Some applications of second-order differential subordination for a class of analytic function defined by the lambda operator

B. Venkateswarlu¹∗, P. Thirupathi Reddy², S. Sridevi¹ and Sujatha¹

¹ Department of Mathematics, GSS, GITAM University, Doddaballapur- 562 163, Bengaluru Rural, Karnataka, India.
² Department of Mathematics, Kakatiya University, Warangal- 506 009, Telangana, India.
* Correspondence: bvlmaths@gmail.com

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Abstract: In this paper, we introduce a new class of analytic functions by using the lambda operator and obtain some subordination results.

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1. Introduction

Let \( \mathbb{C} \) be complex plane and let \( \mathbb{U} = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} = \mathbb{U} \setminus \{ 0 \} \) be an open unit disc in \( \mathbb{C} \). Also let \( H(\mathbb{U}) \) be a class of analytic functions in \( \mathbb{U} \). For \( n \in \mathbb{N} = \{ 1, 2, 3, \cdots \} \) and \( a \in \mathbb{C} \), let \( H[a, n] \) be a subclass of \( H(\mathbb{U}) \) formed by the functions of the form

\[
f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \cdots
\]

with \( H_0 \equiv H[0,1] \) and \( H \equiv H[1,1] \). Suppose that \( A_n \) is a class of all analytic functions of the form

\[
f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k
\]

(1)

in the open unit disk \( \mathbb{U} \) with \( A_1 = A \). A function \( f \in H(\mathbb{U}) \) is univalent if it is a one-to-one function in \( \mathbb{U} \). By \( S \), we denote a subclass of \( A \) formed by functions univalent in \( \mathbb{U} \). If a function \( f \in A \) maps \( \mathbb{U} \) onto a convex domain and \( f \) is univalent, then \( f \) is called a convex function. By

\[
K = \left\{ f \in A : \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, \ z \in \mathbb{U} \right\},
\]

we denote a class of all convex functions defined in \( \mathbb{U} \) and normalized by \( f(0) = 0 \) and \( f'(0) = 1 \).

Let \( f \) and \( F \) be elements of \( H(\mathbb{U}) \). A function \( f \) is said to be subordinate to \( F \), if there exists a Schwartz function \( w \) analytic in \( \mathbb{U} \) with \( w(0) = 0 \) and \( |w(z)| < 1, \ z \in \mathbb{U} \), such that \( f(z) = F(w(z)) \). In this case, we write \( f(z) \prec F(z) \) or \( f \prec F \). Furthermore, if the function \( F \) is univalent in \( \mathbb{U} \), then we get the following equivalence [1,2]:

\[
f(z) \prec F(z) \iff f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \prec F(\mathbb{U}).
\]

The method of differential subordinations (also known as the method of admissible functions) was first introduced by Miller and Mocanu in 1978 [3], and the development of the theory was originated in 1981 [4]. All details can be found in the book by Miller and Mocanu [2]. In recent years, numerous authors studied the properties of differential subordinations (see [5–8], etc.).

Let \( \Psi : \mathbb{C}^3 \setminus \mathbb{U} \rightarrow \mathbb{C} \) and let \( h \) be univalent in \( \mathbb{U} \). If \( p \) is analytic in \( \mathbb{U} \) and satisfies the second-order differential subordination:

\[
\Psi (p(z),zp'(z),zp''(z)) \prec h(z),
\]

(2)
then \( p \) is called the solution of differential subordination. The univalent function \( q \) is called a dominant of the solution of the differential subordination or, simply, a dominant if \( p \prec q \) for all \( p \) satisfying (2). The dominant \( q_1 \) satisfying \( q_1 \prec q \) for all dominants \( q \) of (2) is called the best dominant of (2).

Let us recall lambda function [9] defined by:

\[
\lambda(z, s) = \sum_{k=2}^{\infty} \frac{z^k}{(2k+1)^k}
\]

where \( z \in \mathbb{U}, s \in \mathbb{C} \), when \(|z| < 1, \Re(s) > 1\), when \(|z| = 1\) and let \( \lambda^{-1}(z, s) \) be defined such that

\[
\lambda(z, s) * \lambda^{-1}(z, s) = \frac{1}{(1-z)^{\mu + 1}}, \quad \mu > -1.
\]

We now define \( (z\lambda^{-1}(z, s)) \) as:

\[
(z\lambda(z, s)) * (z\lambda^{-1}(z, s)) = \frac{z}{(1-z)^{\mu + 1}} = z + \sum_{k=2}^{\infty} \frac{\mu + 1}{(k-1)!} z^{k-1}, \quad \mu > -1
\]

and obtain the linear operator \( T^s_p f(z) = (z\lambda^{-1}(z, s)) * f(z) \), where \( f \in \mathcal{A}, z \in \mathbb{U} \) and \( (z\lambda^{-1}(z, s)) = z + \sum_{k=2}^{\infty} \frac{(\mu + 1)k!}{(k-1)!} z^{k-1} \). A simple computation gives us

\[
T^s_p f(z) = z + \sum_{k=2}^{\infty} L(k, \mu, s) a_k z^k,
\]

where

\[
L(k, \mu, s) = \frac{(\mu + 1)k!}{(k-1)!} (2k-1)^s
\]

where \((\mu)_k\) is the Pochhammer symbol defined in terms of the Gamma function by:

\[
(\mu)_k = \frac{\Gamma(\mu + k)}{\Gamma(\mu)} = \begin{cases} 1, & \text{if } k = 0; \\ \mu(\mu + 1) \cdots (\mu + k - 1), & \text{if } k \in \mathbb{N}. \end{cases}
\]

**Definition 1.** Let \( \mathcal{L}_{p,s}(\sigma) \) be a class of function \( f \in \mathcal{A} \) satisfying the inequality

\[
\Re\left(T^s_p f(z) \right) \geq \sigma,
\]

where \( z \in \mathbb{U}, 0 \leq \sigma < 1 \) and \( T^s_p f(z) \) is the Lambda operator.

**Lemma 1.** Let \( h \) be a convex function with \( h(0) = a \) and let \( \gamma \in \mathbb{C}^* := \mathbb{C} \setminus \{0\} \) be a complex number with \( \Re\{\} \geq 0 \). If \( p \in \mathcal{H}[a, n] \) and

\[
p(z) + \frac{1}{\gamma} z p'(z) < h(z),
\]

then \( p(z) \prec q(z) < h(z) \), where \( q(z) = \frac{\gamma}{n \pi} \int_0^{\pi/n} h(t) dt, \; z \in \mathbb{U} \). The function \( q \) is convex and is the best dominant for subordination (5).

**Lemma 2.** [10] Let \( \Re\{\} > 0, n \in \mathbb{N} \) and \( w = \frac{n^2 + n - 1}{4n} \). Also, let \( h \) be an analytic function in \( \mathbb{U} \) with \( h(0) = 1 \). Suppose that \( \Re\left\{1 + \frac{zh'(z)}{h(z)}\right\} > -w \). If \( p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots \) is analytic in \( \mathbb{U} \) and

\[
p(z) + \frac{1}{\mu} z p'(z) < h(z),
\]

then \( p(z) \prec q(z) < h(z) \),
then \( p(z) \prec q(z) \), where \( q \) is a solution of the differential equation \( q(z) + \frac{n}{n+1}zq'(z) = h(z) \), \( q(0) = 1 \), given by
\[
q(z) = \frac{1}{n+1} \int_0^z t^n h(t) dt, \quad z \in U.
\]
Moreover, \( q \) is the best dominant for the differential subordination (6).

**Lemma 3.** [11] Let \( r \) be a convex function in \( U \) and let \( h(z) = r(z) + nqz'(z) \), \( z \in U \), where \( q > 0 \) and \( n \in \mathbb{N} \). If \( p(z) = r(0) + pnz^n + pn+1z^{n+1} + \cdots \), \( z \in U \), is holomorphic in \( U \) and \( p(z) + qzp'(z) \prec h(z) \), \( z \in U \), then \( p(z) \prec r(z) \) and this result is sharp.

In the present paper, we use the subordination results from [10] to prove our main results.

2. Main results

**Theorem 1.** The set \( \mathcal{L}_{\mu,s}(q) \) is convex.

**Proof.** Let \( f_j(z) = z + \sum_{k=2}^{\infty} a_{kj} z^k \), \( z \in U \), \( j = 1, \cdots, m \) be in the class \( \mathcal{L}_{\mu,s}(q) \). Then, by Definition 1, we get
\[
\Re \left\{ (I_{\mu}^s f)(z) \right\} = \Re \left\{ 1 + \sum_{k=2}^{\infty} L(k, \mu, s) a_{k} z^{k-1} \right\} > q. \quad (7)
\]

For any positive numbers \( \xi_1, \xi_2, \xi_3, \cdots, \xi_m \) such that \( \sum_{j=1}^{m} \xi_j = 1 \), it is necessary to show that the function
\[
h(z) = \sum_{j=1}^{m} \xi_j f_j(z)
\]
is an element of \( \mathcal{L}_{\mu,s}(q) \), i.e.,
\[
\Re \left\{ (I_{\mu}^s h)(z) \right\} > q. \quad (8)
\]

Thus, we have
\[
I_{\mu}^s h(z) = z + \sum_{k=2}^{\infty} L(k, \mu, s) \left\{ \sum_{j=1}^{m} \xi_j a_{kj} \right\} z^k. \quad (9)
\]

If we differentiate (9) with respect to \( z \), then we obtain
\[
(I_{\mu}^s h(z))' = 1 + \sum_{k=2}^{\infty} k L(k, \mu, s) \left\{ \sum_{j=1}^{m} \xi_j a_{kj} \right\} z^{k-1}. \quad (10)
\]

Thus by using (8), we have
\[
\Re \left\{ (I_{\mu}^s h(z))' \right\} = 1 + \sum_{j=1}^{m} \xi_j \Re \left\{ \sum_{k=2}^{\infty} k L(k, \mu, s) a_{kj} z^{k-1} \right\} > 1 + \sum_{j=1}^{m} \xi_j (q-1) = q.
\]

Hence, inequality (7) is true and we arrive at the desired result. \( \Box \)

**Theorem 2.** Let \( q \) be a convex function in \( U \) with \( q(0) = 1 \) and \( h(z) = q(z) + \frac{1}{\gamma+1}zq'(z) \), \( z \in U \), where \( \gamma \) is a complex number with \( \Re \{ \gamma \} > -1 \). If \( f \in \mathcal{L}_{\mu,s}(q) \) and \( \mathcal{R} = Y_{\gamma} f \), where
\[
\mathcal{R}(z) = Y_{\gamma} f(z) = \frac{\gamma + 1}{z^\gamma} \int_0^z t^{\gamma-1} f(t) dt,
\]
then
\[
(I_{\mu}^s \mathcal{R}(z))' \prec h(z)
\]
implies that \( (I_{\mu}^s \mathcal{R}(z))' \prec q(z) \) and this result is sharp.
Proof. In view of equality (10), we can write
\[
z^{\gamma}N(z) = (\gamma + 1) \int_0^z t^{\gamma-1} f(t) \, dt. \tag{12}
\]

Differentiating (12) with respect to \( z \), we obtain \((\gamma)N(z) + zN'(z) = (\gamma + 1)f(z)\). Further, by applying the operator \( \mathcal{I}_\mu^s \) to the last equation, we get
\[
(\gamma)\mathcal{I}_\mu^s N(z) + z\mathcal{I}_\mu^s N(z)' = (\gamma + 1)\mathcal{I}_\mu^s f(z). \tag{13}
\]

If we differentiate (13) with respect to \( z \), then we find
\[
(\mathcal{I}_\mu^s N(z))' + \frac{1}{\gamma + 1}z(\mathcal{I}_\mu^s f(z))'' = (\mathcal{I}_\mu^s f(z))'. \tag{14}
\]

By using the differential subordination given by (11) in equality (14), we obtain
\[
(\mathcal{I}_\mu^s N(z))' + \frac{1}{\gamma + 1}z(\mathcal{I}_\mu^s f(z))'' < h(z). \tag{15}
\]

We define
\[
p(z) = (\mathcal{I}_\mu^s N(z))'. \tag{16}
\]

Hence, as a result of simple computations, we get
\[
p(z) = \left\{ z + \sum_{k=1}^\infty L(k, \mu, s) \frac{z^{\gamma+k}}{\gamma+k} q^k \right\}' = 1 + p_1 z + p_2 z^2 + \cdots, \quad p \in H[1, 1].
\]

By using (16) in subordination (15), we obtain
\[
p(z) + \frac{1}{\gamma + 1} z p'(z) < h(z) = q(z) + \frac{1}{\gamma + 1} z q'(z), \quad z \in U.
\]

If we use Lemma 2, then we write \( p(z) < q(z) \). Thus, we obtained the desired result and \( q \) is the best dominant. \( \square \)

Example 1. If we choose \( \gamma = i + 1 \) and \( q(z) = \frac{1+i+2}{1+i+2} \), in Theorem 2, then we get \( h(z) = \frac{(i+2)-(i+2)(z+2)z}{(i+2)(1-z)^2} \). If \( f \in \mathcal{L}_{\mu;\beta}(q) \) and \( N \) is given as \( N(z) = \gamma f(z) = \frac{i+2}{1+i+2} \int_0^z f(t) \, dt \), then, by virtue of Theorem 2, we find \( (\mathcal{I}_\mu^s f(z))' < h(z) = \frac{(i+2)-(i+2)(z+2)z}{(i+2)(1-z)^2} \), implies \( (\mathcal{I}_\mu^s f(z))' < \frac{1+i+2}{1+i+2} \).

Theorem 3. Let \( \Re\{\gamma\} > -1 \) and \( w = \frac{1+|\gamma+1|^2-|\gamma|^2+2\gamma}{4\Re\{\gamma+1\}} \). Suppose that \( h \) is an analytic function in \( U \) with \( h(0) = 1 \) and that \( \Re\left\{ 1 + \frac{\Re\{z\}}{\Re\{z\}} \right\} > -w \). If \( f \in \mathcal{L}_{\mu;\beta}(q) \) and \( N = \gamma f, \) where \( \gamma \) is defined by (10), then
\[
(\mathcal{I}_\mu^s f(z))' < h(z) \tag{17}
\]

implies that \( (\mathcal{I}_\mu^s N(z))' < q(z) \), where \( q \) is the solution of the differential equation \( h(z) = q(z) + \frac{1}{\gamma+1} z q'(z), \quad q(0) = 1, \)
given by \( q(z) = \frac{\gamma+1}{\gamma+1} \int_0^z f(t) \, dt \). Moreover, \( q \) is the best dominant for subordination (17).

Proof. If we choose \( n = 1 \) and \( \mu = \gamma + 1 \) in Lemma 1, then the proof is obtained by means of the proof of Theorem 3. \( \square \)

Theorem 4. Let
\[
h(z) = \frac{1+(2\varphi-1)z}{1+z}, \quad 0 \leq \varphi < 1 \tag{18}
\]
be convex in \( U \) with \( h(0) = 1 \). If \( f \in A \) and verifies the differential subordination \((T^\mu_\gamma f(z))' \prec h(z)\), then \((T^\mu_\gamma h(z))' \prec q(z) = (2\eta - 1) + {2(1-\eta)(\gamma + 1)\tau(\gamma)}{z^{\gamma+1}}\), where \( \tau \) is given by the formula

\[
\tau(\gamma) = \int_0^\gamma \frac{t^\gamma}{1 + t} dt
\]  

and \( \mathcal{N} \) is given by equation (10). The function \( q \) is convex and is the best dominant.

**Proof.** If \( h(z) = \frac{1 + (2\eta - 1)z}{1 + z}, \ 0 \leq \eta < 1 \), then \( h \) is convex and, in view of Theorem 3, we can write \((T^\mu_\gamma h(z))' \prec q(z)\). Now, by using Lemma 1, we get

\[
q(z) = \frac{\gamma + 1}{z^{\gamma+1}} \int_0^\gamma t^\gamma h(t) dt = \frac{\gamma + 1}{z^{\gamma+1}} \int_0^\gamma \left\{ \frac{1 + (2\eta - 1)t}{1 + t} \right\} dt = (2\eta - 1) + \frac{2(1-\eta)(\gamma + 1)\tau(\gamma)}{z^{\gamma+1}},
\]

where \( \tau \) is given by (19). Hence, we obtain

\[
(T^\mu_\gamma h(z))' \prec q(z) = (2\eta - 1) + \frac{2(1-\eta)(\gamma + 1)\tau(\gamma)}{z^{\gamma+1}}.
\]

The function \( q \) is convex. Moreover, it is the best dominant. Hence the theorem is proved. \( \square \)

**Theorem 5.** If \( 0 \leq \eta < 1, 0 \leq \mu < 1, \sigma \geq 0, \Re\{\gamma\} > -1 \), and \( \mathcal{N} = Y_\gamma f \) is defined by (10), then \( Y_\gamma (\mathcal{L}_{\mu,\sigma}(\eta)) \subset \mathcal{L}_{\mu,\sigma}(\rho) \), where

\[
\rho = \min_{|z|=1} \Re\{q(z)\} = \rho(\gamma, \eta) = (2\eta - 1) + 2(1-\eta)(\gamma + 1)\tau(\gamma)
\]

and \( \tau \) is given by (19).

**Proof.** Assume that \( h \) is given by equation (18), \( f \in \mathcal{L}_{\mu,\sigma}(\eta) \), and \( \mathcal{N} = Y_\gamma f \) is defined by (10). Then \( h \) is convex and, by Theorem 3, we deduce

\[
(T^\mu_\gamma h(z))' \prec q(z) = (2\eta - 1) + \frac{2(1-\eta)(\gamma + 1)\tau(\gamma)}{z^{\gamma+1}},
\]

where \( \tau \) is given by (19). Since \( q \) is convex, \( q(\mathcal{U}) \) is symmetric about the real axis, and \( \Re\{\gamma\} > -1 \), we find

\[
\Re\{ (T^\mu_\gamma h(z))' \} \geq \min_{|z|=1} \Re\{q(z)\} = \Re\{q(1)\} = \rho(\gamma, \eta) = (2\eta - 1) + 2(1-\eta)(\gamma + 1)(1-\eta)\tau(\gamma).
\]

It follows from inequality (21) that \( Y_\gamma (\mathcal{L}_{\mu,\sigma}(\eta)) \subset \mathcal{L}_{\mu,\sigma}(\rho) \), where \( \rho \) is given by (20). Hence the theorem is proved. \( \square \)

**Theorem 6.** Let \( q \) be a convex function with \( q(0) = 1 \) and \( h \) be a function such that \( h(z) = q(z) + \zeta q'(z) \), \( z \in \mathbb{U} \). If \( f \in A \), then the subordination

\[
(T^\mu_\gamma f(z))' \prec h(z)
\]

implies that \( \frac{T^\mu_\gamma f(z)}{z} \prec q(z) \), and the result is sharp.

**Proof.** Let

\[
p(z) = \frac{T^\mu_\gamma f(z)}{z}.
\]

Differentiating (23), we find \((T^\mu_\gamma f(z))' = p(z) + zp'(z)\). We now compute \( p(z) \). This gives

\[
p(z) = \frac{T^\mu_\gamma f(z)}{z} = \frac{z + \sum_{k=2}^{\infty} L(k, \mu, s) \zeta^k}{z} = 1 + p_1 z + p_2 z^2 + \cdots, \ p \in H[1,1].
\]
By using (24) in subordination (22), we find \( p(z) + zp'(z) \prec h(z) = q(z) + zq'(z) \). Hence, by applying Lemma 3, we conclude that \( p(z) \prec q(z) \) i.e., \( \frac{T^q_f(z)}{z} \prec q(z) \). This result is sharp and \( q \) is the best dominant. Hence the theorem is proved. \( \Box \)

**Example 2.** If we take \( \mu = 0 \) and \( s = 1 \) in equality (4) and \( q(z) = \frac{1}{1-z} \) in Theorem 5, then \( h(z) = \frac{1}{1-z} \) and

\[
I_0^h f(z) = z + \sum_{k=2}^{\infty} \frac{(2k-1)}{(k-1)!} a_k z^k.
\]  

(25)

Differentiating (25) with respect to \( z \), we get

\[
(I_0^h f(z))' = 1 + \sum_{k=2}^{\infty} \frac{(2k-1)}{(k-1)!} a_k z^{k-1} = 1 + p_1 z + p_2 z^2 + \cdots, \quad p \in H[1,1].
\]

By using Theorem 5, we find \( (I_0^h f(z))' \prec h(z) = \frac{1}{1-z} \). This yields \( \frac{I_0^h f(z)}{z} \prec q(z) = \frac{1}{1-z} \).

**Theorem 7.** Let \( h(z) = \frac{1+(2\epsilon-1)z}{1+z} \), \( z \in \mathbb{U} \) be convex in \( \mathbb{U} \) with \( h(0) = 1 \) and \( 0 \leq \epsilon < 1 \). If \( f \in A \) satisfies the differential subordination

\[
(T^q_f(z))' \prec h(z),
\]

then \( \frac{T^q_f(z)}{z} \prec q(z) = (2\epsilon - 1) + \frac{2(1-\epsilon)\ln(1+z)}{z} \). The function \( q \) is convex and, in addition, it is the best dominant.

**Proof.** Let

\[
p(z) = \frac{T^q_f(z)}{z} = 1 + p_1 z + p_2 z^2 + \cdots, \quad p \in H[1,1].
\]  

(27)

Differentiating (27), we find

\[
(T^q_f(z))' = p(z) + zp'(z).
\]  

(28)

In view of (28), the differential subordination (26) becomes \( (T^q_f(z))' \prec h(z) = \frac{1+(2\epsilon-1)z}{1+z} \), and by using Lemma 1, we deduce \( p(z) \prec q(z) = \frac{1}{z} \int h(t)dt = (2\epsilon - 1) + \frac{2(1-\epsilon)\ln(1+z)}{z} \). Now, by virtue of relation (27) we obtained the desired result. \( \Box \)

**Corollary 1.** If \( f \in \mathcal{L}_{\mu,s}(q) \), then \( \Re \left( \frac{T^q_f(z)}{z} \right) > (2\epsilon - 1) + 2(1-\epsilon)\ln(2) \).

**Proof.** If \( f \in \mathcal{L}_{\mu,s}(q) \), then it follows from Definition 1 that \( \Re \left\{ (T^q_f(z))' \right\} > q, \quad z \in \mathbb{U} \), which is equivalent to \( (T^q_f(z))' \prec h(z) = \frac{1+(2\epsilon-1)z}{1+z} \). Now, by using Theorem 7, we obtain

\[
\frac{T^q_f(z)}{z} \prec q(z) = (2\epsilon - 1) + \frac{2(1-\epsilon)\ln(1+z)}{z}.
\]

Since \( q \) is convex and \( q(\mathbb{U}) \) is symmetric about the real axis, we conclude that

\[
\Re \left( \frac{T^q_f(z)}{z} \right) > \Re(q(1)) = (2\epsilon - 1) + 2(1-\epsilon)\ln(2).
\]

(29)

**Theorem 8.** Let \( q \) be a convex function such that \( q(0) = 1 \) and \( h \) be the function given by the formula \( h(z) = q(z) + zq'(z) \), \( z \in \mathbb{U} \). If \( f \in A \) and verifies the differential subordination

\[
\left\{ \frac{zT^q_f(z)}{T^q_h(z)} \right\}' \prec h(z), \quad z \in \mathbb{U},
\]  

(29)
then \( \frac{T_\mu^s f(z)}{T_\mu^s N(z)} < q(z), \ z \in \mathbb{U}, \) and this result is sharp.

**Proof.** For function \( f \in A \), given by Equation (1), we get

\[
T_\mu^s N(z) = z + \sum_{k=2}^{\infty} L(k, \mu, s) \frac{\gamma + 1}{k + \gamma} a_k b_k z^k, \ z \in \mathbb{U}.
\]

We now consider the function

\[
p(z) = \frac{T_\mu^s f(z)}{T_\mu^s N(z)} = \frac{z + \sum_{k=2}^{\infty} L(k, \mu, s) a_k b_k z^k}{z + \sum_{k=2}^{\infty} L(k, \mu, s) \frac{\gamma + 1}{k + \gamma} a_k b_k z^k} = \frac{1 + \sum_{k=2}^{\infty} L(k, \mu, s) a_k b_k z^k}{1 + \sum_{k=2}^{\infty} L(k, \mu, s) \frac{\gamma + 1}{k + \gamma} a_k b_k z^{k-1}}.
\]

In this case, we get

\[
(p(z))' = \frac{(T_\mu^s f(z))'}{T_\mu^s N(z)} - p(z) \frac{(T_\mu^s N(z))'}{T_\mu^s N(z)}.
\]

Then

\[
p(z) + zp'(z) = \left\{ z \frac{T_\mu^s f(z)}{T_\mu^s N(z)} \right\}', \ z \in \mathbb{U}.
\]

By using relation (30) in inequality (29), we obtain \( p(z) + zp'(z) < h(z) = q(z) + zq'(z) \) and, by virtue of Lemma 3, \( p(z) < q(z) \), i.e., \( \frac{T_\mu^s f(z)}{T_\mu^s N(z)} < q(z) \). Hence the theorem is proved. \( \square \)

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**References**


