Open Journal of Mathematical Analysis

Article

Boundary value problems for a class of stochastic nonlinear fractional order differential equations

McSylvester Ejighikeme Omaba[1,*] and Louis O. Omenyi[2]

1 Department of Mathematics, College of Science, University of Hafr Al Batin, P. O Box 1803 Hafr Al Batin 31991, KSA.
2 Department of Mathematics/Computer Science/Statistics/Informatics, Alex Ekwueme Federal University, Ndufu-Alike, Ikwo, Nigeria.
* Correspondence: mcomaba@uhb.edu.sa

Received: 3 November 2020; Accepted: 10 December 2020; Published: 21 December 2020.

Abstract: Consider a class of two-point Boundary Value Problems (BVP) for a stochastic nonlinear fractional order differential equation $D^\alpha u(t) = \lambda\sqrt{I^\beta[\sigma^2(t,u(t))]}\dot{w}(t),\ 0 < t < 1$ with boundary conditions $u(0) = 0,\ u'(0) = u'(1) = 0$, where $\lambda > 0$ is a level of the noise term, $\sigma : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $\dot{w}(t)$ is a generalized derivative of Wiener process (Gaussian white noise), D^α is the Riemann–Liouville fractional differential operator of order $\alpha \in (3,4)$ and I^β, $\beta > 0$ is a fractional integral operator. We formulate the solution of the equation via a stochastic Volterra–type equation and investigate its existence and uniqueness under some precise linearity conditions using contraction fixed point theorem. A case of the above BVP for a stochastic nonlinear second order differential equation for $\alpha = 2$ and $\beta = 0$ with $u(0) = u(1) = 0$ is also studied.

Keywords: Boundary value problems, existence and uniqueness result, fractional integral, Riemann–Liouville fractional derivative, stochastic volterra–type equation.

MSC: 26A33, 34A08, 60H15, 82B44.

---

1. Introduction

Fractional derivatives and fractional order differential equations have received an increasingly high interest and attention due to its physical and modeling applications in science, engineering and mathematics[1–4]. On the other hand, researches have shown that two-point boundary value problems have found their applications in theoretical physics, applied mathematics, optimization and control theory and engineering [5]. The use of second-order Boundary Value Problem (BVP) arises in several areas of engineering and applied sciences such as celestial mechanics, circuit theory, astrophysics, chemical kinetics, and biology [6].

They are particularly encountered in the study of following natural phenomena: in the study of surface-tension–induced flows of a liquid metal or semiconductor [7,8], used in modeling biological materials (elastic and hyperelastic materials) [9]. In [10], authors studied the existence and uniqueness of a nontrivial solution of a two-point boundary value problems. Stochastic nonlinear fractional order differential equation and its associated BVPs, on the other hand, will undoubtedly give more realistic models for the above natural occurrences.

Motivated by the above applications and the results of the papers [2–4], we study the white noise pertubation of a nonlinear BVP and consider the following BVP for a stochastic nonlinear fractional differential equation

$$\begin{cases} D^\alpha u(t) = \lambda\sqrt{I^\beta[\sigma^2(t,u(t))]}\dot{w}(t),\ 0 < t < 1 \\ u(0) = 0,\ u'(0) = u'(1) = 0, \end{cases} \tag{1}$$

where $\lambda > 0$ is a noise level parameter, $\sigma : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $\dot{w}(t)$ is a Gaussian white noise, $D^\alpha = D^\alpha_{0_+}$ is the Riemann–Liouville (R–L) fractional derivative of order $\alpha \in (3,4)$ and I^β is the fractional integral of order $\beta > 0$. Existence and uniqueness results are very crucial in the study of BVP since they are not well behaved like the initial value problems, therefore, we aim to establish the existence and uniqueness of solution to Equation (1). To the best of our knowledge, the above model has not been studied before and we therefore, try to make sense of the solution to the above equation as follows:

Definition 1. We say that $\{u(t)\}_{0\leq t\leq 1}$ is a mild solution to Equation (1) if $a.s$, the following is satisfied

$$u(t) = \lambda \int_0^1 G(t,s)\sqrt{I^\beta[\sigma^2(s,u(s))]}\dot{w}(s)ds = \lambda \int_0^1 G(t,s)\sqrt{I^\beta[\sigma^2(s,u(s))]}dw(s),$$

where $G(t,s)$ is as defined in (3).

If $\{u(t)\}_{0<t<1}$ satisfies the additional condition $\sup_{0\leq t\leq 1}\mathbb{E}|u(t)|^2 < \infty$, then we say that $\{u(t)\}_{0\leq t\leq 1}$ is a random field solution to Equation (1).

Definition 2. The R–L fractional integral of order $\beta > 0$ of a given continuous function $f : (0,\infty) \to \mathbb{R}$ is defined by

$$I^\beta f(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s)ds,$$

provided the right integral exists. Denote $I^0 f(t) = f(t)$.

Definition 3. The R–L fractional derivative of order $\alpha > 0$ of a given continuous function $f : (0,\infty) \to \mathbb{R}$ is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n = [\alpha]+1$, provided the right integral converges.

Many authors studied different boundary value problems of nonlinear fractional order differential equations, see [1,11–14] and their references. Xu in [3] considered the following boundary value problem of a nonlinear fractional differential equation:

Lemma 1 ([3]). *Given $h \in C[0,1]$ and $3 < \alpha \leq 4$, the unique solution of*

$$\begin{cases} D^\alpha u(t) = h(t),\ 0 < t < 1 \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases}$$

is $u(t) = \displaystyle\int_0^1 G(t,s)h(s)ds$ where

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}+(1-s)^{\alpha-2}t^{\alpha-2}[(s-t)+(\alpha-2)(1-t)s]}{\Gamma(\alpha)}, & 0\leq s\leq t\leq 1 \\ \frac{t^{\alpha-2}(1-s)^{\alpha-2}[(s-t)+(\alpha-2)(1-t)s]}{\Gamma(\alpha)}, & 0\leq t\leq s\leq 1. \end{cases} \tag{2}$$

Lemma 2 ([4]). *The Green function $G(t,s)$ in (2) has the following properties*

(1) $G(t,s) = G(1-s,1-t)$ for $t,s \in [0,1]$;
(2) $t^{\alpha-2}(1-t)^2 q(s) \leq G(t,s) \leq (\alpha-1)q(s)$ and $G(t,s) \leq \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}t^{\alpha-2}(1-t)^2$ for $t,s \in [0,1]$ where $q(s) = \frac{(\alpha-2)}{\Gamma(\alpha)}s^2(1-s)^{\alpha-2}$.

Xu in [3] gave more properties of the function $G(t,s)$ in (2) as follows:

Lemma 3 ([3]). *The Green function $G(t,s)$ defined by (2) satisfies the following conditions:*

(1) $G(t,s) = G(1-s,1-t)$ for $t,s \in (0,1)$;
(2) $(\alpha-2)t^{\alpha-2}(1-t)^2 s^2 \leq \Gamma(\alpha)G(t,s) \leq M_0 s^2(1-s)^{\alpha-2}$, for $t,s \in (0,1)$;
(3) $G(t,s) > 0$, for $t,s \in (0,1)$;

(4) $(\alpha-2)s^2(1-s)^{\alpha-2}t^{\alpha-2}(1-t)^2 \le \Gamma(\alpha)G(t,s) \le M_0 t^{\alpha-2}(1-t)^2$, *for* $t,s \in (0,1)$, *where* $M_0 = \max\{\alpha-1,(\alpha-2)^2\}$.

Stanek in [2] studied the following fractional boundary value problems:

Lemma 4 ([2]). *Suppose that* $\rho \in L^1[0,1]$. *Then* $u(t) = \int_0^1 G(t,s)\rho(s)ds$ *for* $t \in [0,1]$ *is the unique solution of the following equation in* $C^1[0,1]$:

$$\begin{cases} D^\alpha u(t) + \rho(t) = 0, & \alpha \in (2,3) \\ u(0) = 0,\ u'(0) = u'(1) = 0, \end{cases}$$

where

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1 \\ \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \tag{3}$$

Lemma 5 ([2]). *The function* $G(t,s)$ *given in* (3) *has the following properties:*

(1) $G \in C([0,1]\times[0,1])$ *and* $G^\alpha > 0$ *on* $(0,1)\times(0,1)$;
(2) $G(t,s) \le \frac{1}{\Gamma(\alpha)}$ *for* $(t,s) \in [0,1]\times[0,1]$;
(3) $\displaystyle\int_0^1 G(t,s)ds \ge \frac{t^{\alpha-1}}{(\alpha-1)\Gamma(\alpha+1)}$ *for* $t \in [0,1]$;
(4) $\frac{\partial}{\partial t}G(t,s) \in C([0,1]\times[0,1])$ *and* $\frac{\partial}{\partial t}G(t,s) > 0$ *on* $(0,1)\times(0,1)$;
(5) $\frac{\partial}{\partial t}G(t,s) \le \frac{1}{\Gamma(\alpha-1)}$ *for* $(t,s) \in [0,1]\times[0,1]$;
(6) $\displaystyle\int_0^1 \frac{\partial}{\partial t}G(t,s)ds \ge \frac{t(1-t)}{\Gamma(\alpha)}$ *for* $t \in [0,1]$.

The paper is outlined as follows. A short preliminary on second order boundary problems is given in Section 2. In Section 3, we gave the main results and their proofs while Section 4 contains a concise summary of the paper.

2. Preliminaries

Let $(a,b) \in \mathbb{R}$ be an interval and $p,q,f : (a,b) \to \mathbb{R}$ be continuous functions. Now, consider the linear second order equation given by $y'' + p(x)y' + q(x) = f(x)$ subject to the following boundary conditions:

(a.) Dirichlet or first kind: $y(a) = \eta_1,\ y(b) = \eta_2$,
(b.) Neuman or second kind: $y'(a) = \eta_1,\ y'(b) = \eta_2$,
(c.) Robin or third or mixed kind: $\alpha_1 y(a) + \alpha_2 y'(a) = \eta_1,\ \alpha_1 y(b) + \alpha_2 y'(b) = \eta_2$,
(d.) Periodic: $y(a) = y(b),\ y'(a) = y'(b)$.

Remark 1. 1. Unlike the initial value problems, the BVPs do not behave nicely because there are BVPs for which their solutions fail to exist and even when a solution does exist, there might be many of them. This of course makes existence and uniqueness for BVPs to fail generally.
2. For example, $y'' + y = 0$ has a unique solution $y(x) = \sin x + \cos x$ with the boundary conditions $y(0) = y(\frac{\pi}{2}) = 1$. It has no solution with the boundary conditions $y(0) = y(\pi) = 1$ and has infinitely many solutions with the boundary conditions $y(0) = y(2\pi) = 1$.

Thus, for the linear boundary value problem with the Dirichlet boundary conditions:

$$\begin{cases} L[u(x)] = f(x,u), \\ u(a) = u(b) = 0, \end{cases} \tag{4}$$

where L is a second-order differential operator given by $L = \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x)$. The Equation (4) has the following integro-differential equation as its solution:

$$u(x) = \int_a^b G(x,\xi) f(\xi, u(\xi)) d\xi,$$

with $G(x,\xi)$ the Green's function.

Lemma 6 ([10]). *Let $y(t) \in X$, then the Boundary Value Problem*

$$\begin{cases} u''(t) - y(t) = 0, & 0 < t < 1 \\ u(0) = u(1) = 0, \end{cases}$$

has a unique solution $\int_0^1 G(t,s)y(s)ds$, where

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1, \end{cases} \tag{5}$$

is the Green's function of BVP

$$\begin{cases} u''(t) = 0, & 0 < t < 1 \\ u(0) = u(1) = 0. \end{cases}$$

Remark 2. The Green function has the following bound: $\sup\limits_{0 \le t,s \le 1} G(t,s) \le \dfrac{1}{2}$.

3. Main results

Now, consider the following global Lipschitz continuity condition on σ:

Condition 1. *Let there exist a nonnegative function $p \in L^2[0,1]$, such that $|\sigma(t,x) - \sigma(t,y)| \le p(t)|x-y|$, $\forall\, t \in [0,1]$, $x, y \in \mathbb{R}$, with $\sigma(t,0) = 0$ for convenience, and there exists $t_0 \in [0,1]$ such that $p(t_0) \neq 0$.*

In particular, for $p(t) = Lip_\sigma$ and for all $t \in [0,1]$, we have the following:

Condition 2. *There exist a finite positive constant Lip_σ, such that for all x, $y \in \mathbb{R}$, we have $|\sigma(t,x) - \sigma(t,y)| \le Lip_\sigma |x-y|$, with $\sigma(t,0) = 0$ for convenience.*

Now, define $L^2(\mathbb{P})$ norm of the solution u by

$$\|u\|_2 := \left\{ \sup_{0 \le t \le 1} \mathbb{E}|u(t)|^2 \right\}^{1/2}.$$

3.1. Proof of results for Equation (1)

Using Condition 2, we obtain the following results for Equation (1):

Theorem 1. *Suppose $\lambda < \frac{\sqrt{\beta(\beta+1)\Gamma(\beta)\Gamma^2(\alpha)}}{Lip_\sigma}$, where $\alpha \in (3,4)$ and $\beta > 0$. For a positive constant Lip_σ together with Condition 2, there exists a solution u for Equation (1) that is unique up to modification.*

To proof the Theorem 2, let $u(t) = \mathcal{A}u(t)$, where the operator $\mathcal{A}$ is given by

$$\mathcal{A}u(t) = \lambda \int_0^1 G(t,s) \sqrt{I^\beta[\sigma^2(s, u(s))]} dw(s),$$

and we will use the fixed point of $\mathcal{A}$. The proof follows using the Lemma(s) below:

Lemma 7. *Given a random solution u such that $\|u\|_2 < \infty$ and Condition 2 holds. Then*

$$\|\mathcal{A}u\|_2^2 \leq c_{\alpha,\beta}\lambda^2 Lip_\sigma^2 \|u\|_2^2,$$

where $c_{\alpha,\beta} = \frac{1}{\beta(\beta+1)\Gamma(\beta)\Gamma^2(\alpha)}$.

Proof. Take second moment of both sides and use Itó isometry together with Lemma 5(2) to obtain

$$\begin{aligned}
\mathbb{E}|\mathcal{A}u(t)|^2 &= \lambda^2 \int_0^1 G^2(t,s)\mathbb{E}|\sqrt{I^\beta[\sigma^2(s,u(s))]}|^2 ds \\
&\leq \lambda^2 \int_0^1 G^2(t,s)\left[\frac{1}{\Gamma(\beta)}\int_0^s (s-r)^{\beta-1}\mathbb{E}|\sigma^2(r,u(r))|dr\right]ds \\
&\leq \lambda^2 Lip_\sigma^2 \int_0^1 G^2(t,s)\left[\frac{1}{\Gamma(\beta)}\int_0^s (s-r)^{\beta-1}\mathbb{E}|u(r)|^2 dr\right]ds \\
&\leq \lambda^2 Lip_\sigma^2 \sup_{0\leq s\leq 1} G^2(t,s)\|u\|_2^2 \int_0^1 \left[\frac{1}{\Gamma(\beta)}\int_0^s (s-r)^{\beta-1}dr\right]ds \\
&\leq \lambda^2 Lip_\sigma^2 \left[\sup_{0\leq s\leq 1} G(t,s)\right]^2 \|u\|_2^2 \int_0^1 \left[\frac{1}{\Gamma(\beta)}\int_0^s (s-r)^{\beta-1}dr\right]ds \\
&\leq \frac{\lambda^2 Lip_\sigma^2}{\Gamma^2(\alpha)\Gamma(\beta)}\|u\|_2^2 \int_0^1 \left[\int_0^s (s-r)^{\beta-1}dr\right]ds = \frac{\lambda^2 Lip_\sigma^2}{\beta(\beta+1)\Gamma(\beta)\Gamma^2(\alpha)}\|u\|_2^2.
\end{aligned}$$

Now, by taking supremum over $t \in [0,1]$, the result follows. □

Remark 3. The operator $\mathcal{A}$ is a contraction for $\frac{\lambda^2 Lip_\sigma^2}{\beta(\beta+1)\Gamma(\beta)\Gamma^2(\alpha)} < 1$.

Lemma 8. *Suppose u and v are random solutions such that $\|u\|_2 + \|v\|_2 < \infty$ and Condition 2 holds. Then*

$$\|\mathcal{A}u - \mathcal{A}v\|_2^2 \leq c_{\alpha,\beta}\lambda^2 Lip_\sigma^2 \|u-v\|_2^2.$$

Proof. The proof follows similarly to the proof of Lemma 7. □

Now, we present the proof of Theorem 1.

Proof of Theorem 1. From Lemma 7, we have

$$\|u\|_2^2 = \|\mathcal{A}u\|_2^2 \leq c_{\alpha,\beta}\lambda^2 Lip_\sigma^2 \|u\|_2^2,$$

which follows that

$$\|u\|_2^2\left[1 - c_{\alpha,\beta}\lambda^2 Lip_\sigma^2\right] \leq 0,$$

and this shows that $\|u\|_2 < \infty$ whenever $c_{\alpha,\beta}\lambda^2 Lip_\sigma^2 < 1$ or $\lambda < \frac{1}{\sqrt{c_{\alpha,\beta}}Lip_\sigma}$.

Similarly from Lemma 8, we have

$$\|u-v\|_2^2 = \|\mathcal{A}u - \mathcal{A}v\|_2^2 \leq c_{\alpha,\beta}\lambda^2 Lip_\sigma^2 \|u-v\|_2^2,$$

and therefore

$$\|u-v\|_2^2\left[1 - c_{\alpha,\beta}\lambda^2 Lip_\sigma^2\right] \leq 0.$$

This implies that $\|u-v\|_2 < 0$ if and only if $c_{\alpha,\beta}\lambda^2 Lip_\sigma^2 < 1$ and thus $\|u-v\|_2 = 0 \Rightarrow u = v$. This shows the existence and uniqueness result by Banach's contraction principle. □

3.2. Existence and uniqueness for stochastic second order BVP

Here, we consider the following stochastic second order boundary value problem:

$$\begin{cases} \frac{d^2}{dt^2}u(t) = \lambda\sigma(t,u(t))\dot{w}(t), \quad 0 < t < 1 \\ u(0) = u(1) = 0, \end{cases} \tag{6}$$

where $\lambda > 0$ is a noise level parameter, $\sigma : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and $\dot{w}(t)$ is a Gaussian white noise.

Definition 4. We say that $\{u(t)\}_{0\le t\le 1}$ is a mild solution of Equation (6) if *a.s*,

$$u(t) = \lambda \int_0^1 G(t,s)\sigma(s,u(s))\dot{w}(s)ds = \lambda \int_0^1 G(t,s)\sigma(s,u(s))dw(s),$$

is satisfied, where $G(t,s)$ is as given in Equation (5).

If $\{u(t)\}_{0<t<1}$ satisfies the additional condition $\sup\limits_{0\le t\le 1} \mathbb{E}|u(t)|^2 < \infty$, then we say that $\{u(t)\}_{0\le t\le 1}$ is a random field solution to Equation (6).

Theorem 2. *Suppose Condition 1 holds and there exists a positive constant λ^* such that for any $0 < \lambda \le \lambda^*$, then Equation (6) has a unique solution.*

To proof the Theorem 2, we define the operator

$$\mathcal{B}u(x,t) = \lambda \int_0^1 G(t,s)\sigma(s,u(s))dw(s),$$

and use the fixed point of the operator $\mathcal{B}$. The proof follows using the Lemma(s) below:

Lemma 9. *Given a random solution u such that $\|u\|_2 < \infty$ and Condition 1 holds. Then there exists a positive constant λ^* such that for $0 < \lambda \le \lambda^*$,*

$$\|\mathcal{B}u\|_2 \le \frac{1}{2}\|u\|_2.$$

Proof. By Itó Isometry, we obtain

$$\begin{aligned} \mathbb{E}|\mathcal{B}u(t)|^2 &\le \lambda^2 \int_0^1 G^2(t,s)\mathbb{E}|\sigma(s,u(s))|^2 ds \\ &\le \lambda^2 \sup_{0\le t,s\le 1} G^2(t,s) \int_0^1 p^2(s)\mathbb{E}|u(s)|^2 ds \\ &\le \lambda^2 \left[\sup_{0\le t,s\le 1} G(t,s)\right]^2 \int_0^1 p^2(s)\mathbb{E}|u(s)|^2 ds \\ &\le \frac{\lambda^2}{4}\|u\|_2^2 \int_0^1 p^2(s)ds. \end{aligned}$$

Taking suprimum of both sides over $t \in [0,1]$ and letting $\lambda^* := \left(\int_0^1 p^2(s)ds\right)^{-2}$ such that $0 < \lambda \le \lambda^*$, we have $\|\mathcal{B}u\|_2^2 \le \frac{1}{4}\|u\|_2^2$. □

Following similar steps of Lemma 9, we obtain the following result.

Lemma 10. *Suppose u and v are random solutions such that $\|u\|_2 + \|v\|_2 < \infty$ and Condition 1 holds. Then there exists a positive constant λ^* such that for $0 < \lambda \le \lambda^*$,*

$$\|\mathcal{B}u - \mathcal{B}v\|_2 \le \frac{1}{2}\|u - v\|_2.$$

Proof of Theorem 2. Following Lemma 9 and Lemma 12, it is clear that $\mathcal{B}$ is a contraction. Thus by Banach fixed point theorem, the existence of a unique solution for Equation (6) follows: Let $u = \mathcal{B}$ and

$$\|u\|_2^2 = \|\mathcal{B}u\|_2^2 \leq \frac{1}{4}\|u\|_2^2 \Rightarrow \|u\|_2^2\left[1 - \frac{1}{4}\right] \leq 0 \Rightarrow \|u\|_2 = 0 \Rightarrow u = 0.$$

Assume a nontrivial solution u of Equation (6) and we show that it is unique. Suppose for contradiction that there exists another solution v of Equation (6) such that

$$\|u - v\|_2^2 = \|\mathcal{B}u - \mathcal{B}v\|_2^2 \leq \frac{1}{4}\|u - v\|_2^2,$$

then $\|u - v\|_2^2[1 - \frac{1}{4}] \leq 0$, which follows that $\|u - v\| = 0$. Thus $u = v$, a unique solution. □

Next, we seek to establish the existence and uniqueness of solution for Equation (6) using Condition 2 as follows:

Theorem 3. *Suppose $\lambda < \frac{2}{Lip_\sigma}$, for positive constant Lip_σ together with Condition 2. Then there exists solution u that is unique up to modification.*

Lemma 11. *Given a random solution u such that $\|u\|_2 < \infty$ and Condition 2 holds. Then $\|\mathcal{B}u\|_2 \leq \frac{\lambda Lip_\sigma}{2}\|u\|_2$.*

Lemma 12. *Suppose u and v are random solutions such that $\|u\|_2 + \|v\|_2 < \infty$ and Condition 2 holds. Then $\|\mathcal{B}u - \mathcal{B}v\|_2 \leq \frac{\lambda Lip_\sigma}{2}\|u - v\|_2$.*

Proof of Theorem 3. By fixed point theorem we have $u(t) = \mathcal{A}u(t)$ and $\|u\|_2^2 = \|\mathcal{B}u\|_2^2 \leq \frac{\lambda^2 Lip_\sigma^2}{4}\|u\|_2^2$, which follows that $\|u\|_2^2\left[1 - \frac{\lambda^2 Lip_\sigma^2}{4}\right] \leq 0 \Rightarrow \|u\|_2 < \infty \Leftrightarrow \lambda < \frac{2}{Lip_\sigma}$.

Similarly, $\|u - v\|_2^2 = \|\mathcal{B}u - \mathcal{B}v\|_2^2 \leq \frac{\lambda^2 Lip_\sigma^2}{4}\|u - v\|_2^2$, thus $\|u - v\|_2^2\left[1 - \frac{\lambda^2 Lip_\sigma^2}{4}\right] \leq 0$ and therefore $\|u - v\|_2 < 0$ if and only if $\lambda < \frac{2}{Lip_\sigma}$. Hence, the existence and uniqueness result follows by Banach's contraction principle. □

4. Conclusion

We studied the boundary value problems for both stochastic nonlinear fractional order differential equation and stochastic nonlinear second order equation. The existence and uniqueness result for both boundary value problems were given under different linearity conditions on σ using contraction fixed point theorem.

Acknowledgments: The first author wishes to acknowledge the continuous support of the University of Hafr Al Batin, Saudi Arabia.

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: "The authors declare no conflict of interest."

References

[1] Li, C. F., Luo, X. N., & Zhou, Y. (2010). Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations. *Computers & Mathematics with Applications, 59*(3), 1363-1375.

[2] Stanek, S. (2011). The existence of positive solutions of singular fractional boundary value problems. *Computers & Mathematics with Applications, 62*(3), 1379-1388.

[3] Xu, X., Jiang, D., & Yuan, C. (2009). Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation. *Nonlinear Analysis: Theory, Methods & Applications, 71*(10), 4676-4688.

[4] Yuan, C., Jiang, D., & Xu, X. (2009). Singular positone and semipositone boundary value problems of nonlinear fractional differential equations. *Mathematical Problems in Engineering, 2009*, Article ID 535209.

[5] Ha, S. N., & Lee, C. R. (2002). Numerical study for two-point boundary value problems using Green's functions. *Computers & Mathematics with Applications, 44*(12), 1599-1608.

[6] Biala, T. A., & Jator, S. N. (2017). A family of boundary value methods for systems of second-order boundary value problems. *International Journal of Differential Equations, 2017*, Article ID 2464759.
[7] Yongdong, S., & Liangsheng, D. (2001). On the existence of solution of a two-point boundary value problem in a cylindrical floating zone. *International Journal of Mathematics and Mathematical Sciences, 25*, Article ID 641523.
[8] Shi, Y., Zhou, Q., & Li, Y. (1997). A note on a two-point boundary value problem arising from a liquid metal flow. *SIAM Journal on Mathematical Analysis, 28*(5), 1086-1093.
[9] Horgan, C. O., Saccomandi, G., & Sgura, I. (2002). A two-point boundary-value problem for the axial shear of hardening isotropic incompressible nonlinearly elastic materials. *SIAM Journal on Applied Mathematics, 62*(5), 1712-1727.
[10] Xin, L., Guo, Y., & Zhao, J. (2019). Nontrivial Solutions Of Second-Order Nonlinear Boundary Value Problems. *Applied Mathematics E-Notes, 19*, 668-674.
[11] Bai, Z., & Lü, H. (2005). Positive solutions for boundary value problem of nonlinear fractional differential equation. *Journal of Mathematical Analysis and Applications, 311*(2), 495-505.
[12] Bai, C. (2008). Triple positive solutions for a boundary value problem of nonlinear fractional differential equation. *Electronic Journal of Qualitative Theory of Differential Equations, 2008*(24), 1-10.
[13] Kaufmann, E., & Mboumi, E. (2008). Positive solutions of a boundary value problem for a nonlinear fractional differential equation. *Electronic Journal of Qualitative Theory of Differential Equations, 2008*(3), 1-11.
[14] Kosmatov, N. (2009). A singular boundary value problem for nonlinear differential equations of fractional order. *Journal of Applied Mathematics & Computing, 29*, 125–135.