In this paper, we consider the following problem

\[
\begin{aligned}
    &u_{tt} + \mu \partial_1^{1+\alpha} u + \Delta^2 u + a(x,y,t)u = |u|^{p-1}u, \quad \text{in } \Omega \times (0,T), \\
    &u(x,y,0) = u_0(x,y), \quad u_t(x,y,0) = u_1(x,y), \quad \text{in } \Omega
\end{aligned}
\]

with partially hinged boundary condition

\[
\begin{aligned}
    &u(0,y,t) = u_{xx}(0,y,t) = 0, \quad \text{for } (y,t) \in (-\ell,\ell) \times (0,T), \\
    &u(L,y,t) = u_{xx}(L,y,t) = 0, \quad \text{for } (y,t) \in (-\ell,\ell) \times (0,T), \\
    &u_{yy}(x,\pm \ell,t) + \nu u_{xx}(x,\pm \ell,t) = 0, \quad \text{for } (x,t) \in (0,L) \times (0,T), \\
    &u_{yy}(x,\pm \ell,t) + (2-\nu) u_{xxy}(x,\pm \ell,t) = 0, \quad \text{for } (x,t) \in (0,L) \times (0,T),
\end{aligned}
\]

where \( \Omega = (0,L) \times (-\ell,\ell) \subset \mathbb{R}^2 \) represent a thin rectangular plate as a model of a suspension bridge and \( u = u(x,y,t) \) is the downward displacement of the rectangular plate, see [1,2] for detail description of suspension bridge models. The function \( a = a(x,y,t) \) is bounded, continuous and sign changing. For instance, if \( h : [0,\infty) \to (-\infty,\infty) \) be any function and \( g : \Omega \to (-\infty,\infty) \) be a bounded function, then \( a(x,t) = (\text{signh})(t)g(x) \) is example of a sign changing function. Furthermore, \( \mu > 0, 0 < \nu < 1, 1 < p < \infty \) and \( -1 < \alpha < 1 \). The notation \( \partial_1^{1+\alpha} \) stand for the Capito’s fractional derivative (see [3,4]) of order \( 1+\alpha \) with respect to \( t \) defined by

\[
\partial_1^{1+\alpha} u(t) = \begin{cases} 
    I^{-\alpha} \frac{du(t)}{dt}, & \text{if } -1 < \alpha < 0, \\
    I^{1-\alpha} \frac{d^2u(t)}{dt^2}, & \text{if } 0 < \alpha < 1,
\end{cases}
\]

where \( I^\beta (\beta > 0) \) is the fractional derivative defined by

\[
I^\beta \frac{du(t)}{dt} = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} u(\tau) d\tau.
\]
For $-1 < \alpha < 0$, the term $\partial_t^{1+\alpha}u$ is called the fractional damping while for $\alpha = -1$ and $\alpha = 0$, it represent respectively the weak and strong damping. We should mention here that the fractional damping plays a dissipative role that is sandwich between the weak and the strong damping (see [5]). Concerning blow up results for plate equations, we mention among others the result of Messaoudi [6], where he studied the Petrovsky equation

$$u_{tt} + \Delta^2 u + a |u_t|^{m-2} u_t = b |u|^{p-2} u,$$

where $a, b > 0$ and $\Omega \subset \mathbb{R}^N$, $N \geq 1$ is a bounded domain with a smooth boundary $\partial \Omega$. He established local existence and uniqueness of a weak local solution and that for negative initial energy ($E(0) < 0$) the local solution blows up in finite time when $p > m$. In addition, established the existence of global solution when $m \geq p$. The result in [6] was later improved by Chen and Zhou in [7]. Li et al. [8] considered

$$u_{tt} + \Delta^2 u - \Delta u + |u_t|^{m-1} u_t = |u|^{p-1} u$$

and established global existence and blow up of solutions. Piskin and Polat [9] considered (6) and investigated the decay of solutions. Alaimia and Tatar [10] studied

$$\begin{cases}
    u_{tt} - \Delta u + \partial_t^{1+\alpha} u = |u|^{p-1} u, & x \in \Omega, t > 0 \\
    u = 0, \partial \Omega, t > 0, \\
    u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega
\end{cases}$$

and proved blow up of the solutions for negative initial energy. For related results with fractional damping, we refer the reader to [11–18] and references therein. The article is organized as follows: In Section 2, we recall some fundamental materials and useful assumptions on the relaxation function $g$. In Section 3, we state and prove some technical lemmas. Finally, in section 4, we establish a blow-up result for problem 1.

2. Preliminaries

Throughout the paper, $C_i$, $i = 1, 2, 3, \ldots$ or $c$ are generic positive constants that may change within lines and $(,)_2$ and $||.||_2$ denote respectively the inner product and norm in $L^2(\Omega)$. We recall some useful materials. We consider the Hilbert space (see [1])

$$H^2_r(\Omega) = \left\{ w \in H^2(\Omega) : w = 0 \text{ on } \{0,L\} \times (-\ell, \ell) \right\},$$

together with the inner product

$$(u,v)_{H^2_r} = \int_{\Omega} \left( (\Delta u \Delta v + (1-v)(2u_{xy}v_{xy} - u_{x}v_{y} - u_{y}v_{x}) \right) dxdy,$$

and denote by $H(\Omega)$ the dual of $H^2_r(\Omega)$.

**Lemma 1.** (Embedding, see [19]) Suppose $1 < p < +\infty$. Then for any $u \in H^2_r(\Omega)$, there exists an embedding constant $S_p = S_p(\Omega, p) > 0$ such that

$$||u||_{L^p(\Omega)} \leq S_p ||u||_{H^2_r(\Omega)},$$

where $S_p = \left( \frac{L^2}{2} + \frac{\ell^2}{2} \right) (2L \ell)^{\frac{p+2}{2}} \left( \frac{1}{1-p} \right)^\frac{1}{2}$.

The eigenvalue problem

$$\begin{cases}
    \Delta^2 u = \lambda u, & (x,y) \in \Omega, \\
    u(0,y) = u_x(0,y) = u(L,y) = u_{xx}(L,y) = 0, \text{ for } y \in (-\ell, \ell), \\
    u_{yy}(x,\pm \ell) + v u_{xx}(x,\pm \ell) = 0, \text{ for } x \in (0, L), \\
    u_{yy}(x,\pm \ell) + (2-v) u_{xy}(x,\pm \ell) = 0, \text{ for } x \in (0, L)
\end{cases}$$

(9)
which has been studied in [1], has a unique eigenvalue $\lambda_1 \in (1 - \nu, 1)$, $0 < \nu < \frac{1}{2}$ and $\lambda = \lambda_1^2$ is the least eigenvalue. As a consequent, we have the following lemma

**Lemma 2.** Suppose $-\lambda_1 < a_1 \leq a \leq a_2$. Then, the following inequality holds

$$A_1 \|u\|_{H^2_0(\Omega)}^2 \leq \|u\|_{H^2_0(\Omega)}^2 + (au, u)_2 \leq A_2 \|u\|_{H^2_0(\Omega)}^2,$$

where $A_1 = \left\{1 + \frac{a_1}{a}, a_1 < 0, 1, a_1 \geq 0 \right\}$ and $A_2 = \left\{1, a_2 < 0, 1 + \frac{a_2}{a}, a_2 \geq 0 \right\}$ which has been proved in [19].

For completeness, we state without proof a local existence result for problem (1)-(2) (see [19,20] for more on existence).

**Theorem 1.** Let $(u_0, u_1) \in H^2_0(\Omega) \times L^2(\Omega)$ be given and assume $-\lambda_1 < a_1 \leq a \leq a_2$. Then, there exists a weak unique local solution to problem (1) - (2) in the class

$$u \in L^\infty \left([0, T), H^2_0(\Omega)\right), u_t \in L^\infty \left([0, T), L^2(\Omega)\right), u_{tt} \in L^\infty([0, T), \mathcal{H}(\Omega)),$$

for some $T > 0$.

**Definition 1.** A function $u$ satisfying (11) is called a weak solution of (1) if

$$\frac{d}{dt}(u_t(t), w)_2 + \frac{\mu}{\Gamma(-a)} \int_\Omega w \int_0^t (t-s)^{-(a+1)} u_s(s) ds dxdy + (u(t), w)_{H^2_0(\Omega)} + (au(t), w)_2 = \int_\Omega |u|^{p-1} w dxdy$$

a.e $t \in (0, T)$ and $\forall w \in H^2_0(\Omega)$.

We consider the energy functional $E(t)$ defined by

$$E(t) = \frac{1}{2} \|u_t(t)\|^2 + \frac{1}{2} \|u(t)\|^2_{H^2_0(\Omega)} + \frac{1}{2} (au(t), u(t))_2 - \frac{1}{p+1} \int_\Omega |u|^{p+1} dx dy.$$

Multiplying (1) by $u_t$ and integrating over $\Omega$, using integration by part, definition of fractional derivative (4) and recalling that $a_1 \leq a \leq a_2$, we obtain

$$E'(t) = -\frac{\mu}{\Gamma(-a)} \int_\Omega u_t \int_0^t (t-s)^{-(a+1)} u_s(s) ds dxdy$$

for almost all $t \in [0, T)$. The result in (14) is for any regular solutions. However, this result remains valid for weak solutions by simple density argument. We define a modify energy functional:

$$E_\epsilon(t) = E(t) - \epsilon (u, u_t)_2,$$

for some $\epsilon$ to be specified later. Differentiating (15) and making use of (1) and (14), we arrive at

$$E'_\epsilon(t) = -\frac{\epsilon \mu}{\Gamma(-a)} \int_\Omega u_t \int_0^t (t-s)^{-(a+1)} u_s(s) ds dxdy - \epsilon \|u_t(t)\|^2 + \epsilon \|u(t)\|^2_{H^2_0(\Omega)}$$

$$+ \frac{\epsilon \mu}{\Gamma(-a)} \int_\Omega u_t \int_0^t (t-s)^{-(a+1)} u_s(s) ds dxdy + \epsilon (au, u)_2 - \epsilon \int_\Omega |u|^{p+1} dx dy.\tag{16}$$

Also, we define the functional

$$H(t) = - (e^{-\gamma t} E_\epsilon(t) + \theta F(t) + \lambda),$$

where

$$F(t) = \int_\Omega \int_0^t M(t-s) e^{-\gamma s} u^2(x, y, s) ds dxdy$$

and (17).
with

\[ M(t) = e^{\beta t} \int_{t}^{+\infty} e^{-\beta s} s^{-(\alpha+1)} ds, \tag{19} \]

where \( \gamma = \frac{\mu+1}{2} \) and \( \theta, \lambda, \beta \) are positive constants to be specified later. The differentiation of \((18)\) gives the relation

\[ F'(t) = \beta^2 \Gamma(-\alpha) e^{-\gamma t} ||u(t)||_2^2 + \beta F(t) - \int_{t_0}^{t} (t-s)^{-(\alpha+1)} \epsilon \gamma e u_0^2(s) ds dx dy. \tag{20} \]

In the next section, we state and prove some useful Lemmas.

3. Technical lemma

**Lemma 3.** Suppose \( E_0 < 0 \) and \( p \) is sufficiently large, then \( H(t) \) and \( H'(t) \) are strictly positive.

**Proof.** Differentiating \((17)\) with respect to \( t \) and using \((15)\) yields

\[ H'(t) = \gamma e^{-\gamma t} E_0(t) - e^{-\gamma t} E'_0(t) - \theta F'(t) \]

\[ = \gamma e^{-\gamma t} E(t) - \gamma e^{2} e^{-\gamma t}(u,v)_2 - e^{-\gamma t} E'_0(t) - \theta F'(t). \tag{21} \]

Substituting \((13),(16)\) and \((20)\) into \((21)\), we arrive at

\[ H'(t) = \left[ \frac{\gamma e}{2} + e - \beta^2 \Theta \Gamma(-\alpha) \right] e^{-\gamma t} \|u(t)\|_2^2 + \left[ \frac{\gamma e}{2} - e \right] e^{-\gamma t} \|u(t)\|_2^2 \]

\[ + \left[ \frac{\gamma e}{2} - e \right] e^{-\gamma t} (u,v)_2 + \left[ e - \frac{\gamma e}{p+1} \right] e^{-\gamma t} \int_{t_0}^{t} |u|^{p+1} dx dy \]

\[ - \gamma e^{2} e^{-\gamma t}(u,v)_2 + \left[ \frac{\mu e^{-\gamma t}}{\Gamma(-\alpha)} \right] \int_{t_0}^{t} (t-s)^{-(\alpha+1)} u_0 u(s) ds dx dy \]

\[ + \theta \int_{t_0}^{t} (t-s)^{-(\alpha+1)} e^{-\gamma e u_0^2(s)} ds dx dy - \beta \theta F(t). \tag{22} \]

Using Young’s inequality and Lemma 1, we obtain

\[ (u,v)_2 \leq \delta_1 S_2^2 \|u(t)\|_{H^2_0}^2 + \frac{1}{\delta_1} \|u(t)\|_2^2, \quad \delta_1 > 0. \tag{23} \]

Again, Young’s and Cauchy-Schwarz inequalities, we get

\[ e^{-\gamma t} \int_{t_0}^{t} (t-s)^{-(\alpha+1)} u_0 u(s) ds dx dy \]

\[ \leq \delta_2 e^{-\gamma t} \|u(t)\|_2^2 + \frac{\gamma e \Gamma(-\alpha)}{4 \delta_2} \int_{t_0}^{t} (t-s)^{-(\alpha+1)} u_0^2(s) ds dx dy \leq \delta_2 e^{-\gamma t} \|u(t)\|_2^2 + \frac{\gamma e \Gamma(-\alpha)}{4 \delta_2} \int_{t_0}^{t} (t-s)^{-(\alpha+1)} e^{-\gamma e u_0^2(s)} ds dx dy, \quad \delta_2 > 0. \tag{24} \]

In a similar way, with the help of lemma 1, we find

\[ e^{-\gamma t} \int_{t_0}^{t} (t-s)^{-(\alpha+1)} u_0 u(s) ds dx dy \]

\[ \leq \delta_3 \|u(t)\|_{H^2_0}^2 + \frac{\gamma e \Gamma(-\alpha)}{4 \delta_3} \int_{t_0}^{t} (t-s)^{-(\alpha+1)} e^{-\gamma e u_0^2(s)} ds dx dy, \quad \delta_3 > 0. \tag{25} \]

Substitution of \((23)-(25)\) into \((22)\) and using lemma 2, we obtain
Adding $C_1 H(t) - C_1 H(t)$ to the right hand of (26), for some $C_1$ to be precise, we arrive

$$H'(t) \geq C_1 H(t) + \left[ C_1 \tfrac{\gamma e}{2} + \tfrac{\gamma e}{2} + \epsilon - \beta^a \theta \Gamma(-a) - \tfrac{\gamma e^2}{4 \delta_1} - \frac{\delta_2}{\Gamma(-a)} - \frac{C_1 e}{4 \delta_1} \right] e^{-\gamma t} \| u_i(t) \|_2^2$$

Applying (23) to (27), we arrive at

$$H'(t) \geq \left[ \frac{C_1 e}{2} + \frac{\gamma e}{2} + \epsilon - \beta^a \theta \Gamma(-a) - \frac{\gamma e^2}{4 \delta_1} - \frac{\delta_2}{\Gamma(-a)} - \frac{C_1 e}{4 \delta_1} \right] e^{-\gamma t} \| u_i(t) \|_2^2$$

Recalling that $\gamma = \frac{p+1}{2}$ and choosing $\delta_1 = \frac{1}{2}$, $\delta_2 = \frac{\Gamma(-a) \epsilon}{2}$ and $C_1 = \frac{(p+1) \epsilon}{2}$, we get

$$H'(t) \geq \frac{p+1}{2} H(t) + \left[ \frac{p+1}{2} \epsilon (1 - \epsilon) - \beta^a \theta \Gamma(-a) \right] e^{-\gamma t} \| u_i(t) \|_2^2$$

Now, choosing

$$\epsilon < \epsilon_1 := \min \left\{ 1, \frac{A_1 (p-1)}{2 S_2 ((p+1) + \mu)} \right\},$$

we get that

$$\frac{\epsilon}{2} \left[ A_1 (p-1) - \epsilon S_2 ((p+1) + \mu) \right] > \frac{A_1 (p-1) \epsilon}{4}.$$
Next, we select $\beta = 1$, we see that for sufficiently large values of $p$
\[
\frac{(p+1)e}{2} - \beta > 0.
\]

Finally, we choose $\theta$ such that the coefficient of the second term is non-negative and the coefficient of the last term is greater than $\frac{\mu(p+1)^a}{2^{a+1}e^{1-a}(-\alpha)}$. Thus, we arrive at
\[
H'(t) \geq \frac{(p+1)e}{2}H(t) + \frac{A_1(p-1)e}{4}e^{-\gamma \epsilon t}\|u(t)\|^2_{H^2(\Omega)}
+ \frac{\mu(p+1)^a}{2^{a+1}e^{1-a}(-\alpha)} \int_0^t (t-s)^{-(\alpha+1)}e^{-\gamma \epsilon s}u^2(s)dsdxdy. \tag{31}
\]

If we choose $\lambda < -E_\epsilon(0)$, then $H(0) > 0$. Consequently, it follows from (31) that $H(t) > 0$ and $H'(t) > 0$. This completes the proof. \hfill \square

4. Main results

In this section, we show that the solutions of 1-2 blows up in finite time for negative initial energy.

**Theorem 2.** Assume that $-\alpha_1 < a_1 \leq a \leq a_2$, $-1 < \alpha \leq 0$, $E(0) < 0$ and $(u_0, u_1)_2 \geq 0$. Then the solutions of 1-2 blows up in finite time for sufficiently large values of $p$.

**Proof.** We begin by defining the functional $G$ by
\[
G(t) = H^{1-\sigma}(t) + \eta e^{-\gamma \epsilon t}(u, u)_2, \tag{32}
\]
where $\sigma = \frac{p-1}{2(p+1)}$ and $\eta > 0$ to be specified later. Then differentiating $G(t)$ and using (1) yields
\[
G'(t) = (1-\sigma)H^{-\sigma}(t)H'(t) - \eta \gamma \epsilon e^{-\gamma \epsilon t}(u, u)_2 + \eta e^{-\gamma \epsilon t}\|u(t)\|^2_2 + \eta e^{-\gamma \epsilon t}(u, u)_2
= (1-\sigma)H^{-\sigma}(t)H'(t) - \eta \gamma \epsilon e^{-\gamma \epsilon t}(u, u)_2 + \eta e^{-\gamma \epsilon t}\|u(t)\|^2_2 + \eta e^{-\gamma \epsilon t}\int_\Omega |u|^{p+1}dxdy
- \eta e^{-\gamma \epsilon t}(au, u)_2 - \eta e^{-\gamma \epsilon t}\|u(t)\|^2_{H^2(\Omega)} - \frac{\mu \eta e^{-\gamma \epsilon t}}{\Gamma(-\alpha)} \int_\Omega \int_0^t (t-s)^{-(\alpha+1)}u_\alpha(s)dxdy. \tag{33}
\]
Similarly as in the inequalities (23) and (25), we have that
\[
(u, u)_2 \leq \delta_4 S^2_2\|u(t)\|^2_{H^2(\Omega)} + \frac{1}{4\delta_4}\|u_\epsilon(t)\|^2_2, \quad \delta_4 > 0. \tag{34}
\]
and
\[
e^{-\gamma \epsilon t}\int_\Omega \int_0^t (t-s)^{-(\alpha+1)}u_\alpha(s)dxdy
\leq \delta_5 e^{-\gamma \epsilon t}\|u(t)\|^2_2 + \frac{(\gamma \epsilon)^\alpha \Gamma(-\alpha)}{4\delta_5} \int_\Omega \int_0^t (t-s)^{-(\alpha+1)}e^{-\gamma \epsilon s}u^2_\alpha(s)dxdy, \quad \delta_5 > 0. \tag{35}
\]
From Lemma 2, we get
\[
A_1 e^{-\gamma \epsilon t}\|u(t)\|^2_{H^2(\Omega)} \leq e^{-\gamma \epsilon t}\left(\|u(t)\|^2_{H^2(\Omega)} + (au, u)_2\right). \tag{36}
\]
Substituting (34)-(36) into (33), we obtain
\[
G'(t) \geq (1-\sigma)H^{-\sigma}(t)H'(t) + \eta \left(1 - \frac{\gamma \epsilon}{4\delta_4}\right)e^{-\gamma \epsilon t}\|u(t)\|^2_2 - \eta \left(A_1 + \delta_4 \gamma \epsilon S^2_2\right)e^{-\gamma \epsilon t}\|u(t)\|^2_{H^2(\Omega)}
+ \eta e^{-\gamma \epsilon t}\int_\Omega |u|^{p+1}dxdy - \frac{\eta \mu \delta_5}{\Gamma(-\alpha)} e^{-\gamma \epsilon t}\|u(t)\|^2_2 - \frac{\mu \eta (\gamma \epsilon)^\alpha}{4\delta_5} \int_\Omega \int_0^t (t-s)^{-(\alpha+1)}e^{-\gamma \epsilon s}u^2_\alpha(s)dxdy. \tag{37}
\]
Using (31), we obtain
\[ G'(t) \geq (1 - \sigma)H^{-\sigma}(t)H'(t) + \eta \left( 1 - \frac{\gamma e}{4\delta} \right) e^{-\gamma e t} \| u(t) \|_{H^{2}\Omega}^2 \]
\[ - \eta \left( A_1 + \delta_4 \gamma e S_2 \right) e^{-\gamma e t} \| u(t) \|_{H^{2}\Omega}^2 + \frac{\eta_2 \mu_2}{\Gamma(-\alpha)} e^{-\gamma e t} \| u(t) \|_{H^{2}\Omega}^2 \]
\[ + \frac{2^{n-1} \gamma e \mu_1}{\delta_5(p+1)^{n-1}} \left( -H(t) + \frac{p+1}{2} e H(t) + \frac{\eta e - \gamma e t}{4} e^{-\gamma e t} \| u(t) \|_{H^{2}\Omega}^2 \right). \]

From the last inequality, we get

\[ G'(t) \geq \left[ (1 - \sigma)H^{-\sigma}(t) - \frac{\eta 2^{n-1} \gamma e \mu_1}{\delta_5(p+1)^{n-1}} \right] H'(t) + \eta_2 \mu_2 \frac{\eta e - \gamma e t}{\Gamma(-\alpha)} e^{-\gamma e t} \| u(t) \|_{H^{2}\Omega}^2 \]
\[ + \frac{\eta e - \gamma e t}{4} e^{-\gamma e t} \| u(t) \|_{H^{2}\Omega}^2 \]
\[ + \frac{\eta e - \gamma e t}{4} e^{-\gamma e t} \| u(t) \|_{H^{2}\Omega}^2 + \eta_2 \mu_2 \frac{\eta e - \gamma e t}{\Gamma(-\alpha)} e^{-\gamma e t} \| u(t) \|_{H^{2}\Omega}^2. \]

Now, we choose \( \delta_5 = BH^\sigma(t) \) for some \( B \) positive to be precise later. Then, (39) becomes

\[ G'(t) \geq \left[ (1 - \sigma) - \frac{\eta 2^{n-1} \gamma e \mu_1}{\delta_5(p+1)^{n-1}} \right] H^{-\sigma}(t)H'(t) + \eta_2 \mu_2 \frac{\eta e - \gamma e t}{\Gamma(-\alpha)} e^{-\gamma e t} \| u(t) \|_{H^{2}\Omega}^2 \]
\[ + \frac{\eta e - \gamma e t}{4} e^{-\gamma e t} \| u(t) \|_{H^{2}\Omega}^2. \]

Adding and subtracting \( H(t) \) on the right hand side of (40) and making use of lemma 2 leads to

\[ G'(t) \geq \left[ (1 - \sigma) - \frac{\eta 2^{n-1} \gamma e \mu_1}{B(p+1)^{n-1}} \right] H^{-\sigma}(t)H'(t) + \left[ 1 + \frac{\eta 2^{n-1} \gamma e \mu_1}{B(p+1)^{n-1}} H^{-\sigma}(t) \right] H(t) \]
\[ + \left[ \frac{A_1}{2} + \frac{\eta_2 \mu_2}{\Gamma(-\alpha)} H^{-\sigma}(t) \right] H^{-\sigma}(t) - \frac{A_1 + \delta_4 \gamma e S_2}{B(p+1)^{n-1}} e^{-\gamma e t} \| u(t) \|_{H^{2}\Omega}^2 \]
\[ + \left[ \frac{1}{2} + \frac{\eta e - \gamma e t}{4} \right] e^{-\gamma e t} \| u(t) \|_{H^{2}\Omega}^2 + \left[ \frac{\eta e - \gamma e t}{4} \right] e^{-\gamma e t} \| u(t) \|_{H^{2}\Omega}^2 \]
\[ + \eta_2 \mu_2 \frac{\eta e - \gamma e t}{\Gamma(-\alpha)} e^{-\gamma e t} H^\sigma(t) \| u(t) \|_{H^{2}\Omega}^2. \]

The term \( (u,u)_{2} \) is estimated similarly as in (23) as

\[ (u,u)_{2} \leq \delta_6 S_2^2 \| u(t) \|_{H^{2}\Omega}^2 + \frac{1}{4\delta_6} \| u(t) \|_{H^{2}\Omega}^2, \quad \delta_6 > 0. \]

For the term \( H^\sigma(t) \| u(t) \|_{H^{2}\Omega}^2 \), we use the definition of \( H(t) \) in (17) and the choice of \( e \) in (30) to get (see [10] page 141 for detail computations)

\[ H^\sigma(t) \| u(t) \|_{H^{2}\Omega}^2 \leq \frac{C_2}{(p+1)^{n-1}} \left( 1 + \int \| u \|_{H^{2}\Omega}^2 d\Omega \right) \]
\[ \frac{1}{(p+1)^{n-1}} \int \| u \|_{H^{2}\Omega}^2 d\Omega \]

for some constant \( C_2 > 0 \). Substituting (42) and (43) into (41) yields

\[ G'(t) \geq \left[ (1 - \sigma) - \frac{\eta 2^{n-1} \gamma e \mu_1}{B(p+1)^{n-1}} \right] H^{-\sigma}(t)H'(t) + \left[ 1 + \frac{\eta 2^{n-1} \gamma e \mu_1}{B(p+1)^{n-1}} H^{-\sigma}(t) \right] H(t) \]
\[ + \left[ \frac{A_1}{2} - \eta A_1 \right] \gamma e t \| u(t) \|_{H^{2}\Omega}^2. \]
\[ + \varepsilon \left[ \eta^{2a-3}\gamma^a A_1(p-1) \varepsilon \Gamma(-\alpha) \right. \\
+ \frac{\eta^{2a-3}\gamma^a A_1(p-1) \varepsilon \Gamma(-\alpha)}{B(p+1)^a} - \left( \eta \delta_4 \gamma S_2^2 + \delta_6 S_2^2 \right) \frac{\varepsilon}{4\delta_4} e^{-\gamma \varepsilon t} \left\| u(t) \right\|^2_{H^2(\Omega)} \]
\[ + \left( \frac{1}{2} + \eta \left( 1 - \frac{\varepsilon}{4\delta_4} \right) - \frac{\varepsilon}{4\delta_6} \right) e^{-\gamma \varepsilon t} \left\| u(t) \right\|^2_{2} + \left[ \eta - \frac{1}{p+1} - \frac{\eta \mu B C_2}{(p+1) \varepsilon \Gamma(-\alpha)} \right] e^{-\gamma \varepsilon t} \int_{\Omega} |u|^{p+1} \, dx \, dy \tag{44} \]
\[ + \left( \lambda - \frac{\eta \mu B C_2}{(p+1) \varepsilon \Gamma(-\alpha)} \right) + \theta F(t). \]

Now, we choose parameters carefully. First, recalling \( \sigma = \frac{p-1}{2(p+1)} \) and selecting \( \varepsilon \) so small such that
\[ \varepsilon \leq \varepsilon_2 := \frac{1}{2} B(p+1)^a(1-\sigma) \]
we see that the coefficient of the first term is positive. By choosing \( \eta = \frac{p+3}{4(p+1)} \), \( \delta_4 = \delta_6 = \frac{1}{2} \), and \( \varepsilon \) small enough so that
\[ \varepsilon \leq \varepsilon_3 := \frac{4(p-1)}{(p+1)^2 S_2^2}, \]
we find that the coefficient of \( \left\| u(t) \right\|^2_{H^2(\Omega)} \) is positive. Next, we pick \( \varepsilon \) small enough such that
\[ \varepsilon \leq \varepsilon_4 := \frac{2(3p+5)}{(p+1)(p+11)}, \]
to get the coefficient of \( \left\| u(t) \right\|^2_{2} \) greater or equal to \( \frac{1}{2} \). We select \( B \) such that
\[ B < \frac{(p+1)^2 \Gamma(-\alpha)}{(p+1) \mu C_2} \min \left\{ \frac{p-1}{2}, 4\lambda(p+1) \right\}, \tag{48} \]
to see that the coefficient of \( \int_{\Omega} |u|^{p+1} \, dx \, dy \) is greater than \( \frac{p-1}{4(p+1)} \) and the term
\[ \left( \lambda - \frac{\eta \mu B C_2}{(p+1) \varepsilon \Gamma(-\alpha)} \right) > 0. \]

Thus, for any \( \varepsilon \) positive small enough such that
\[ \varepsilon < \min \{ \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \}, \tag{49} \]
we arrive at
\[ \frac{\partial}{\partial t} G(t) \geq H(t) + \frac{1}{2} \left\| u(t) \right\|^2_{2} + \frac{p-1}{4(p+1)} \int_{\Omega} |u|^{p+1} \, dx \, dy \quad \forall \ t \geq 0. \tag{50} \]

Using Cauchy-Schwarz and Young’s inequalities, we have
\[ \left| (u, u_t) \right| \leq \left\| u(t) \right\|^\frac{1}{p} \left\| u_t(t) \right\|^\frac{1}{p} \leq C_2 \left\| u(t) \right\|^\frac{1}{p+1} \left\| u_t(t) \right\|^\frac{1}{p} \leq C_3 \left( \left\| u(t) \right\|^\frac{2}{p+1} + \left\| u_t(t) \right\|^\frac{2}{p} \right), \tag{51} \]
where \( C_2 = C_2(|\Omega|, p) > 0, \ C_3 = C_3(|\Omega|, p, \sigma) > 0 \) are constants and \( \frac{1}{r_1} + \frac{1}{r_2} = 1 \). We recall that \( \sigma = \frac{p-1}{2(p+1)} \), therefore we select \( r_1 = \frac{2(1-\sigma)}{p-2\sigma}, \ r_2 = 2(1-\sigma) \), and arrive at
\[ \left| (u, u_t) \right| \leq C_3 \left( \left\| u(t) \right\|^\frac{2}{p+1} + \left\| u_t(t) \right\|^\frac{2}{p} \right). \tag{52} \]
We observe that \( \frac{2}{p+1(1-2\sigma)} = 1 \), so
\[ \|u(t)\|_{p+1}^{\frac{2}{p+1}} = \int_\Omega |u|^{p+1} dx dy. \]

From the definition of \( G(t) \), we have
\[
G(t)^{\frac{1}{1+\sigma}} = \left( H^{1-\sigma}(t) + \eta e^{-\gamma t}(u, u_t)_2 \right)^{\frac{1}{1+\sigma}} \leq 2^{\frac{1}{1+\sigma}} \left( H(t) + \frac{\eta}{1+\sigma} |(u, u_t)_2|^{\frac{1}{1+\sigma}} \right) \\
\leq 2^{\frac{1}{1+\sigma}} \left( H(t) + C_3 \eta^{\frac{1}{1+\sigma}} \left( \|u(t)\|_{p+1}^{\frac{2}{p+1}} + \|u_t(t)\|_2^2 \right) \right) \\
= 2^{\frac{1}{1+\sigma}} \left( H(t) + C_3 \eta^{\frac{1}{1+\sigma}} \left( \int_\Omega |u|^{p+1} dx dy + \|u_t(t)\|_2^2 \right) \right) \\
\leq C \left( H(t) + \frac{1}{2} \|u_t(t)\|_2^2 + \frac{p}{4(p+1)} \int_\Omega |u|^{p+1} dx dy \right),
\]
for some positive constant \( C \) such that
\[
C \geq 2^{\frac{1}{1+\sigma}} \max \left\{ 1, 2C_3 \eta^{\frac{1}{1+\sigma}}, C_3 \eta^{\frac{1}{1+\sigma}} \frac{4(p+1)}{p-1} \right\}. \]

A combination of (50) and (53) leads to
\[
\left( G(t) \right)^{\frac{1}{1+\sigma}} \leq CG'(t), \ \forall t \geq 0. \tag{54}
\]

From (50), we see clearly that \( G'(t) \geq 0 \). It follows from the definition of \( G(t) \) and the assumption on \( u_0 \) and \( u_1 \) that
\[
G(t) \geq G(0) > \eta(u_0, u_1)_2 \geq 0. \tag{55}
\]

Hence, \( G(t) > 0 \). Integrating (54) over \((0, t)\) yields
\[
\left( G(t) \right)^{\frac{1}{1+\sigma}} \leq \left( G(0) \right)^{\frac{1}{1+\sigma}} - \frac{\sigma}{C(1-\sigma)} t
\]
which gives
\[
\left( G(t) \right)^{\frac{1}{1+\sigma}} \geq \frac{1}{\left( G(0) \right)^{\frac{1}{1+\sigma}}} - \frac{\sigma}{C(1-\sigma)} t. \tag{56}
\]

From (56), we obtain that \( G(t) \) blows up in time
\[
T^* \leq \frac{C(1-\sigma)}{\sigma \left( G(0) \right)^{\frac{1}{1+\sigma}}}. \tag{57}
\]

This completes the proof. \( \square \)

5. Conclusion

In this paper, we have studied a plate equation supplemented with partially hinged boundary conditions as model for suspension bridge in the presence of fractional damping and non-linear source terms. We showed that the solution blows up in finite time. We saw that, even in the present of a weaker damping, the bridge will collapse in infinite time when the power \( p \) of the non-linear source term is sufficiently large. This is a very important factor for engineers to consider when constructing such types of bridges.

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References


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