On some new subclass of bi-univalent functions associated with the Opoola differential operator

Timilehin Gideon Shaba

Department of Mathematics, University of Ilorin, P. M. B. 1515, Ilorin, Nigeria.; shabatimilehin@gmail.com

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Abstract: By applying Opoola differential operator, in this article, two new subclasses $M_{H,\sigma}(m, \psi, k, \tau)$ and $M_{H,\sigma}(m, \xi, k, \tau)$ of bi-univalent functions class $H$ defined in $\bigtriangledown$ are introduced and investigated. The estimates on the coefficients $|l_2|$ and $|l_3|$ for functions of the classes are also obtained.

Keywords: Univalent function, bi-univalent function, coefficient bounds, Opoola differential operator.

MSC: 30C45, 30C50.

1. Introduction

Let $J$ denote the subclass of $G$ which is of the form

$$\Im(z) = z + \sum_{k=2}^{\infty} l_k z^k$$

(1)

consisting of functions which are holomorphic and univalent in the unit disk $\bigtriangledown$. Let $\Im^{-1}$ be inverse of the function $\Im(z)$, then we have

$$\Im^{-1}(\Im(z)) = z$$

and

$$\Im^{-1}(\Im(b)) = b, \quad |b| < r_0(\Im); r_0(\Im) \geq \frac{1}{4}$$

where

$$\Im^{-1}(\Im(b)) = b - l_2 b^2 + (2 l_2^2 - l_3) b^3 - (5 l_2^3 - 5 l_2 l_3 + l_4) b^4 + \cdots$$

(2)

A function $\Im(z) \in G$ denoted by $H$ is said to be bi-univalent in $\bigtriangledown$ if both $\Im(z)$ and $\Im^{-1}(z)$ are univalent in $\Delta$ [1]. Subclasses of $H$, such as class of bi-convex and starlike functions and bi-strongly convex and starlike function similar to the well known subclasses $L^*(\vartheta)$ and $K(\vartheta)$ of starlike and convex functions of order $\vartheta(0 < \vartheta < 1)$ respectively [2].

Recently, numerous researchers [1,3,4] obtained the coefficient $|l_2|$ and $|l_3|$ of bi-univalent functions for the several subclasses of functions in the class $H$. Motivated by the work of Darus and Singh [5], we introduce the subclasses $M_{H,\sigma}(m, \psi, k, \tau)$ and $M_{H,\sigma}(m, \xi, k, \tau)$ of the function class $H$, which are associated with the Opoola differential operator and to obtain estimates on the coefficients $|l_2|$ and $|l_3|$ for functions in these new subclasses of the function class $H$ applying the techniques used earlier by Darus and Singh [5], Frasin and Aouf [4] and Srivastava et al., [1].

Lemma 1. [6] Suppose $u(z) \in P$ and $z \in \bigtriangledown$, then $|w_k| \leq 2$ for each $k$, where $P$ is the family of all function $u$ analytic in $\bigtriangledown$ for which $\Re(u(z)) > 0$,

$$u(z) = 1 + w_1 z + w_2 z^2 + \cdots$$

Definition 1. A function \( \Im(z) \in \mathcal{G} \) is in the class \( \mathcal{M}_{H,\psi}^{\mu,\beta}(m, \psi, \tau) \) if the following condition are fulfilled:

\[
\left| \arg \left( \frac{(1 - \sigma)D_{r,\beta}^{m,\mu} \Im(z) + \sigma D_{r,\beta}^{m+1,\mu} \Im(z)}{z} \right) \right| < \frac{\psi \tau}{2},
\]

(3)

\[
\left| \arg \left( \frac{(1 - \sigma)D_{r,\beta}^{m,\mu} h(b) + \sigma D_{r,\beta}^{m+1,\mu} h(b)}{b} \right) \right| < \frac{\psi \tau}{2},
\]

(4)

where \( 0 < \psi \leq 1, \sigma \geq 1, \tau \geq 0, z \in \triangle, b \in \triangle, 0 \leq \mu \leq \beta, m \in \mathbb{N}_0 \) and

\[
h(b) = b - l_2 b^2 + (2l_2^2 - l_3) b^3 - (5l_2^3 - 5l_2l_3 + l_4) b^4 + \cdots
\]

(5)

and

\[
D_{r,\beta}^{m,\mu} \Im(z) = z + \sum_{k=2}^{\infty} (1 + (k + \mu - \beta - 1) \tau)^m l_k z^k
\]

(6)

where \( 0 \leq \mu \leq \beta, \tau \geq 0 \) and \( m \in \mathbb{N}_0 = \{0, 1, 2, 3 \ldots \} \) is the generalized Al-oboudi derivative defined by Opoola [7].

Remark 1.

1. \( \mathcal{M}_{H,\psi}^{\mu,\beta}(0, \psi, \tau) = \mathcal{M}_{H}(\psi) \) which Srivastava et al., [1] presented and studied.
2. \( \mathcal{M}_{H,\psi}^{\mu,\beta}(0, \psi, \tau) = \mathcal{M}_{H,\psi}(\psi) \) which Frasin and Aouf [4] presented and studied.
3. \( \mathcal{M}_{H,\psi}^{1,\beta}(m, \psi, 1) = \mathcal{M}_{H,\psi}(m, \psi) \) which Porwal and Darus [8] presented and studied.
4. \( \mathcal{M}_{H,\psi}^{1,\beta}(m, \psi, \tau) = \mathcal{M}_{H,\psi}(m, \psi, \tau) \) which Darus and Singh [5] presented and studied.

2. Coefficient Bounds For The Function Class \( \mathcal{M}_{H}^{\mu,\beta}(m, \psi, k, \tau) \)

Theorem 1. Let \( \Im(z) \in \mathcal{G} \) be in the class \( \mathcal{M}_{H}^{\mu,\beta}(m, \psi, k, \tau) \), \( 0 < \psi \leq 1, \sigma \geq 1, \tau \geq 0, z \in \triangle, b \in \triangle, 0 \leq \mu \leq \beta, m \in \mathbb{N}_0 \), then

\[
|l_2| < \frac{2\psi}{\sqrt{2\psi^2(1 - \sigma)(1 + \tau(2 + \mu - \beta))^m + \sigma(1 + \tau(2 + \mu - \beta))^{m+1}}} - \psi(\psi - 1)(1 - \sigma)(1 + \tau(1 + \mu - \beta))^m + \sigma(1 + \tau(1 + \mu - \beta))^{m+1}]^2
\]

(7)

and

\[
|l_3| \leq \frac{2\psi}{[(1 - \sigma)(1 + \tau(2 + \mu - \beta))^m + \sigma(1 + \tau(2 + \mu - \beta))^{m+1}]^{4\psi^2} + [(1 - \sigma)(1 + \tau(2 + \mu - \beta))^m + \sigma(1 + \tau(2 + \mu - \beta))^{m+1}]^{4\psi^2}}
\]

(8)

Proof. It follows from (3) and (4) that

\[
\frac{(1 - \sigma)D_{r,\beta}^{m,\mu} \Im(z) + \sigma D_{r,\beta}^{m+1,\mu} \Im(z)}{z} = \left( q(z) \right) \psi,
\]

(9)

and

\[
\frac{(1 - \sigma)D_{r,\beta}^{m,\mu} h(b) + \sigma D_{r,\beta}^{m+1,\mu} h(b)}{b} = \left( t(b) \right) \psi,
\]

(10)

where \( q(z) = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \cdots \) and \( t(b) = 1 + t_1 b + t_2 b^2 + t_3 b^3 + \cdots \) are in \( \mathcal{P} \). Equating the coefficient in (9) and (10), we have

\[
[(1 - \sigma)(1 + \tau(1 + \mu - \beta))^m + \sigma(1 + \tau(1 + \mu - \beta))^{m+1}] l_2 = \psi q_1,
\]

(11)
\[(1 - \sigma)(1 + \tau(2 + \mu - \beta))^m + \sigma(1 + \tau(2 + \mu - \beta))^{m+1}l_3 = \psi q_2 + \frac{\psi(\psi - 1)}{2} q_1^2, \quad (12)\]
\[- [(1 - \sigma)(1 + \tau(2 + \mu - \beta))^m + \sigma(1 + \tau(1 + \mu - \beta))^{m+1}]_l_2 = \psi q_1, \quad (13)\]
\[[(1 - \sigma)(1 + \tau(2 + \mu - \beta))^m + \sigma(1 + \tau(2 + \mu - \beta))^{m+1}]_2 = \psi t_2 + \frac{\psi(\psi - 1)}{2} t_1^2. \quad (14)\]

From (11) and (13), we get
\[q_1 = -t_1, \quad (15)\]

and
\[2[(1 - \sigma)(1 + \tau(1 + \mu - \beta))^m + \sigma(1 + \tau(1 + \mu - \beta))^{m+1}]_l_2^2 = \psi^2(q_1^2 + t_1^2). \quad (16)\]

From (12),(14) and (16), we get
\[2[(1 - \sigma)(1 + \tau(1 + \mu - \beta))^m + \sigma(1 + \tau(1 + \mu - \beta))^{m+1}]_l_2^2 - [(1 - \sigma)(1 + \tau(1 + \mu - \beta))^m + \sigma(1 + \tau(1 + \mu - \beta))^{m+1}]_l_3 = \psi t_2 + \frac{\psi(\psi - 1)}{2} t_1^2 \]

implies

\[2[(1 - \sigma)(1 + \tau(1 + \mu - \beta))^m + \sigma(1 + \tau(1 + \mu - \beta))^{m+1}]_l_2^2 - [(1 - \sigma)(1 + \tau(1 + \mu - \beta))^m + \sigma(1 + \tau(1 + \mu - \beta))^{m+1}]_l_3 = \psi t_2 + \frac{\psi(\psi - 1)}{2} t_1^2. \quad (17)\]

Then from (12), we have
\[2[(1 - \sigma)(1 + \tau(1 + \mu - \beta))^m + \sigma(1 + \tau(1 + \mu - \beta))^{m+1}]_l_2^2 = \psi q_2 + \frac{\psi(\psi - 1)}{2} q_1^2 + \psi t_2 + \frac{\psi(\psi - 1)}{2} t_1^2. \]

implies
\[2[(1 - \sigma)(1 + \tau(1 + \mu - \beta))^m + \sigma(1 + \tau(1 + \mu - \beta))^{m+1}]_l_2^2 = \psi(q_2 + t_2) + \frac{\psi(\psi - 1)}{2}(q_1^2 + t_1^2). \quad (18)\]

Then from (16), we get
\[2[(1 - \sigma)(1 + \tau(1 + \mu - \beta))^m + \sigma(1 + \tau(1 + \mu - \beta))^{m+1}]_l_2^2 = \psi(q_2 + t_2) + \frac{\psi(\psi - 1)}{2} q_2^2 + \psi t_2 + \frac{\psi(\psi - 1)}{2} t_1^2. \quad (19)\]

implies
\[l_2^2 = \frac{\psi^2(q_2 + t_2)}{2\psi[(1 - \sigma)(1 + \tau(2 + \mu - \beta))^m + \sigma(1 + \tau(2 + \mu - \beta))^{m+1}] - \psi(\psi - 1)[(1 - \sigma)(1 + \tau(1 + \mu - \beta))^m + \sigma(1 + \tau(1 + \mu - \beta))^{m+1}]^2. \quad (20)\]

Applying Lemma 1 for (17), we get
\[|l_2| \leq \frac{2\psi}{\sqrt{2\psi[(1 - \sigma)(1 + \tau(2 + \mu - \beta))^m + \sigma(1 + \tau(2 + \mu - \beta))^{m+1}] - \psi(\psi - 1)[(1 - \sigma)(1 + \tau(1 + \mu - \beta))^m + \sigma(1 + \tau(1 + \mu - \beta))^{m+1}]^2}} \]

which gives the desired estimate on |l_2| in (7). Hence in order to find the bound on |l_3|,
\[[(1 - \sigma)(1 + \tau(2 + \mu - \beta))^m + \sigma(1 + \tau(2 + \mu - \beta))^{m+1}]_l_3 - [(1 - \sigma)(1 + \tau(2 + \mu - \beta))^m + \sigma(1 + \tau(2 + \mu - \beta))^{m+1}]_l_3 = \psi q_2 + \frac{\psi(\psi - 1)}{2} q_1^2 - \psi t_2 + \frac{\psi(\psi - 1)}{2} t_1^2, \]

implies
\[l_3^2 = \frac{\psi^2(q_2 + t_2)}{2\psi[(1 - \sigma)(1 + \tau(2 + \mu - \beta))^m + \sigma(1 + \tau(2 + \mu - \beta))^{m+1}] - \psi(\psi - 1)[(1 - \sigma)(1 + \tau(1 + \mu - \beta))^m + \sigma(1 + \tau(1 + \mu - \beta))^{m+1}]^2. \quad (21)\]
implies
\[ 2[(1 - \sigma)(1 + \tau(2 + \mu - \beta)) + \sigma(1 + \tau(2 + \mu - \beta))]l_3 \]
\[ = [(1 - \sigma)(1 + \tau(2 + \mu - \beta)) + \sigma(1 + \tau(2 + \mu - \beta))]2l_2^2 + \psi(q_2 - t_2) + \frac{\psi(\psi - 1)}{2}(q_1^2 - t_1^2). \]

Since \((q_1)^2 = (-t_1)^2 \Rightarrow q_1^2 = t_1^2\), then we have
\[ 2[(1 - \sigma)(1 + \tau(2 + \mu - \beta)) + \sigma(1 + \tau(2 + \mu - \beta))]l_3 \]
\[ = \psi(q_2 - t_2) + [(1 - \sigma)(1 + \tau(2 + \mu - \beta)) + \sigma(1 + \tau(2 + \mu - \beta))]2l_2^2. \]

From (16), we have
\[ l_3 = \frac{\psi(q_2 - t_2)}{2[(1 - \sigma)(1 + \tau(2 + \mu - \beta)) + \sigma(1 + \tau(2 + \mu - \beta))]2l_2^2} \]
\[ + \frac{\psi q_2^2 + t_2^2}{2[(1 - \sigma)(1 + \tau(1 + \mu - \beta)) + \sigma(1 + \tau(1 + \mu - \beta))]2l_2^2}. \]

Applying Lemma 1 for coefficient \(q_1, q_2, t_1\) and \(t_2\), we have
\[ |l_3| \leq \frac{2\psi}{[(1 - \sigma)(1 + \tau(2 + \mu - \beta)) + \sigma(1 + \tau(2 + \mu - \beta))]2l_2^2} \]
\[ + \frac{4\psi^2}{[(1 - \sigma)(1 + \tau(1 + \mu - \beta)) + \sigma(1 + \tau(1 + \mu - \beta))]2l_2^2}. \]

\[ \square \]

3. Coefficient bounds for the function class \(\mathcal{M}_{H,\sigma}^{\mu,\beta}(m, \xi, k, \tau)\)

**Definition 2.** A function \(H(z) \in \mathcal{G}\) is said to be in the class \(\mathcal{M}_{H,\sigma}^{\mu,\beta}(m, \xi, k, \tau)\) if the following condition are fulfilled:
\[ \Re \left[ \frac{(1 - \sigma)D_{\tau,\beta}^{\mu,\mu} H(z) + \sigma D_{\tau,\beta}^{\mu+1,\mu} H(z)}{z} \right] > \xi, \tag{18} \]
\[ \Re \left[ \frac{(1 - \sigma)D_{\tau,\beta}^{\mu,\mu} h(b) + \sigma D_{\tau,\beta}^{\mu+1,\mu} h(b)}{b} \right] > \xi, \tag{19} \]

where \(H(z) \in \mathcal{H}, 0 \leq \xi < 1, \sigma \geq 1, \tau \geq 0, z \in \Delta, b \in \Delta, 0 \leq \mu \leq \beta, m \in \mathbb{N}_0,\) and
\[ h(b) = b - l_2 b^2 + (2l_2^2 - l_3)b^3 - (5l_2^3 - 5l_2 l_3 + l_4)b^4 + \cdots, \tag{20} \]
and
\[ D_{\tau,\beta}^{\mu,\mu} H(z) = z + \sum_{k=2}^{\infty} (1 + (k + \mu - \beta - 1)\tau)^m l_k z^k, \tag{21} \]

where \(0 \leq \mu \leq \beta, \tau \geq 0\) and \(m \in \mathbb{N}_0 = \{0, 1, 2, 3 \cdots \}\) is the generalized Al-oboudi derivative defined by Opoola [7].

**Remark 2.**

1. \(\mathcal{M}_{H,\tau}^{\mu,\beta}(0, \xi, \tau) = \mathcal{M}_{\tau}(\xi)\) which Srivastava et al., [1] presented and studied.
2. $\mathcal{M}^{H_{\beta}}(0,\xi,\tau)=\mathcal{M}_{H_{\psi}}(\xi)$ which Frasin and Aouf [4] presented and studied.
3. $\mathcal{M}^{H_{\psi}}(m,\xi,1)=\mathcal{M}_{H_{\beta}}(m,\xi)$ which Porwal and Darus [8] presented and studied.
4. $\mathcal{M}^{H_{\beta}}(m,\xi,\tau)=\mathcal{M}_{H_{\beta}}(m,\xi,\tau)$ which Darus and Singh [5] presented and studied.

Theorem 2. Let $\mathfrak{F}(z) \in \mathcal{G}$ be in the class $\mathcal{M}^{H_{\beta}}(m,\xi,\psi,\tau)$, $0 \leq \xi < 1$, $\sigma \geq 1$, $\tau \geq 0$, $z \in \Delta$, $b \in \Delta$, $0 \leq \mu \leq \beta$, $m \in \mathbb{N}$, then

$$|l_2| \leq \sqrt{\frac{2(1-\xi)}{[(1-\sigma)(1+\tau(2+\mu-\beta))^{m} + \sigma(1+\tau(2+\mu-\beta))^{m+1}]}}$$

(22)

and

$$|l_3| \leq \frac{4(1-\xi)^2}{[(1-\sigma)(1+\tau(2+\mu-\beta))^{m} + \sigma(1+\tau(2+\mu-\beta))^{m+1}]^2}$$

$$+ \frac{2(1-\xi)}{[(1-\sigma)(1+\tau(2+\mu-\beta))^{m} + \sigma(1+\tau(2+\mu-\beta))^{m+1}]}.$$

(23)

Proof. From (18) and (19), where $q(z), t(z) \in \mathcal{P}$,

$$(1-\sigma)D_{v,\beta}^{m,\mu} \mathfrak{F}(z) + \sigma D_{v,\beta}^{m+1,\mu} \mathfrak{F}(z)$$

$$= \xi + (1-\xi)q(z),$$

(24)

and

$$(1-\sigma)D_{v,\beta}^{m,\mu} h(b) + \sigma D_{v,\beta}^{m+1,\mu} h(b)$$

$$= \xi + (1-\xi)t(b),$$

(25)

where $q(z) = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \cdots$ and $t(b) = 1 + t_1 b + t_2 b^2 + t_3 b^3 \cdots$. Now on equating the coefficient in (24) and (25), we have

$$[(1-\sigma)(1+\tau(1+\mu-\beta))^{m} + \sigma(1+\tau(1+\mu-\beta))^{m+1}]l_2 = (1-\xi)q_1,$$

(26)

$$[(1-\sigma)(1+\tau(1+\mu-\beta))^{m} + \sigma(1+\tau(1+\mu-\beta))^{m+1}]l_3 = (1-\xi)q_2,$$

(27)

$$- [(1-\sigma)(1+\tau(1+\mu-\beta))^{m} + \sigma(1+\tau(1+\mu-\beta))^{m+1}]l_2 = (1-\xi)t_1,$$

(28)

$$[(1-\sigma)(1+\tau(2+\mu-\beta))^{m} + \sigma(1+\tau(2+\mu-\beta))^{m+1}]l_2 = (1-\xi)t_2.$$  

(29)

From (26) and (28), we have

$$q_1 = -t_1,$$

(30)

and

$$2[(1-\sigma)(1+\tau(1+\mu-\beta))^{m} + \sigma(1+\tau(1+\mu-\beta))^{m+1}]l_2^2 = (1-\xi)^2(q_1^2 + t_1^2).$$

(31)

From (27) and (29), we have

$$2[(1-\sigma)(1+\tau(1+\mu-\beta))^{m} + \sigma(1+\tau(1+\mu-\beta))^{m+1}]l_2^2 = (1-\xi)(q_2 + t_2),$$

(32)

or we have

$$l_2^2 = \frac{(1-\xi)(q_2 + t_2)}{2[(1-\sigma)(1+\tau(1+\mu-\beta))^{m} + \sigma(1+\tau(1+\mu-\beta))^{m+1}]},$$

implies

$$|l_2^2| \leq \frac{2(1-\xi)}{[(1-\sigma)(1+\tau(1+\mu-\beta))^{m} + \sigma(1+\tau(1+\mu-\beta))^{m+1}]},$$

which is the bound on $|l_2|$ as given in (22). Hence in order to find the bound on $|l_3|$, we subtract (27) and (29) and get

$$[(1-\sigma)(1+\tau(2+\mu-\beta))^{m} + \sigma(1+\tau(2+\mu-\beta))^{m+1}]l_3$$

$$- [(1-\sigma)(1+\tau(2+\mu-\beta))^{m} + \sigma(1+\tau(2+\mu-\beta))^{m+1}]l_2 = (1-\xi)q_2 - [(1-\xi)t_2].$$
implies
\[
2[(1-\sigma)(1+\tau(2+\mu-\beta))^m + \sigma(1+\tau(2+\mu-\beta))^{m+1}]l_3
= [(1-\sigma)(1+\tau(2+\mu-\beta))^m + \sigma(1+\tau(2+\mu-\beta))^{m+1}]2l_2^2 + (1-\xi)(q_2-t_2).
\]
imples
\[
l_3 = l_2^2 + \frac{(1-\xi)(q_2-t_2)}{2[(1-\sigma)(1+\tau(2+\mu-\beta))^m + \sigma(1+\tau(2+\mu-\beta))^{m+1}]}.
\]
Then from (31), we have
\[
l_3 = \frac{(1-\xi)^2(q_2^2+t_2^2)}{2[(1-\sigma)(1+\mu-\beta)]^m + \sigma(1+\tau(2+\mu-\beta))^{m+1}} + \frac{(1-\xi)(q_2-t_2)}{2[(1-\sigma)(1+\tau(2+\mu-\beta))^m + \sigma(1+\tau(2+\mu-\beta))^{m+1}]}.
\]
Applying Lemma 1 for the coefficient \(q_1, q_2, l_1\) and \(l_2\), we get
\[
|l_3| \leq \frac{4(1-\xi)^2}{[(1-\sigma)(1+\tau(2+\mu-\beta))^m + \sigma(1+\tau(2+\mu-\beta))^{m+1}]} + \frac{2(1-\xi)}{2[(1-\sigma)(1+\tau(2+\mu-\beta))^m + \sigma(1+\tau(2+\mu-\beta))^{m+1}]}.
\]
which is the bond on \(|l_3|\) in (23).

4. Conclusion

In this present paper, two new subclasses of bi-univalent functions associated with Opoola differential operator \(D_{r,\beta}^{m,\mu}\) were introduced and worked on. Furthermore, the coefficient bounds for \(|l_2|\) and \(|l_3|\) of functions in these classes are obtained.

Conflicts of Interest: “The author declares no conflict of interest.”

References


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