Certain new subclasses of \( m \)-fold symmetric bi-pseudo-starlike functions using \( Q \)-derivative operator

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Abstract: In this current study, we introduced and investigated two new subclasses of the bi-univalent functions associated with \( q \)-derivative operator; both \( f \) and \( f^{-1} \) are \( m \)-fold symmetric holomorphic functions in the open unit disk. Among other results, upper bounds for the coefficients \( |\rho_m + 1| \) and \( |\rho_{2m} + 1| \) are found in this study. Also certain special cases are indicated.

Keywords: \( m \)-fold symmetric bi-univalent functions, analytic functions, univalent function.

MSC: 30C45.

1. Introduction

Let \( \mathcal{A} \) be the family of holomorphic functions, normalized by the conditions \( f(0) = f'(0) - 1 = 0 \) which is of the form

\[
f(z) = z + \rho z^2 + \rho z^3 + \cdots
\]

in the open unit disk \( \Omega = \{ z; z \in \mathbb{C} \text{ and } |z| < 1 \} \). We denote by \( \mathcal{G} \) the subclass of functions in \( \mathcal{A} \) which are univalent in \( \Omega \) (see for details [1]).

The Keobe-One Quarter Theorem [1] state that the image of \( \Omega \) under all univalent function \( f \in \mathcal{A} \) contains a disk of radius \( \frac{1}{4} \). Hence all univalent function \( f \in \mathcal{A} \) has an inverse \( f^{-1} \) satisfy \( f^{-1}(f(z)) \) and \( f(f^{-1}(v)) = v \) \(|v| < r_0(f), r_0(f) \geq \frac{1}{4} \), where

\[
g(v) = f^{-1}(v) = v - \rho v^2 + (2\rho - \rho) v^3 - (5\rho^2 - 5\rho^2 v + \rho) v^4 + \cdots
\]

A function \( f \in \mathcal{A} \) denoted by \( \Sigma \) is said to be bi-univalent in \( \Omega \) if both \( f^{-1} \) and \( f \) are univalent in \( \Omega \) (see for details [2–11]).

A domain \( \Psi \) is said to be \( m \)-fold symmetric if a rotation of \( \Psi \) about the origin through an angle \( 2\pi/m \) carries \( \Psi \) on itself. Therefore, a function \( f(z) \) holomorphic in \( \Omega \) is said to be \( m \)-fold symmetric if

\[
f\left(e^{2\pi i/m} z\right) = e^{2\pi i/m} f(z).
\]

A function is said to be \( m \)-fold symmetric if it has the following normalized form

\[
f(z) = z + \sum_{\phi=1}^{\infty} \rho_{m\phi+1} z^{m\phi+1} \quad (z \in \Omega, \ m \in \mathbb{N} = \{1, 2, 3, \cdots \}).
\]

Let \( \mathcal{S}_m \) the class of \( m \)-fold symmetric univalent functions in \( \Omega \), that are normalized by (3), in which, the functions in the class \( \mathcal{S} \) are one-fold symmetric. Similar to the concept of \( m \)-fold symmetric univalent functions, we introduced the concept of \( m \)-fold symmetric bi-univalent functions which is denoted by \( \Sigma_m \). Each of the function \( f \in \Sigma \) produces \( m \)-fold symmetric bi-univalent function for each integer \( m \in \mathbb{N} \).
The normalized form of \( f(z) \) is given as in (3) and the series expansion for \( f^{-1}(z) \), which has been investigated by Srivastava et al., [12], is given below:

\[
g(v) = f^{-1}(v) = v - \rho_{m+1}v^{m+1} + \left[ (m + 1)\rho_{m-1}^2 - \rho_{2m+1} \right]v^{2m+1} - \left[ \frac{1}{2}(m + 1)(3m + 2)\rho_{m+1}^3 - (3m + 2)\rho_{m+1}\rho_{2m+1} + \rho_{3m+1} \right].
\]

(4)

Some of the examples of \( m \)-fold symmetric bi-univalent functions are

\[
\left\{ \frac{z^m}{1 - z^m} \right\}^\frac{1}{m},
\]

\[
[- \log(1 - z^m)]^\frac{1}{m},
\]

and

\[
\left\{ \frac{1}{2} \log \left( \frac{1 + z^m}{1 - z^m} \right) \right\}^\frac{1}{m}.
\]

For more details on \( m \)-fold symmetric analytic bi-univalent functions (see [5,12–17]).

Jackson [18,19] introduced the \( q \)-derivative analytic bi-univalent functions as follows;

\[
D_qf(z) = \frac{f(qz) - f(z)}{(q - 1)z}
\]

(5)

and \( D_qf(0) = f'(0) \). In case of \( g(z) = z^k \) for \( k \) is a positive integer, the \( q \)-derivative of \( f(z) \) is given by

\[
D_qz^k = \frac{z^k - (qz)^k}{(q - 1)z} = [k_q]z^{k-1}.
\]

As \( q \rightarrow 1^- \) and \( k \in \mathbb{N} \), we get

\[
[k_q] = \frac{1 - q^k}{1 - q} = 1 + q + \cdots + q^k \rightarrow k,
\]

(6)

where \( z \neq 0, q \neq 0 \). For more details on the concepts of \( q \)-derivative (see [5,20–27]).

**Definition 1.** [28] Let \( f(z) \in A \), \( 0 \leq \chi < 1 \) and \( \sigma \geq 1 \) is real. Then \( f(z) \in L_{\nu}(\chi) \) of \( \sigma \)-pseodu-starlike function of order \( \chi \) in \( \Omega \) if and only if

\[
\Re \left( \frac{z|f'(z)|^\nu}{f(z)} \right) > \chi.
\]

(7)

Babalola [28] verified that, all pseodu-starlike function are Bazilevic of type \( 1 - \frac{1}{\sigma} \), order \( \chi^2 \) and univalent in \( \Omega \).

**Lemma 1.** [1] Let the function \( \omega \in \mathcal{P} \) be given by the following series \( \omega(z) = 1 + \omega_1z + \omega_2z^2 + \cdots \) \( (z \in \Omega) \). The sharp estimate given by \( |\omega_n| \leq 2 \) \( (n \in \mathbb{N}) \) holds true.

In [29] Girgaonkar et al., introduced a new subclasses of holomorphic and bi-univalent functions as follows:

**Definition 2.** A function \( f(z) \) given by (1) is said to be in the class \( M_{\Sigma}(\chi) \) \( (0 < \chi \leq 1, (z, v) \in \Omega) \) if \( f \in \mathcal{E} \), \( |\arg(f'(z))| < \frac{\pi}{\chi^2} \) and \( |\arg(g'(v))| < \frac{\pi}{\chi} \), where \( g(v) \) is given by (2).
Definition 3. A function \( f(z) \) given by (1) is said to be in the class \( \mathcal{M}_\Sigma(\psi) \) \( (0 \leq \psi < 1, (z, v) \in \Omega) \) if \( \theta \in \Sigma, \Re[(f'(z))^{\psi}] > \psi \) and \( \Re[(g'(v))^\psi] > \psi \), where \( g(v) \) is given by (2).

In this current research, we introduced two new subclasses denoted by \( \mathcal{M}_\Sigma^q(\chi) \) and \( \mathcal{M}_\Sigma^q(\psi) \) of the function class \( \Sigma_m \) and obtain estimates coefficient \( |\rho_{m+1}| \) and \( |\rho_{2m+1}| \) for functions in these two new subclasses.

2. Main Results

Definition 4. A function \( f(z) \) given by (3) is said to be in the class \( \mathcal{M}_\Sigma^q(\chi) \) \( (m \in \mathcal{N}, 0 < q < 1, \sigma \geq 1, 0 < \chi \leq 1, (z, v) \in \Omega) \) if

\[
 f \in \Sigma \quad \text{and} \quad |\arg(D_qf(z))^{\chi}| < \frac{\chi\pi}{2},
\]

and

\[
 |\arg(D_qg(v))^{\chi}| < \frac{\chi\pi}{2},
\]

where \( g(v) \) is given by (2).

Remark 1. We have the class \( \lim_{q \to 1^{-1}} \mathcal{M}_\Sigma^q(\chi) = \mathcal{M}_\Sigma^0(\chi) \) which was introduced and studied by Girgaonkar et al., [29].

Remark 2. We have the class \( \lim_{q \to 1^{-1}} \mathcal{M}_\Sigma^q(\chi) = \mathcal{M}_\Sigma^0(\chi) \) which was introduced and studied by Srivastava et al., [11].

Theorem 1. Let \( f(z) \in \mathcal{M}_\Sigma^q(\chi) \), \( (m \in \mathcal{N}, 0 < q < 1, \sigma \geq 1, 0 < \chi \leq 1, (z, v) \in \Omega) \) be given (3). Then

\[
|\rho_{m+1}| \leq \frac{2\chi}{\sqrt{(m+1)\sigma\chi[2m+1]_q} - (\chi - \sigma)\sigma[m+1]^2_\eta},
\]

and

\[
|\rho_{2m+1}| \leq \frac{2\chi}{\sigma[2m+1]_q} + \frac{2(m+1)\chi^2}{\sigma^2[m+1]^2_\eta}.
\]

Proof. Using inequalities (1) and (9), we get

\[
(D_qf(z))^{\chi} = [\tau(z)]^\chi,
\]

and

\[
(D_qg(v))^{\chi} = [\zeta(v)]^\chi
\]

respectively, where \( \tau(z) \) and \( \zeta(v) \) in \( \mathcal{P} \) are given by the following series

\[
\tau(z) = 1 + \tau_mz^m + \tau_{2m}z^{2m} + \tau_{3m}z^{3m} + \cdots,
\]

and

\[
\zeta(v) = 1 + \zeta_mv^m + \zeta_{2m}v^{2m} + \zeta_{3m}v^{3m} + \cdots.
\]

Clearly,

\[
[\tau(z)]^\chi = 1 + \chi\tau_mz^m + \left(\chi^2\tau_{2m} + \frac{\chi(\chi-1)}{2}\tau_m^2\right)z^{2m} + \cdots,
\]

and

\[
[\zeta(v)]^\chi = 1 + \chi\zeta_mv^m + \left(\chi^2\zeta_{2m} + \frac{\chi(\chi-1)}{2}\zeta_m^2\right)v^{2m} + \cdots.
\]

Also

\[
(D_qf(z))^{\chi} = 1 + \sigma[m+1]q\rho_{m+1}z^m + \left(\sigma[2m+1]q\rho_{2m+1} + \frac{\sigma(\sigma-1)}{2}[m+1]^2q\rho_{m+1}^2\right)z^{2m} + \cdots,
\]
and

\[
(D_qg(v))^m = 1 - \sigma[m + 1]q\rho_{m+1}v^m - \sigma[2m + 1]q\rho_{2m+1}v^{2m} + \left(\sigma(m + 1)[2m + 1]q\rho_{m+1}^2 + \frac{\sigma(\sigma - 1)}{2}[m + 1]q\rho_{m+1}^2\right)v^{2m} + \ldots
\]

Comparing the coefficients in (12) and (13), we have

\[
\sigma[m + 1]q\rho_{m+1} = \chi\tau_m, \quad \sigma[2m + 1]q\rho_{2m+1} + \frac{\sigma(\sigma - 1)}{2}[m + 1]q\rho_{m+1}^2 = \chi\tau_{2m} + \frac{\chi(\chi - 1)}{2}\tau_m, \quad -\sigma[m + 1]q\rho_{m+1} = \chi\xi_m, \quad -\sigma[2m + 1]q\rho_{2m+1} + \left(\sigma(m + 1)[2m + 1]q + \frac{\sigma(\sigma - 1)}{2}[m + 1]q\right)\rho_{m+1}^2 = \chi\xi_{2m} + \frac{\chi(\chi - 1)}{2}\xi_m.
\]

From (16) and (18), we obtain

\[
\tau_m = -\xi_m,
\]

and

\[
2\sigma[m + 1]q\rho_{m+1}^2 = \chi^2(\tau_m^2 + \xi_m^2).
\]

Further from (17), (19) and (21), we obtain that

\[
\sigma(\sigma - 1)[m + 1]q\rho_{m+1}^2 + (m + 1)\sigma\chi[2m + 1]q\rho_{2m+1}^2 - (\chi - 1)\sigma^2[m + 1]q\rho_{m+1}^2 = \chi^2(\tau_{2m} + \xi_{2m}).
\]

Therefore, we have

\[
\rho_{m+1}^2 = \frac{\chi^2(\tau_{2m} + \xi_{2m})}{\sigma[m + 1]q\chi[2m + 1]q - (\chi - \sigma)\sigma^2[m + 1]q}.
\]

By applying Lemma 1 for the coefficients \(\tau_{2m}\) and \(\xi_{2m}\), then we have

\[
|\rho_{m+1}| \leq \frac{2\chi}{\sqrt{(m + 1)\sigma\chi[2m + 1]q - (\chi - \sigma)\sigma^2[m + 1]q}}.
\]

Also, to find the bound on \(|\rho_{2m+1}|\), using the relation (19) and (17), we obtain

\[
2\sigma[2m + 1]q\rho_{2m+1} - (m + 1)\sigma\chi[2m + 1]q\rho_{m+1}^2 = \chi(\tau_{2m} - \xi_{2m}) + \frac{\chi(\chi - 1)}{2}(\tau_m^2 - \xi_m^2).
\]

It follows from (20), (21) and (23),

\[
\rho_{2m+1} = \frac{(m + 1)\chi^2\tau_{2m}^2}{2\sigma^2[m + 1]q^2} + \frac{\chi(\tau_{2m} - \xi_{2m})}{2\sigma[2m + 1]q}.
\]

Applying Lemma 1 for the coefficients \(\tau_m\), \(\tau_{2m}\), \(\xi_m\), \(\xi_{2m}\), then we have

\[
|\rho_{2m+1}| \leq \frac{2\chi}{\sigma[2m + 1]q} + \frac{2(m + 1)\chi^2}{\sigma^2[m + 1]q^2}.
\]

Choosing \(q \to 1^{-1}\) in Theorem 1, we get the following result:

**Corollary 1.** Let \(f(z) \in \mathcal{M}_{n,m}^\sigma(\chi), (m \in \mathcal{N}, \sigma \geq 1, 0 < \chi \leq 1, (z, v) \in \Omega)\) be given (3). Then

\[
|\rho_{m+1}| \leq \frac{2\chi}{\sqrt{(m + 1)(\sigma\chi m + \sigma^2 m + \sigma^2)}},
\]
and

\[ |\rho_{2m+1}| \leq \frac{2\chi}{\sigma(2m+1)} + \frac{2\chi^2}{\sigma^2(m+1)}. \]  
(26)

Choosing \( m = 1 \) (one-fold case) in Theorem 1, we get the following result:

**Corollary 2.** Let \( f(z) \in \mathcal{M}_{\Sigma}^{q,\sigma}(\chi), (0 < q < 1, \sigma \geq 1, 0 < \chi \leq 1, (z, v) \in \Omega) \) be given \((1)\). Then

\[ |\rho_2| \leq \frac{2\chi}{\sqrt{2\sigma(2\sigma + \chi)}}, \]  
(27)

and

\[ |\rho_3| \leq \frac{2\chi}{\sigma[3]_q} + \frac{4\chi^2}{\sigma^2[2]_q}. \]  
(28)

Choosing \( q \to 1^{-1} \) in Corollary 2, we get the following result:

**Corollary 3.** \([29]\) Let \( f(z) \in \mathcal{M}_{\Sigma}^{q,\sigma}(\chi), (\sigma \geq 1, 0 < \chi \leq 1, (z, v) \in \Omega) \) be given \((1)\). Then

\[ |\rho_2| \leq \frac{2\chi}{\sqrt{2\sigma(2\sigma + \chi)}}, \]  
(29)

and

\[ |\rho_3| \leq \frac{\chi(2\sigma + 3\chi)}{3\sigma}. \]  
(30)

**Remark 3.** For one-fold case, we have \( \lim_{q \to 1^{-1}} \mathcal{M}_{\Sigma,1}^{q,1}(\chi) = \mathcal{M}_{\Sigma}(\chi) \), and we can get the results of Srivastava et al., \([11]\).

**Definition 5.** A function \( f(z) \) given by \((3)\) is said to be in the class \( \mathcal{M}_{\Sigma,m}^{q,\sigma}(\psi) \) \((m \in \mathbb{N}, 0 < q < 1, \sigma \geq 1, 0 \leq \psi < 1, (z, v) \in \Omega) \) if

\[ f \in \Sigma \quad \text{and} \quad \Re([D_qf(z)]^{\psi}) > \psi, \]  
(31)

and

\[ \Re([D_qg(v)]^{\psi}) > \psi, \]  
(32)

where \( g(v) \) is given by \((2)\).

**Remark 4.** We have the class \( \lim_{q \to 1^{-1}} \mathcal{M}_{\Sigma,1}^{q,1}(\psi) = \mathcal{M}_{\Sigma}^{q,1}(\chi) \) which was introduced and studied by Girgaonkar et al., \([29]\).

**Remark 5.** We have the class \( \lim_{q \to 1^{-1}} \mathcal{M}_{\Sigma,1}^{q,1}(\psi) = \mathcal{M}_{\Sigma}^{q,1}(\chi) \) which was introduced and studied by Srivastava et al., \([11]\).

**Theorem 2.** Let \( f(z) \in \mathcal{M}_{\Sigma,m}^{q,\sigma}(\psi), (m \in \mathbb{N}, 0 < q < 1, \sigma \geq 1, 0 \leq \psi < 1, (z, v) \in \Omega) \) be given \((3)\). Then

\[ |\rho_{m+1}| \leq \min \left\{ \frac{2(1 - \psi)}{\sigma[m + 1]_q}, \sqrt{\frac{1 - \psi}{(\sigma - 1)[m + 1]_q^2 + (m + 1)\sigma[2m + 1]_q}} \right\}, \]  
(33)

and

\[ |\rho_{2m+1}| \leq \frac{2(m + 1)(1 - \psi)}{\sigma(\sigma - 1)[m + 1]_q^2 + (m + 1)\sigma[2m + 1]_q} + \frac{2(1 - \psi)}{\sigma[2m + 1]_q}. \]  
(34)

**Proof.** Using inequalities \((31)\) and \((32)\), we get

\[ (D_qf(z))^{\psi} = \psi + (1 - \psi)\tau(z), \]  
(35)
and

\[(D_q g(v))'' = \psi + (1 - \psi)\zeta(v),\]  

(36)

Here \(\tau(z)\) and \(\zeta(v)\) in \(P\) are given by the following series

\[\tau(z) = 1 + \tau_m z^m + \tau_2 z^{2m} + \tau_3 z^{3m} + \cdots,\]

and

\[\zeta(v) = 1 + \zeta_m v^m + \zeta_2 v^{2m} + \zeta_3 v^{3m} + \cdots.\]

Clearly,

\[\psi + (1 - \psi)\tau(z) = 1 + (1 - \psi)\tau_m z^m + (1 - \psi)\tau_2 z^{2m} + \cdots,\]

and

\[\psi + (1 - \psi)\zeta(v) = 1 + (1 - \psi)\zeta_m v^m + (1 - \psi)\zeta_2 v^{2m} + \cdots.\]

Also

\[(D_q f(z))'' = 1 + \sigma(m + 1) q \rho_{m+1} z^m + \left(\sigma(2m + 1) q \rho_{2m+1} + \frac{\sigma(\sigma - 1)}{2} [m + 1] \rho_{m+1}^2\right) z^{2m} + \cdots,\]

and

\[(D_q g(v))'' = 1 - \sigma(m + 1) q \rho_{m+1} v^m - \sigma(2m + 1) q \rho_{2m+1} v^{2m} + \left(\sigma(m + 1) q \rho_{m+1}^2 + \frac{\sigma(\sigma - 1)}{2} [m + 1] \rho_{m+1}^2\right) v^{2m} + \cdots.\]

Now comparing the coefficients in (35) and (36), we get

\[\sigma(m + 1) q \rho_{m+1} = (1 - \psi) \tau_m,\]  

(37)

\[\sigma(2m + 1) q \rho_{2m+1} + \frac{\sigma(\sigma - 1)}{2} [m + 1] \rho_{m+1}^2 = (1 - \psi) \tau_2 m,\]  

(38)

\[-\sigma(m + 1) q \rho_{m+1} = (1 - \psi) \zeta_m,\]  

(39)

\[-\sigma(2m + 1) q \rho_{2m+1} + \left(\sigma(m + 1) q \rho_{m+1}^2 + \frac{\sigma(\sigma - 1)}{2} [m + 1] \rho_{m+1}^2\right) \rho_{m+1}^2 = (1 - \psi) \zeta_2 m.\]  

(40)

From (37) and (39), we obtain

\[\tau_m = -\zeta_m,\]  

(41)

and

\[2\sigma(m + 1) \rho_{m+1}^2 = (1 - \psi)^2 (\zeta_2 m + \zeta_2 m).\]  

(42)

Also, from (38) and (40), we get

\[\sigma(\sigma - 1) q [m + 1] \rho_{m+1}^2 + (m + 1) \sigma(2m + 1) q \rho_{m+1}^2 = (1 - \psi) (\tau_2 m + \zeta_2 m).\]  

(43)

Applying the Lemma 1 for the coefficients \(\tau_m, \tau_2 m, \zeta_m, \zeta_2 m,\) we find that

\[|\rho_{m+1}| \leq \sqrt{\frac{(1 - \psi)}{\sigma(\sigma - 1) [m + 1] q^2 + (m + 1) \sigma(2m + 1) q}}.\]

Also, to find the bound on \(|\rho_{2m+1}|\), using the relation (40) and (38), we obtain

\[-(m + 1) \sigma(2m + 1) q \rho_{m+1}^2 + 2\sigma(2m + 1) q \rho_{2m+1} = (1 - \psi) (\tau_2 m - \zeta_2 m),\]  

(44)
or equivalently
\[
\rho^{2m+1} = \frac{(1-\psi)(\tau_{2m} - \zeta_{2m})}{2\sigma[2m+1]q} + \frac{(m+1)(1-\psi)^2(\tau_m^2 + \zeta_m^2)}{4\sigma^2[2m+1]^2q}. \tag{45}
\]
By substituting the value of \(\rho^{2m+1}\) from (42), we have
\[
\rho^{2m+1} = \frac{(1-\psi)(\tau_{2m} - \zeta_{2m})}{2\sigma[2m+1]q} + \frac{(m+1)(1-\psi^2 (\tau_m^2 + \zeta_m^2))}{4\sigma^2[2m+1]^2q}. \tag{46}
\]
Applying the Lemma 1 for the coefficients \(\tau_m, \tau_{2m}, \zeta_m, \zeta_{2m}\), we get
\[
|\rho^{2m+1}| \leq \frac{2(1-\psi)}{\sigma[2m+1]q} + \frac{2(m+1)(1-\psi)^2}{2\sigma^2[2m+1]^2q}.
\]
Also, by using (43) and (45), and applying Lemma 1 we obtain
\[
|\rho^{2m+1}| \leq \frac{2(m+1)(1-\psi)}{\sigma(\sigma-1)[m+1]^2 + (m+1)\sigma[2m+1]q} + \frac{2(1-\psi)}{\sigma[2m+1]q}.
\]
This complete the proof. \( \Box \)

Choosing \(q \rightarrow 1^{-1}\) in Theorem 2, we get the following result:

**Corollary 4.** Let \( f(z) \in \mathcal{M}_{\Sigma, m}^\sigma(\psi), (m \in \mathcal{N}, \sigma \geq 1, 0 \leq \psi < 1, (z, v) \in \Omega) \) be given (3). Then
\[
|\rho^{m+1}| \leq \begin{cases}
2 \left( \sqrt{\frac{(1-\psi)}{\sigma(\sigma-1)[m+1]^2 + (m+1)\sigma[2m+1]q}} ight) & 0 \leq \psi \leq \frac{m}{1+2m}, \\
\frac{2(1-\psi)}{\sigma[2m+1]} & \frac{m}{1+2m} \leq \psi < 1,
\end{cases}
\]
and
\[
|\rho^{2m+1}| \leq \frac{2(m+1)(1-\psi)}{\sigma(\sigma-1)[m+1]^2 + (m+1)\sigma[2m+1]q} + \frac{2(1-\psi)}{\sigma[2m+1]q}.
\]

For one fold case, Corollary 4, yields the following Corollary:

**Corollary 5.** Let \( f(z) \in \mathcal{M}_{\Sigma}^\sigma(\psi), (\sigma \geq 1, 0 \leq \psi < 1, (z, v) \in \Omega) \) be given (1). Then
\[
|\rho^{2}| \leq \begin{cases}
\sqrt{\frac{2(1-\psi)}{\sigma(2\sigma+1)}} & 0 \leq \psi \leq \frac{1}{2}, \\
\frac{1-\psi}{\sigma} & \frac{1}{2} \leq \psi < 1,
\end{cases}
\]
and
\[
|\rho^{3}| \leq \frac{(1-\psi)(2\sigma - 3\psi + 3)}{3\sigma^2}.
\]

**Remark 6.** Corollary 5 gives above is the improvement of the estimates for coefficients on \(|\rho^{2}|\) and \(|\rho^{3}|\) investigated by Girgaonkar et al., [29].

**Corollary 6.** [29] Let \( f(z) \in \mathcal{M}_{\Sigma}^\sigma(\psi), (\sigma \geq 1, 0 \leq \psi < 1, (z, v) \in \Omega) \) be given (1). Then
\[
|\rho^{2}| \leq \sqrt{\frac{2(1-\psi)}{\sigma(2\sigma+1)}},
\]
and
\[
|\rho^{3}| \leq \frac{(1-\psi)(2\sigma - 3\psi + 3)}{3\sigma^2}.
\]

Taking \(\sigma = 1\) in Corollary 7, we get the following result:
Corollary 7. [11] Let $f(z) \in M^{\sigma}_\Sigma(\psi)$, $(\sigma \geq 1, 0 \leq \psi < 1, (z, \nu) \in \Omega)$ be given (1). Then
\[
|\rho_2| \leq \sqrt{\frac{2(1-\psi)}{3}},
\]
and
\[
|\rho_3| \leq \frac{(1-\psi)(5-3\psi)}{3}.
\]

3. Conclusion

In this present paper, two new subclasses indicated by $M^{\sigma}_\Sigma(\chi)$ and $M_{\Sigma m}(\psi)$ of function class of $E_m$ was obtained and worked on. Also, the estimates coefficients for $|\rho_{m+1}|$ and $|\rho_{2m+1}|$ of functions in these classes are determined.

Conflicts of Interest: “The author declares no conflict of interest.”

References


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