonup u(0) \text{ weakly in } L^2(\Omega), \\ u_k'(0) \rightharpoonup u'(0) \text{ weakly star in } L^1(\Omega). \end{cases} \tag{35}$$

We get from (18) and (35) that $u(0) = u_0$, $u'(0) = u_1$. Thus, the proof is complete.

Lemma 3. *Assume that $p > m$ and $u_0 \in \mathcal{N}^+$, $u_1 \in L^2(\Omega)$. If $0 < E(0)$ and satisfy (14) then the local solution of the problem (1)-(3) is global in time.*

Proof. Since the map $t \longmapsto E(t)$ is a decreasing of the time t, we have

$$E(0) \geq E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{p-m}{mp}\int_\Omega \sqrt{1+|\nabla u|^{2m}}dx + \frac{1}{p}I(t) \tag{36}$$

which give

$$E(0) \geq E(t) \geq \frac{1}{2}\|u_t\|_2^2 + \frac{p-m}{mp}\int_\Omega \sqrt{1+|\nabla u|^{2m}}dx \tag{37}$$

thus, $\forall t \in [0, T)$, $\|u_t\|_2^2 + \int_\Omega \sqrt{1+|\nabla u|^{2m}}dx$ is uniformly bounded by a constant depending only on $E(0)$, p and m then the solution is global, so $T_{\max} = \infty$. $\square$

Theorem 2. *Assume that $p > m$. Let $u_0 \in \mathcal{N}^+$ and $u_1 \in L^2(\Omega)$. Moreover, assume that $0 < E(0)$ and satisfy (14). Then there exists two positive constants α and β independent of t such that: $0 < E(t) \leq \beta e^{-\alpha t}$, $\forall t > 0$.*

Proof. Since we have proved that $t \geq 0$, $u(t) \in \mathcal{N}^+$, we already have

$$0 < E(t), \ \forall t \geq 0.$$

We define a Lyaponov function, for $\epsilon > 0$.

$$L(t) = E(t) + \epsilon\int_\Omega u_t u dx + \frac{\epsilon\omega}{2}\|\nabla u\|_2^2. \tag{38}$$

We prove that $L(t)$ and $E(t)$ are equivalent in the sens that there exist two constants B_1 and B_2 depending on ϵ such that for $t \geq 0$

$$B_1 E(t) \leq L(t) \leq B_2 E(t). \tag{39}$$

By the Lemma 1, we have

$$L(t) = E(t) + \epsilon\int_\Omega u_t u dx + \frac{\epsilon\omega}{2}\|\nabla u\|_2^2 \leq E(t) + \epsilon\left(\frac{1}{4\delta}\|u_t\|_2^2 + \delta\|u\|_2^2\right) + \frac{\epsilon\omega}{2}\|\nabla u\|_2^2.$$

Thanks of the Poincaré inequality and since δ is an arbitrary constant, we choose δ small suffisant for that,

$$\delta \|u\|_2^2 \leq \delta C \|\nabla u\|_2^2 \leq \int_\Omega \sqrt{1+|\nabla u|^{2m}}dx \tag{40}$$

Then, we get

$$L(t) \leq E(t) + \epsilon \frac{1}{4\delta} \|u_t\|_2^2 + \epsilon \left(\delta C + \frac{\omega}{2}\right) \|\nabla u\|_2^2 \leq E(t) + \epsilon \frac{1}{4\delta} \|u_t\|_2^2 + \epsilon \left(1 + \frac{\omega}{2}\right) \int_\Omega \sqrt{1+|\nabla u|^{2m}}dx.$$

By (37), we get

$$L(t) \leq E(t) + \epsilon \frac{1}{2\delta} E(t) + \epsilon \left(1 + \frac{\omega}{2}\right) \frac{mp}{p-m} E(t) \leq B_2 E(t), \tag{41}$$

where $B_2 = \left(1 + \epsilon \frac{1}{2\delta} + \epsilon \left(1 + \frac{\omega}{2}\right) \frac{mp}{p-m}\right)$.

On the other hand, we have

$$\begin{aligned} L(t) &\geq E(t) - \epsilon \left(\frac{1}{4\delta} \|u_t\|_2^2 + \delta \|u\|_2^2\right) + \frac{\epsilon\omega}{2} \|\nabla u\|_2^2 \\ &\geq E(t) - \epsilon \frac{1}{4\delta} \|u_t\|_2^2 - \epsilon\delta \|u\|_2^2 \\ &\geq E(t) - \epsilon \frac{1}{2\delta} E(t) - \epsilon\delta \|u\|_2^2 \\ &\geq \left(1 - \epsilon \frac{1}{2\delta}\right) E(t) - \epsilon\delta \|u\|_2^2. \end{aligned}$$

From (37) and (40), we obtain

$$L(t) \geq \left(1 - \epsilon \frac{1}{2\delta} - \epsilon \frac{mp}{p-m}\right) E(t) = B_1 E(t), \tag{42}$$

where $B_1 = \left(1 - \epsilon \frac{1}{2\delta} - \epsilon \frac{mp}{p-m}\right)$.

Now, we have

$$\begin{aligned} \frac{d}{dt} L(t) &= -\omega \|\nabla u_t\|_2^2 - \mu \|u_t\|_2^2 + \epsilon \|u_t\|_2^2 + \epsilon \int_\Omega div \left(\frac{|\nabla u|^{2m-2} \nabla u}{\sqrt{1+|\nabla u|^{2m}}}\right) u dx + \epsilon \|u\|_p^p - \epsilon\mu \int_\Omega u_t u dx \\ &= -\omega \|\nabla u_t\|_2^2 - \mu \|u_t\|_2^2 + \epsilon \|u_t\|_2^2 - \epsilon \int_\Omega \frac{|\nabla u|^{2m}}{\sqrt{1+|\nabla u|^{2m}}} dx + \epsilon \|u\|_p^p - \epsilon\mu \int_\Omega u_t u dx. \end{aligned}$$

So that

$$\frac{d}{dt} L(t) \leq -\omega \|\nabla u_t\|_2^2 + \left(\epsilon \left(\frac{\mu}{4\delta} + 1\right) - \mu\right) \|u_t\|_2^2 + \epsilon\mu\delta \|u\|_2^2 - \epsilon \int_\Omega \frac{|\nabla u|^{2m}}{\sqrt{1+|\nabla u|^{2m}}} dx + \epsilon \int_\Omega \sqrt{1+|\nabla u|^{2m}} dx. \tag{43}$$

So

$$\frac{d}{dt} L(t) \leq \left(\epsilon \left(\frac{\mu}{4\delta} + 1\right) - \mu\right) \|u_t\|_2^2 + \epsilon (1+\mu) \int_\Omega \sqrt{1+|\nabla u|^{2m}} dx. \tag{44}$$

Using the inequality (37) and (44), we deduce

$$\begin{aligned} \frac{d}{dt} L(t) &\leq 2\left(\epsilon \left(\frac{\mu}{4\delta} + 1\right) - \mu\right) E(t) + \epsilon (1+\mu) \frac{mp}{p-m} E(t) \\ &\leq -\left(2\mu - \epsilon \left(\left(\frac{\mu}{2\delta} + 2\right) + (1+\mu) \frac{mp}{p-m}\right)\right) E(t). \end{aligned}$$

We choosing ϵ small enough such that

$$-\left(2\mu-\epsilon\left(\left(\frac{\mu}{2\delta}+2\right)+(1+\mu)\,\frac{mp}{p-m}\right)\right)=\zeta<0. \tag{45}$$

So

$$\frac{d}{dt}L\left(t\right)\leq\zeta E\left(t\right). \tag{46}$$

From (39), we have

$$\frac{d}{dt}L\left(t\right)\leq\frac{\zeta}{B_2}L\left(t\right). \tag{47}$$

Integrating the provious differential inequality (47) between 0 and t gives the following estimate for the function L :

$$L\left(t\right)\leq ce^{\frac{\zeta}{B_2}t},\quad\forall t\geq0. \tag{48}$$

Consequently, by using (39) once again, we conclude

$$E\left(t\right)\leq ke^{\frac{\zeta}{B_2}t},\quad\forall t\geq0. \tag{49}$$

By using (26) and (37) we easily have

$$\|u\|_p^p\leq k_1e^{\frac{\zeta}{B_2}t},\quad\forall t\geq0. \tag{50}$$

The proof is complete. □

4. Conclusion

In this paper, we have studied a class of hyperbolic equation supplemented with Dirichlet boundary conditions as a model of wave equation with damping and source nonlinear terms. We showed that the solution with positive initial energy exponentially decay, this is mainly due to the presence of one of term of weak or strong damping.

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