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General decay of the double dispersive wave equation with memory and source terms

Mohamed Mellah

Faculty of Exact Sciences and Computer Science, Hassiba Benbouali University of Chlef, Chlef Algeria.; m.mellah@univ-chlef.dz

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Abstract: The double dispersive wave equation with memory and source terms $u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau - \Delta u_t = |u|^{p-2}u$ is considered in bounded domain. The existence of global solutions and decay rates of the energy are proved.

Keywords: Double dispersive wave equation, small data global Solution, general decay.

MSC: 35L35, 35L82, 35B40.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$. We consider the initial-boundary value problem:

$$\begin{cases} u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u - g * \Delta^2 u - \Delta u_t = |u|^{p-2}u, & x \in \Omega,\ t > 0, \\ u = 0, \quad \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1}$$

where $p > 2$ and ν represents the unit outward normal to $\partial\Omega$. Here, $g(t)$ is a positive function that represents the kernel of the memory term, which will be specified in Section 2 and

$$g * \Delta^2 u(t) = \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau.$$

The motivation of our work is due to the initial boundary problem of the double dispersive-dissipative wave equation with nonlinear damping and source terms

$$\begin{cases} u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u - \Delta u_t + a|u_t|^{m-2}u_t = b|u|^{p-2}u, & x \in \Omega,\ t > 0, \\ u = 0, \quad \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ a, b > 0, \end{cases} \tag{2}$$

which has been discussed by Di and Shang [1] by considering the existence of global solutions and the asymptotic behavior of global solutions with $m \geq p$.

In the absence of the dispersive term and the nonlinear damping term, model (2) reduces to the following wave equation

$$u_{tt} - \Delta u - \Delta u_{tt} - \Delta u_t = f(u). \tag{3}$$

Shang [2] studied the well-posedness, asymptotic behavior, and the finite time blow-up of the solutions under some suitable conditions on f and for $N = 1, 2, 3$. Zhang and Hu [3] showed the existence and the stability of global weak solutions. Xie and Zhong [4] obtained the existence of global attractors in $H_0^1(\Omega) \times H_0^1(\Omega)$, where the nonlinear term f satisfies a critical exponential growth assumption. Xu *et al.*, [5] used the multiplier method to investigate the asymptotic behavior of solutions for (3).

Mellah [6] considered the following initial-boundary value problem

$$\begin{cases} u_{tt} - \Delta u + \Delta^2 u - g * \Delta^2 u + u_t = |u|^{p-1}u, & x \in \Omega,\ t > 0, \\ u = 0, \quad \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases}$$

in a bounded domain and $p > 1$. He investigated the small data global weak solutions and general decay of solutions, respectively.

Motivated by previous works, it is interesting to prove that problem (1) has a global weak solution assuming small initial data. In addition, we show the general decay of solutions. The global solutions are constructed by means of the Galerkin approximations and the general decay is obtained by employing the technique used in [7].

2. Preliminaries

In this section, we present some materials needed in the proof of our main result. We use the following abbreviations; $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$ $(1 \le p \le +\infty)$ denotes usual L^p norm, $(\cdot,\cdot)$ denotes the L^2-inner product, and consider the Sobolev spaces $H_0^1(\Omega)$ and $H_0^2(\Omega)$ with their usual scalar products and norms. We also use the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ for $2 < p \le \frac{2N}{N-2}$ if $N \ge 3$ or $2 < p < \infty$ if $N = 1,2$. In this case, the embedding constant is denoted by C_*, that is $\|u\|_p \le C_* \|\nabla u\|_2$. We define

$$Q(z) = \frac{1}{2}z^2 - \frac{C_*^p}{p} z^p.$$

By the direct computation, we deduce that Q is increasing in $[0, z_0]$, where $z_0 = C_*^{\frac{p}{2-p}}$ is its unique local maximum.

Next, we give the assumptions for problem (1).

(G1) The relaxation function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded C^1 function such that

$$g(0) > 0, \quad 0 < \eta = 1 - \int_0^\infty g(\tau)d\tau \le 1 - \int_0^t g(\tau)d\tau = \eta(t).$$

(G2) There exist positive constants ξ_1 and ξ_2 such that

$$-\xi_1 g(t) \le g'(t) \le -\xi_2 g(t) \quad \forall t \ge 0.$$

(G3) We also assume that

$$2 < p \le \frac{2N}{N-2} \text{ if } N \ge 3 \text{ and } p > 2 \text{ if } N = 1,2,$$

where λ_1 is the first eigenvalue of the following problem

$$\Delta^2 u = \lambda_1 u \quad \text{in } \Omega, \quad u = \frac{\partial u}{\partial \nu} = 0 \quad \text{in } \partial\Omega. \tag{4}$$

Remark 1. [8] Assuming λ_1 is the first eigenvalue of the problem (4), we have

$$\|\Delta u\|_2^2 \ge \lambda_1 \|\nabla u\|_2^2. \tag{5}$$

The energy associated with problem (1) is given by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|\Delta u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \Delta u)(t) - \frac{1}{p}\|u\|_p^p, \tag{6}$$

for $u \in H_0^2(\Omega)$, where

$$(g \circ \Delta u)(t) = \int_0^t g(t-\tau)\|\Delta u(\tau) - \Delta u(t)\|_2^2 d\tau.$$

Now, we are in a position to state our main results.

3. Main results

In this section, we are going to obtain the existence of global weak solutions for problem (1) with the initial conditions $\|\nabla u_0\|_2 < z_0$ and $E(0) < Q(z_0)$.

Theorem 1. *Assume that $(G1)-(G3)$ hold, and that $\{u_0, u_1\}$ belong to $H_0^2(\Omega) \times H_0^1(\Omega)$. Further assume that $\|\nabla u_0\|_2 < z_0$ and $E(0) < Q(z_0)$. Then, problem (1) admits a global weak solution, which satisfies*

$$u \in L^\infty(0,\infty; H_0^2(\Omega)), \quad u_t \in L^\infty(0,\infty; H_0^1(\Omega)).$$

Moreover, the identity

$$E(t) + \int_0^t \|\nabla u_t(\tau)\|_2^2 d\tau - \frac{1}{2}\int_0^t (g' \circ \Delta u)(\tau) d\tau + \frac{1}{2}\int_0^t g(\tau)\|\Delta u(\tau)\|_2^2 d\tau = E(0), \tag{7}$$

holds for $0 \le t < \infty$. Also, for an increasing C^2 function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\zeta(0) = 0, \quad \zeta_t(0) > 0, \quad \lim_{t\to+\infty} \zeta(t) = +\infty, \quad \zeta_{tt}(t) < 0 \quad \forall t \ge 0, \tag{8}$$

and, if $\|g\|_{L^1(0,\infty)}$ is sufficiently small, we have for $\kappa > 0$

$$E(t) \le E(0)e^{-\kappa\zeta(t)}, \quad \forall t \ge 0.$$

Remark 2. From (8) and $(G2)$, we obtain

$$\begin{aligned} \frac{d}{dt}E(t) &= -\|\nabla u_t(t)\|_2^2 + \frac{1}{2}(g' \circ \Delta u)(t) - \frac{1}{2}g(t)\|\Delta u(t)\|_2^2 \\ &\le -\|\nabla u_t(t)\|_2^2 - \frac{1}{2}\xi_2(g \circ \Delta u)(t) - \frac{1}{2}g(t)\|\Delta u(t)\|_2^2 \le 0. \end{aligned} \tag{9}$$

Proof of Theorem 1 (Main result)

We divide the proof into two steps. In step 1, we prove the small data global existence of weak solutions by using the Faedo-Galerkin approximation and in step 2, we establish the general decay of energy employing the method used in [7].

Step 1: Global existence of weak solutions

Let $\{\omega_j\}_{j=1}^\infty$ be an orthogonal basis of $H_0^2(\Omega)$ with ω_j being the eigenfunction of the following problem:

$$-\Delta\omega_j = \lambda_j\omega_j, \quad x \in \Omega, \quad \omega_j = 0, \quad x \in \partial\Omega.$$

Let $V^n = \text{Span}\{\omega_1, \omega_2, \cdots, \omega_n\}$. By the standard method of ODE, we know that there exists only one local solution

$$u^n(t) = \sum_{j=1}^n b_j^n(t)\omega_j$$

of the Cauchy problem as follows:

$$\begin{aligned} &\int_\Omega u_{tt}^n \omega dx + \int_\Omega \nabla u^n \cdot \nabla\omega dx + \int_\Omega \nabla u_{tt}^n \cdot \nabla\omega dx + \int_\Omega \Delta u^n \cdot \Delta\omega dx \\ &- \int_0^t g(t-\tau)\int_\Omega \Delta u^n(\tau) \cdot \Delta\omega dx d\tau + \int_\Omega \nabla u_t^n \cdot \nabla\omega dx - \int_\Omega |u^n|^{p-2}u^n \omega dx = 0, \end{aligned} \tag{10}$$

$$u^n(0) = u_0^n \to u_0, \text{ in } H_0^2(\Omega), \quad u_t^n(0) = u_1^n \to u_1 \text{ in } H_0^1(\Omega). \tag{11}$$

By the standard theory of ODE system, we prove the existence of solutions of problem (10)-(11) on some interval $[0, t_n)$, $0 < t_n < T$ for arbitrary $T > 0$, then, this solution can be extended to the whole interval $[0, T]$ using the first estimate given below.

A Priori Estimates

Setting $\omega = u_t^n(t)$ in (10), we have

$$\frac{1}{2}\frac{d}{dt}\|u_t^n\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u_t^n\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u^n\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\Delta u^n\|_2^2 - \frac{1}{p}\frac{d}{dt}\|u^n\|_p^p + \|\nabla u_t^n\|_2^2$$
$$- \int_0^t g(t-\tau)\int_\Omega \Delta u^n(\tau)\cdot \Delta u_t^n(t)dxd\tau = 0. \tag{12}$$

A direct computation shows that

$$-\int_0^t g(t-\tau)\int_\Omega \Delta u^n(\tau)\cdot \Delta u_t^n(t)dxd\tau$$
$$= \frac{1}{2}\frac{d}{dt}(g\circ \Delta u^n)(t) - \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(\tau)d\tau\right)\|\Delta u^n(t)\|_2^2 - \frac{1}{2}(g'\circ \Delta u^n)(t) + \frac{1}{2}g(t)\|\Delta u^n(t)\|_2^2. \tag{13}$$

Inserting (13) into (12) and integrating over $[0, t] \subset [0, T]$, we obtain

$$\frac{1}{2}\|u_t^n\|_2^2 + \frac{1}{2}\|\nabla u_t^n\|_2^2 + \frac{\eta(t)}{2}\|\Delta u^n(t)\|_2^2 + \frac{1}{2}\|\nabla u^n\|_2^2 - \frac{1}{p}\|u^n\|_p^p + \int_0^t \|\nabla u_t^n(\tau)\|_2^2 d\tau + \frac{1}{2}(g\circ \Delta u^n)(t)$$
$$-\frac{1}{2}\int_0^t (g'\circ \Delta u^n)(\tau)d\tau + \frac{1}{2}\int_0^t g(\tau)\|\Delta u^n(\tau)\|_2^2 d\tau = E^n(0). \tag{14}$$

From assumption $(G3)$ and the Sobolev embedding, we have

$$\|u^n\|_p^p \le C_*^p \|\nabla u^n\|_2^p,$$

and then we have

$$\frac{1}{2}\|u_t^n\|_2^2 + \frac{1}{2}\|\nabla u_t^n\|_2^2 + \frac{\eta(t)}{2}\|\Delta u^n(t)\|_2^2 + \mathcal{Q}(\|\nabla u^n\|_2^2) + \int_0^t \|\triangle u_t^n(\tau)\|_2^2 d\tau + \frac{1}{2}(g\circ \Delta u^n)(t)$$
$$-\frac{1}{2}\int_0^t (g'\circ \Delta u^n)(\tau)d\tau + \frac{1}{2}\int_0^t g(\tau)\|\Delta u^n(\tau)\|_2^2 d\tau \le E^n(0). \tag{15}$$

By using the fact that

$$-\int_0^t (g'\circ \Delta u^n)(\tau)d\tau + \int_0^t g(\tau)\|\Delta u^n(\tau)\|_2^2 d\tau \ge 0,$$

estimate (15) yields

$$\frac{1}{2}\|u_t^n\|_2^2 + \frac{1}{2}\|\nabla u_t^n\|_2^2 + \frac{\eta(t)}{2}\|\Delta u^n(t)\|_2^2 + \frac{1}{2}(g\circ \Delta u^n)(t) + \mathcal{Q}(\|\nabla u^n\|_2^2) + \int_0^t \|\nabla u_t^n(\tau)\|_2^2 d\tau \le E^n(0). \tag{16}$$

From $E(0) < \mathcal{Q}(z_0)$ and (11), it follows that

$$E^n(0) < \mathcal{Q}(z_0) \tag{17}$$

for sufficiently large n. We claim that there exists an integer N such that

$$\|\nabla u^n(t)\|_2^2 < z_0 \quad \forall t \in [0, t_n) \quad n > N. \tag{18}$$

Suppose the claim is proved, then $\mathcal{Q}(\|\nabla u^n\|_2^2) \ge 0$ and from (16) and (17),

$$\frac{1}{2}\|u_t^n\|_2^2 + \frac{1}{2}\|\nabla u_t^n\|_2^2 + \frac{\eta(t)}{2}\|\Delta u^n(t)\|_2^2 + \frac{1}{2}(g\circ \Delta u^n)(t) + \int_0^t \|\nabla u_t^n(\tau)\|_2^2 d\tau \le E^n(0) < \mathcal{Q}(z_0), \tag{19}$$

for sufficiently large n and $0 \leq t < \infty$.

Proof of the claim

Suppose that (18) false, then for each $n > N$, there exists $t \in [0, t_n)$ such that $\|\nabla u^n(t)\|_2 \geq z_0$. Note that from $\|\nabla u_0\|_2 < z_0$ and (11) there exists N_0 such that

$$\|\nabla u^n(0)\|_2 < z_0 \quad \forall n > N_0.$$

Then by continuity there exits a first $\widetilde{t_n} \in [0, t_n)$ such that

$$\|\nabla u^n(\widetilde{t_n})\|_2 = z_0, \tag{20}$$

from where

$$\mathcal{Q}(\|\nabla u^n(t)\|_2) \geq 0 \quad \forall t \in [0, \widetilde{t_n}].$$

From $E(0) < \mathcal{Q}(z_0)$ and (19), there exists $N > N_0$ and $\gamma \in (0, z_0)$ such that

$$\begin{aligned} 0 &\leq \frac{1}{2}\|u_t^n(t)\|_2^2 + \frac{1}{2}\|\nabla u_t^n(t)\|_2^2 + \frac{\eta(t)}{2}\|\Delta u^n(t)\|_2^2 + \frac{1}{2}(g \circ \Delta u^n)(t) + \mathcal{Q}(\|\nabla u^n(t)\|_2^2) \\ &\leq \mathcal{Q}(\gamma) \quad \forall \quad t \in [0, \widetilde{t_n}] \quad \forall n > N. \end{aligned}$$

The monotonicity of $\mathcal{Q}$ in $[0, z_0]$ implies that

$$0 \leq \|\nabla u^n(t)\|_2^2 \leq \gamma < z_0 \quad \forall t \in [0, \widetilde{t_n}],$$

in particular, $\|\nabla u^n(t)\|_2^2 < z_0$, which is a contradiction to (20). From (19), we have

$$\|\Delta u^n\|_2^2 < \frac{2\mathcal{Q}(z_0)}{\eta}, \qquad 0 \leq t < \infty, \tag{21}$$

$$\|u_t^n\|_2^2 < 2\mathcal{Q}(z_0), \qquad 0 \leq t < \infty, \tag{22}$$

$$\|\nabla u_t^n\|_2^2 < 2\mathcal{Q}(z_0), \qquad 0 \leq t < \infty, \tag{23}$$

$$\int_0^t \|\nabla u_t^n(\tau)\|_2^2 d\tau < \mathcal{Q}(z_0), \qquad 0 \leq t < \infty. \tag{24}$$

Using Sobolev inequality, (5) and (21), it follows that

$$\|u^n\|_p^2 \leq C_*^2\|\nabla u^n\|_2^2 \leq C_*^2\lambda_1^{-1}\|\Delta u^n\|_2^2 < \frac{2C_*^2\lambda_1^{-1}\mathcal{Q}(z_0)}{\eta}, \quad 0 \leq t < \infty. \tag{25}$$

Moreover, by (25), we get

$$|(|u^n|^{p-2}u^n, u^n)| \leq \|u^n\|_p^p < C_*^p\left(\frac{2C_*^2\lambda_1^{-1}\mathcal{Q}(z_0)}{\eta}\right)^{\frac{p}{2}}, \quad 0 \leq t < \infty. \tag{26}$$

Therefore, there exist u, χ and a subsequence still denotes $\{u_n\}$ such that

$$u_n \to u \text{ weak star in } L^\infty(0, \infty; H_0^2(\Omega)), \quad n \to +\infty, \tag{27}$$

$$u_t^n \to u_t \text{ weak star in } L^\infty(0, \infty; H_0^1(\Omega)), \quad n \to +\infty, \tag{28}$$

$$|u^n|^{p-2}u^n \to \chi \text{ weak star in } L^\infty(0, \infty; L^{\frac{p}{p-1}}(\Omega)), \quad n \to +\infty, \tag{29}$$

Besides, from Lions-Aubin Lemma we also have

$$u^n \to u \text{ strongly in } L^2(0, \infty; L^2(\Omega)), \quad n \to +\infty, \tag{30}$$

and consequently, making use of the Lemma 1.3 in [9], we deduce

$$|u^n|^{p-2}u^n \to \chi = |u|^{p-2}u \text{ weak star in } \ L^\infty(0,\infty; L^{\frac{p}{p-1}}(\Omega)), \quad n \to +\infty. \tag{31}$$

Thus, we obtain that u is a global weak of problem (1). In order to prove (7), we use the mean value theorem, we see that there exists $0 < \theta_n < 1$ such that

$$\begin{aligned} \|u^n\|_p^p - \|u\|_p^p &\leq p\left|\int_\Omega |u+\theta_n u^n|^{p-2}(u+\theta_n u^n)(u^n-u)dx\right| \\ &\leq p\|u+\theta_n u^n\|_p^{p-1}\|u^n-u\|_p \\ &\leq c\|u^n-u\|_p \to 0 \ \text{ as } \ n\to+\infty, \end{aligned}$$

and for each fixed $t > 0$, we obtain

$$\begin{aligned} |(g\circ\Delta u)(t)-(g\circ\Delta u^n)(t)| &= \left|\int_0^t g(t-\tau)\|\Delta u(\tau)-\Delta u(t)\|_2^2 d\tau - \int_0^t g(t-\tau)\|\Delta u^n(\tau)-\Delta u^n(t)\|_2^2 d\tau\right| \\ &\leq \int_0^t g(t-\tau)\|\Delta u(\tau)-\Delta u^n(\tau)\|_2\|\Delta u(\tau)+\Delta u^n(\tau)\|_2 d\tau \\ &\quad + \int_0^t g(t-\tau)\|\Delta u(\tau)-\Delta u^n(\tau)\|_2 d\tau\|\Delta u(t)+\Delta u^n(t)\|_2 \\ &\quad + \int_0^t g(t-\tau)\|\Delta u(\tau)+\Delta u^n(\tau)\|_2 d\tau\|\Delta u(t)-\Delta u^n(t)\|_2 \\ &\quad + \int_0^t g(\tau)d\tau\|\Delta u(t)+\Delta u^n(t)\|_2\|\Delta u(t)-\Delta u^n(t)\|_2 \\ &\leq c\int_0^t g(t-\tau)\|\Delta u(\tau)-\Delta u^n(\tau)\|_2 d\tau \\ &\quad + c\int_0^t g(\tau)d\tau\|\Delta u(t)-\Delta u^n(t)\|_2 \to 0 \ \text{ as } \ n\to+\infty. \end{aligned}$$

Thus, we have

$$\lim_{n\to+\infty}\|u^n\|_p^p = \|u\|_p^p, \quad \lim_{n\to+\infty}(g\circ\Delta u^n)(t) = (g\circ\Delta u)(t).$$

From (11), it follows that $E^n(0) \to E(0)$ as $n \to +\infty$. Finally, taking $n \to +\infty$ in (14), we deduce that the energy identity (7) holds for $0 \leq t < \infty$.

Step 2: General decay of the energy

Here, we prove the energy decay estimate of the global solutions obtained in the previous section. To obtain the decay result, we use the following lemmas which are of crucial importance in the proof.

Lemma 1. *Let $u \in L^\infty(0,\infty; H_0^2(\Omega))$ with $u_t \in L^\infty(0,\infty; H_0^1(\Omega))$ be the solution of (1) and $E(0) < \mathcal{Q}(z_0)$, $\|\nabla u_0\|_2 < z_0$, then we have*

$$0 \leq E(t) \leq C_1\|\nabla u_t\|_2^2 + C_2\|\Delta u\|_2^2 + \frac{1}{2}(g\circ\Delta u)(t), \tag{32}$$

where $C_1 = \frac{1}{2}(1+B^2)$, $C_2 = \frac{1}{2}(1+\lambda_1^{-1})$ and B is the optimal constant satisfying the Poincaré inequality $\|u_t\|_2 \leq B\|\nabla u_t\|_2$.

Proof. From $E(0) < \mathcal{Q}(z_0)$ and $\|\nabla u_0\|_2 < z_0$, we can obtain $\mathcal{Q}(\|\nabla u(t)\|_2) \geq 0$ for $0 \leq t < \infty$. Thus we have

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}(g\circ\Delta u)(t) + \frac{1}{2}\left(1-\int_0^t g(\tau)d\tau\right)\|\Delta u\|_2^2 - \frac{1}{p}\|u\|_p^p \\ &\geq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{\eta}{2}\|\Delta u\|_2^2 + \frac{1}{2}(g\circ\Delta u)(t) + \mathcal{Q}(\|\nabla u(t)\|_2) \\ &\geq 0, \end{aligned}$$

and

$$\begin{aligned} E(t) &\leq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{2}(g\circ\Delta u)(t) + \frac{1}{2}\|\nabla u\|_2^2 \\ &\leq \frac{1}{2}B^2\|\nabla u_t\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\lambda_1^{-1}\|\Delta u\|_2^2 + \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{2}(g\circ\Delta u)(t). \end{aligned}$$

Let $C_1 = \frac{1}{2}(1+B^2)$ and $C_2 = \frac{1}{2}(1+\lambda_1^{-1})$, then we have (32). □

Lemma 2. *The energy $E(t)$ satisfies*

$$\frac{dE(t)}{dt} \leq -\|\nabla u_t(t)\|_2^2 - \frac{1}{2}\xi_2(g\circ\Delta u)(t) - \frac{1}{2}\left[g(0) - \xi_1\|g\|_{L^1(0,\infty)}\right]\|\Delta u(t)\|_2^2 \quad \forall \quad t\geq 0. \tag{33}$$

Proof. From (9), we have

$$\frac{dE(t)}{dt} \leq -\|\nabla u_t(t)\|_2^2 - \frac{\xi_2}{2}(g\circ\Delta u)(t) - \frac{1}{2}g(t)\|\Delta u(t)\|_2^2. \tag{34}$$

From assumptions $(G2)$ and since $\int_0^t g'(\tau)d\tau = g(t)-g(0)$, we obtain

$$\begin{aligned} -\frac{1}{2}g(t)\|\Delta u(t)\|_2^2 &= -\frac{1}{2}g(0)\|\Delta u(t)\|_2^2 - \frac{1}{2}\left(\int_0^t g'(\tau)d\tau\right)\|\Delta u(t)\|_2^2 \\ &\leq -\frac{1}{2}g(0)\|\Delta u(t)\|_2^2 + \frac{\xi_1}{2}\|g\|_{L^1(0,\infty)}\|\Delta u(t)\|_2^2 \\ &= -\frac{1}{2}\left[g(0) - \xi_1\|g\|_{L^1(0,\infty)}\right]\|\Delta u(t)\|_2^2. \end{aligned} \tag{35}$$

Then, Combining (34) and (35) our conclusion holds. Multiplying (33) by $e^{\kappa\zeta(t)}$ $(\kappa>0)$ and using (32), we have

$$\begin{aligned} \frac{d}{dt}\left(e^{\kappa\zeta(t)}E(t)\right) &\leq -\|\nabla u_t(t)\|_2^2 e^{\kappa\zeta(t)}E(t) - \frac{1}{2}\xi_2(g\circ\Delta u)(t)e^{\kappa\zeta(t)}E(t) \\ &\quad -\frac{1}{2}\left[g(0)-\xi_1\|g\|_{L^1(0,\infty)}\right]\|\Delta u(t)\|_2^2 e^{\kappa\zeta(t)}E(t) + \kappa\zeta_t(t)e^{\kappa\zeta(t)}E(t) \\ &\leq -\left[1-\kappa C_1\zeta_t(t)\right]\|\nabla u_t(t)\|_2^2 e^{\kappa\zeta(t)}E(t) - \frac{1}{2}\left[\xi_2 - \kappa\zeta_t(t)\right](g\circ\Delta u)(t)e^{\kappa\zeta(t)}E(t) \\ &\quad -\frac{1}{2}\left[g(0)-\xi_1\|g\|_{L^1} - 2C_2\kappa\zeta_t(t)\right]\|\Delta u(t)\|_2^2 e^{\kappa\zeta(t)}E(t). \end{aligned} \tag{36}$$

Using the fact that ζ_t is decreasing by (8), we conclude that

$$\begin{aligned} \frac{d}{dt}\left(e^{\kappa\zeta(t)}E(t)\right) &\leq -\left[1-\kappa C_1\zeta_t(0)\right]\|\nabla u_t(t)\|_2^2 e^{\kappa\zeta(t)}E(t) - \frac{1}{2}\left[\xi_2 - \kappa\zeta_t(0)\right](g\circ\Delta u)(t)e^{\kappa\zeta(t)}E(t) \\ &\quad -\frac{1}{2}\left[g(0)-\xi_1\|g\|_{L^1(0,\infty)} - 2C_2\kappa\zeta_t(0)\right]\|\Delta u(t)\|_2^2 e^{\kappa\zeta(t)}E(t). \end{aligned} \tag{37}$$

Choosing $\|g\|_{L^1(0,\infty)}$ sufficiently small so that

$$g(0) - \xi_1\|g\|_{L^1(0,\infty)} = K > 0$$

and defining

$$\kappa_0 = \min\left\{\frac{1}{C_1\zeta_t(0)}, \frac{\xi_2}{\zeta_t(0)}, \frac{K}{2C_2\zeta_t(0)}\right\},$$

we conclude by taking $\kappa\in(0,\kappa_0]$ in (37) that

$$\frac{d}{dt}\left(e^{\kappa\zeta(t)}E(t)\right) \leq 0, \quad t>0. \tag{38}$$

Integrating (38) over $(0, t)$, it follows that

$$E(t) \leq E(0)e^{-\kappa\zeta(t)}, \quad t > 0. \tag{39}$$

□

Example 1. For $\zeta(t) = t + \frac{t}{t+1}$, we can get the exponential decay rate $E(t) \leq E(0)e^{-\kappa t}, \quad \forall t \geq 0$. For $\zeta(t) = ln(1+t)$, we can get polynomial decay rate $E(t) \leq E(0)(1+t)^{-\kappa}, \quad \forall t \geq 0$.

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