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On the solutions of non-planar oscillations for a nonlinear coupled Kirchhoff beam equations with moving boundary

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Abstract: In this paper, we are concerned with the existence and uniqueness of global strong solution of non-planar oscillations for a nonlinear coupled Kirchhoff beam equations with moving boundary.

Keywords: Non-planar oscillations, Kirchhoff's beam, moving boundary.

MSC: 35A01, 35M11, 35E15.

1. Introduction

In this work, we focus on the existence and uniqueness of strong solution for a nonlinear coupled Kirchhoff beam equations with moving boundary given by

$$\begin{cases} u_{tt} + u_{xxxx} - M(u, v)u_{xx} + F(u) + u_t = 0, & \text{in } Q_t, \\ v_{tt} + v_{xxxx} - M(u, v)v_{xx} + F(v) + v_t = 0, & \text{in } Q_t, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & \forall \alpha(0) < x < \beta(0), \\ u_t(x, t) = u_1(x), \quad v_t(x, t) = v_1(x), & \forall \alpha(0) < x < \beta(0), \\ u(x, t) = u_x(x, t) = v(x, 0) = v_x(x, t) = 0, & \text{on } \Sigma_t, \end{cases} \quad (1)$$

where a and b are real constants, Q_t is the non-cylindrical domain of \mathbb{R}^2 defined by

$$Q_t = \{(x, t) \in \mathbb{R}^2 \mid \alpha(t) < x < \beta(t), 0 < t < T\}, \quad (2)$$

being $\alpha(\cdot)$ and $\beta(\cdot)$ functions of the class C^3 such that

$$\alpha(t) < \beta(t), \quad \forall 0 \leq t \leq T.$$

We denote by Σ_t the lateral boundary of Q_t given by

$$\Sigma_t = \bigcup_{0 < t < T} (\alpha(t) \times \{t\}) \cup (\beta(t) \times \{t\}).$$

The coupled system (1) describes non-planar oscillations of beams with a small thickness and variable limit length. Steady non-planar motions are characterized by each point on the beam centerline tracing an elliptical path perpendicular to the beam axis, then the functions $u(x, t)$ and $v(x, t)$ are the components of the displacement on a point of the beam in the direction of symmetry. The method used here consists in transform an initial boundary-value problem defined in a noncylindrical domain into another defined over a cylindrical domain whose sections are not time-dependent (see Dal Passo and Ughi [1]).

In the last decades, several types of equations have been used as some mathematical models that describe physical, chemical, biological, and engineering systems. Among them, mathematical models of vibrating and flexible structures have been considerably stimulated by an increasing number of questions of practical interest. In this sense, we stick to the study of a strong solution, for the proposed nonlinear model.

A mathematical model for transverse deflection of an extensible beam of length L , with ends attached at a certain fixed distance is given by the equation:

$$u_{tt} + \alpha u_{xxxx} + \left(\beta \int_0^L u_{\xi}^2(\xi, t) d\xi \right) (-u_{xx}) = 0.$$

For cylindrical domain, the stability of nonlinear oscillations of an elastic rod was studied by Haight and King [2]. Later, non-planar and nonlinear oscillations of a beam were considered by Ho *et al.*, [3]. Following the authors, the equations of motion describing the non-planar response of a beam of length L in $(0, L) \times (0, \infty)$ are given by

$$\begin{aligned} u_{tt} + u_{xxxx} - [a + b \int_0^{\pi} (u_x^2(x, t) + v_x^2(x, t)) dx] u_{xx} + u_t &= p(x, t), \\ v_{tt} + v_{xxxx} - [a + b \int_0^{\pi} (u_x^2(x, t) + v_x^2(x, t)) dx] v_{xx} + v_t &= q(x, t), \end{aligned}$$

where $p(x, t), q(x, t)$ are external forces. For non-planar oscillation of beams under periodic forcing, (see Fix and Kannan [4] and references therein).

Moving boundary problems occur in many physical applications like in heat transfer where a phase transition occurs, in moisture transport, such as swelling grains or polymers, and in deformable porous media problems where solid displacement is governed by diffusion. The noncylindrical domain is created by smooth perturbations of a fixed open set which are defined by diffeomorphism and was initially investigated by Lions [5] applying a method, called by himself the penalty method. From then on, problems in noncylindrical domain have been extensively studied by several authors.

To close this section, we mention some references on the partial differential equations in non-cylindrical domains in several contexts. Following these references, the reader will have an overview of the subject.

For the nonlinear vibrations of an elastic string, see for instance: Narasimha [6], Lions [7], Ho *et al.*, [8], Ebihara *et al.*, [9], Feireisl [10], Matsuyama and Ikehata [11], Cavalcanti [12]. Hyperbolic-parabolic equations with the nonlinearity of Kirchhoff-Carrier type in domain with moving boundary was studied Benabidallah and Ferreira [13]. A variational approach to evolution problems with variable domains was considered by Bonaccorsi and Guatteri [14]. Later, Límaco *et al.*, [15] proved the existence, uniqueness, and controllability for parabolic equations in non-cylindrical domains. Asymptotic behaviour for the nonlinear beam equation in the noncylindrical domain was analyzed by Benabidallah and Ferreira [16]. For asymptotic behaviour for wave equations with memory in a non-cylindrical domain see Ferreira and Santos [17]. The stability for a Kirchhoff beam equation with memory in noncylindrical domain was given by Ferreira *et al.*, [18]. The exponential decay for a Kirchhoff wave equations with a nonlocal condition in a non-cylindrical domain was analyzed by Ferreira *et al.*, [19]. Santos *et al.*, in [20] studied the existence and uniform decay for a nonlinear beam equation with nonlinearity of Kirchhoff type in domains with moving boundary. Later, Robalo *et al.*, in [21] considered a reaction diffusion model for a class of nonlinear parabolic equations with moving boundaries. We cite the work of Lopez *et al.*, [22] for remarks on a nonlinear wave equation in a noncylindrical domain. The dynamics of stochastic Boissonade system on the time-varying domain was considered by Zhang and Huang [23]. For Pullback attractors for non-autonomous reaction-diffusion equation in non-cylindrical domains, we mention Xiao [24]. Finally, we refer the recent manuscript of Almeida *et al.*, [25] where was applied the finite element scheme for a class of nonlocal parabolic systems with moving boundaries.

As far as we know, the strong solutions of non-planar oscillations for a nonlinear coupled Kirchhoff beam equations with moving boundary for system (1) were not considered previously. This paper consists of two sections in addition to the introduction. In Section 2, we recall some definitions of Sobolev spaces with their properties and present the technique to deal with our mobile boundary domain. In Section 3, we provide the existence and uniqueness of strong solutions.

2. Preliminaries

In this section we present the notation and the technique to transform the system (1) into another initial boundary-value problem defined over a cylindrical domain whose sections are not time-dependent.

For simplicity of notations hereafter we denote by $|\cdot|$ the Lebesgue Space $L^2(\Omega)$ norm and by $|\cdot|_2$ the Sobolev Space $H_0^2(\Omega)$ norm. Let B be a Banach space and $u : [0, T] \rightarrow B$ a measurable function. We denote

$$L^p(0, T; B) = \left\{ u : \left(\int_0^T |u(t)| dt \right)^{1/p} < \infty, \text{ if } 1 \leq p < \infty \right\},$$

$$L^\infty(0, T; B) = \left\{ u : \sup_{t \in (0, T)} |u(t)|_B < \infty, \text{ if } p = \infty \right\}.$$

To prove the existence of solution for system (1), the technic is transformed the non-cylindrical domain Q_t into an equivalent problem in a fixed rectangular domain $Q = (0, 1) \times (0, T)$, $T > 0$, using the inverse of the transformation, that is, by using the diffeomorphism

$$h(x, t) = (y, t) = \left(\frac{x - \alpha(t)}{\gamma(t)}, t \right) \text{ for } (x, t) \in Q_t, \text{ with } x = \alpha(t) + \gamma(t)y \tag{3}$$

where $\gamma(t) = \beta(t) - \alpha(t)$ and $h^{-1} : Q \rightarrow Q_t$ is defined by

$$h^{-1}(y, t) = (x, t) = (\alpha(t) + \gamma(t)y, t). \tag{4}$$

Then we have

$$\begin{cases} u(y, t) = (u \circ h^{-1})(y, t) = u(\alpha(t) + \gamma(t)y, t) = u(x, t), \\ v(y, t) = (v \circ h^{-1})(y, t) = v(\alpha(t) + \gamma(t)y, t) = v(x, t). \end{cases} \tag{5}$$

The change of variables

$$(x, t) \in Q_t \rightarrow (y, t) \in Q$$

transforms the equation (1) into the following problem in the cylindrical domain Q :

$$\begin{cases} u_{tt} + \frac{1}{\gamma^4} u_{yyyy} - \frac{1}{\gamma^2} M(u, v) u_{yy} + u_t + F(u) + a_1 u_{yy} + a_2 u_{ty} + a_3 u_y = 0, \\ v_{tt} + \frac{1}{\gamma^4} v_{yyyy} - \frac{1}{\gamma^2} M(u, v) v_{yy} + v_t + F(v) + a_1 v_{yy} + a_2 v_{ty} + a_3 v_y = 0, \end{cases} \tag{6}$$

$$u(0, t) = u(1, t) = v(0, t) = v(1, t), \tag{7}$$

$$u_y(0, t) = u_y(1, t) = v_y(0, t) = v_y(1, t), \tag{8}$$

$$u(y, 0) = u_0(y); \quad u_t(y, 0) = u_1(y), \tag{9}$$

$$v(y, 0) = v_0(y); \quad v_t(y, 0) = v_1(y), \tag{10}$$

where

$$a_1(y, t) = \frac{(\alpha'(t) + \gamma''(t)y)^2}{\gamma^2(t)}, \tag{11}$$

$$a_2(y, t) = -\frac{(\alpha'(t) + \gamma''(t)y)}{\gamma(t)}, \tag{12}$$

$$a_3(y, t) = \frac{(\alpha'(t) - \alpha''(t)) + (2\gamma'(t) - \gamma''(t)y)}{\gamma(t)}. \tag{13}$$

In order to simplify the calculation, we will take $a = 0$ and $b = 1$ in the definition of the function $M(u, v)$ that is,

$$M(u, v) = \frac{1}{\gamma(t)} \int_{\alpha(t)}^{\beta(t)} (u_y^2 + v_y^2) dy.$$

We denote by \widehat{M} the following function

$$\widehat{M}(\lambda) = \int_0^\lambda M(u(y, t), v(y, t)) \, dA. \tag{14}$$

To show existence of strong solution we will use the following assumptions

$$\alpha'(t) \leq 0, \quad \beta'(t) \geq 0, \tag{15}$$

$$\alpha, \gamma \in W^{2,\infty}(0, \infty), \tag{16}$$

$$\sup_{0 \leq t \leq \infty} \text{ess } \gamma(t) = \gamma_1 > 0, \tag{17}$$

$$\inf_{0 \leq t \leq \infty} \alpha'(t) = \alpha_0 > 0, \tag{18}$$

$$\sup_{0 \leq t \leq \infty} \text{ess } \gamma(t) = \gamma_1. \tag{19}$$

Observe that (17) implies that Q_t increases, that is, $I_{t_1} =]\alpha(t_1), \beta(t_1)[\subset I_{t_2}$ when ever $t_1 < t_2$. Concerning the functional M and F we assume that

$$t \mapsto M(u(y, t), v(y, t)) \text{ is in } W_{Loc}^{1,\infty}(\mathbb{R}), \forall t \in \mathbb{R}, \tag{20}$$

$$M(u, v) \geq m_0 > 0, \tag{21}$$

$$F \in W_{Loc}^{1,\infty}(\mathbb{R}), \text{ and } sF(s) \geq 0, \forall s \in \mathbb{R}. \tag{22}$$

Definition 1. The functions $u(x, t), v(x, t)$ defined in Q_t are said strong solutions for system (1), if

$$\begin{cases} (u, v) \in [L^\infty(0, \infty; H_0^2(I_t) \cap H^4(I_t))]^2, \\ (u_t, v_t) \in [L^2(\infty; L^2(I_t))]^2, \\ (u_{tt}, v_{tt}) \in [L^2(\infty; L^2(I_t))]^2, \end{cases} \tag{23}$$

and the Equation (1) is satisfied almost everywhere in the corresponding domain.

In the next section we prove the existence of strong solutions for system (1).

3. Existence of global strong solution

To prove the existence of strong solutions for system (1) in Q_t , first we prove the following result in the cylindrical domain Q :

Theorem 1. Let $(u_0, v_0) \in [H_0^2(0,1) \cap H^4(0,1)]^2, (u_1, v_1) \in [H_0^2(0,1)]^2$ and suppose that the assumptions (15)-(22) holds. Then there exists a unique strong solution (u, v) of the problem (6)-(10) satisfying the Equations (6) in the sense $L^\infty(0, T; L^2(0, 1))$.

Proof. Let A be the operator $Aw = w_{xxxx}$ and $D(A) = H^4(0,1) \cap H_0^2(0,1)$. Obviously, A is a positive self adjoint operator in the Hilbert Space $L^2(0,1)$ for which there exist sequences $\{w_n\}_{n \in \mathbb{N}}$ and $\{\lambda_n\}_{n \in \mathbb{N}}$ of eigenfunctions and eigenvalues of A such that the set of linear combinations of $\{w_n\}_{n \in \mathbb{N}}$ is dense in $D(A)$ and $\lambda_1 < \lambda_2 \leq \dots \leq \lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. Let us denote

$$\begin{aligned} u_{0m} &= \sum_{j=1}^m (u_0, w_j) w_j, & v_{0m} &= \sum_{j=1}^m (v_0, w_j) w_j, \\ u_{1m} &= \sum_{j=1}^m (u_1, w_j) w_j, & v_{1m} &= \sum_{j=1}^m (v_1, w_j) w_j. \end{aligned}$$

Note that for any $\{(u_0, u_1), (v_0, v_1)\} \in [D(A) \cap H_0^2(0, 1)]^2$, we have

$$\begin{aligned} (u_{0m}, u_{1m}) &\longrightarrow (u_0, u_1) \text{ strong in } D(A) \cap H_0^2(0, 1), \\ (v_{0m}, v_{1m}) &\longrightarrow (v_0, v_1) \text{ strong in } D(A) \cap H_0^2(0, 1). \end{aligned}$$

Let V_m be the space generated by w_1, w_2, \dots, w_m . Standard results on ordinary differential equation imply the existence of a local solution (u_m, v_m) of the form

$$u_m(t) = \sum_{j=1}^m g_{jm}(t)w_j, \quad v_m(t) = \sum_{j=1}^m h_{jm}(t)w_j$$

to the system

$$\begin{aligned} &(u_{tt}^m, w_j) + \left(\frac{1}{\gamma^4} u_{yyyy}^m, w_j\right) - \left(\frac{1}{\gamma^2} M(u^m, v^m) u_{yy}^m, w_j\right) \\ &+ (u_t^m, w_j) + (F(u^m), w_j) + (a_1 u_{yy}^m, w_j) + (a_2 u_{ty}^m, w_j) + (a_3 u_y^m, w_j) = 0, \end{aligned} \tag{24}$$

$$\begin{aligned} &(v_{tt}^m, w_j) + \left(\frac{1}{\gamma^4} v_{yyyy}^m, w_j\right) - \left(\frac{1}{\gamma^2} M(u^m, v^m) v_{yy}^m, w_j\right) \\ &+ (v_t^m, w_j) + (F(v^m), w_j) + (a_1 v_{yy}^m, w_j) + (a_2 v_{ty}^m, w_j) + (a_3 v_y^m, w_j) = 0, \end{aligned} \tag{25}$$

$$u^m(y, 0) = u_0^m(y), \quad u_t^m(y, 0) = u_1^m(0), \tag{26}$$

$$v^m(y, 0) = v_0^m(0), \quad v_t^m(y, 0) = v_1^m(0), \tag{27}$$

in $[0, t_m)$, with $0 < t_m < T$ for any arbitrary $T > 0$. The extension of the solution on the whole interval $[0, T]$ is a consequence of the priori estimates.

3.1. The first priori estimate

Let $w_j = u_t^m(t)$ in (24) and $w_j = v_t^m(t)$ in (25), we obtain the following identities

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} |u_t^m(t)|^2 + \frac{1}{2\gamma^4} \frac{d}{dt} |u_{tyy}^m(t)|^2 + \frac{1}{\gamma^2} M(u^m(t), v^m(t)) \frac{1}{2} \frac{d}{dt} |u_y^m(t)|^2 + |u_t^m(t)|^2 + (F(u^m(t)), u_t^m(t)) \\ &+ (a_1(t) u_y^m(t), u_{ty}^m(t)) + (a_2(t) u_{ty}^m(t), u_t^m(t)) + (a_3(t) u_y^m(t), u_t^m(t)) = 0, \end{aligned} \tag{28}$$

and

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} |v_t^m(t)|^2 + \frac{1}{2\gamma^4} \frac{d}{dt} |v_{tyy}^m(t)|^2 + \frac{1}{\gamma^2} M(u^m(t), v^m(t)) \frac{1}{2} \frac{d}{dt} |v_y^m(t)|^2 + |v_t^m(t)|^2 + (F(v^m(t)), v_t^m(t)) \\ &+ (a_1(t) v_y^m(t), v_{ty}^m(t)) + (a_2(t) v_{ty}^m(t), v_t^m(t)) + (a_3(t) v_y^m(t), v_t^m(t)) = 0. \end{aligned} \tag{29}$$

Note that

$$\begin{cases} (a_1(t) u_y^m(t), u_{ty}^m(t)) = \frac{1}{2} \frac{d}{dt} [a_1(t) |u_y^m(t)|^2] - \frac{1}{2} a_{1t}(t) |u_y^m(t)|^2, \\ (a_1(t) v_y^m(t), v_{ty}^m(t)) = \frac{1}{2} \frac{d}{dt} [a_1(t) |v_y^m(t)|^2] - \frac{1}{2} a_{1t}(t) |v_y^m(t)|^2, \\ (a_2(t) u_{ty}^m(t), u_t^m(t)) = (a_2(t) v_{ty}^m(t), v_t^m(t)) = 0. \end{cases} \tag{30}$$

Now, summing up the identities (28) and (29) and observing (30), we obtain

$$\begin{aligned}
 & \frac{d}{dt} \left\{ |u_t(t)|^2 + |v_t(t)|^2 + \frac{1}{\gamma^4} (|u_{yy}(t)|^2 + |v_{yy}(t)|^2) \right\} \\
 & + \frac{d}{dt} \left\{ \frac{1}{\gamma^2} \widehat{M} (|u_y(t)|^2 + |v_y(t)|^2) + a_1(t) (|u_y(t)|^2 + |v_y(t)|^2) \right\} + 2(|u_t(t)|^2 + |v_t(t)|^2) \\
 & = a'_1(t) (|u_y(t)|^2 + |v_y(t)|^2) - 2(F(u_t(t)), u_t(t)) - 2(F(v_t(t)), v_t(t)) \\
 & - 2(a_3(t)u_y(t), u_t(t)) - 2(a_3(t)v_y(t), v_t(t)),
 \end{aligned} \tag{31}$$

where the index m was omitted to facility the notation.

Using the hypothesis (22) in (31), we get

$$\begin{aligned}
 & \frac{d}{dt} \left\{ |u_t(t)|^2 + |v_t(t)|^2 + \frac{1}{\gamma^4} (|u_{yy}(t)|^2 + |v_{yy}(t)|^2) \right\} \\
 & + \frac{d}{dt} \left\{ \frac{1}{\gamma^2} \widehat{M} (|u_y(t)|^2 + |v_y(t)|^2) + a_1(t) (|u_y(t)|^2 + |v_y(t)|^2) \right\} + 2(|u_t(t)|^2 + |v_t(t)|^2) \\
 & \leq |a'_1(t)| (|u_y(t)|^2 + |v_y(t)|^2) + 2\alpha_0 |u_t(t)|^2 + 2\alpha_0 |v_t(t)|^2 \\
 & + 2|a_3(t)| |u_y(t)| |u_t(t)| + 2|a_3(t)| |v_y(t)| |v_t(t)|.
 \end{aligned} \tag{32}$$

Integrating (32) over $(0, t)$ where $t \geq t_m$, we have

$$\begin{aligned}
 & |u_t(t)|^2 + |v_t(t)|^2 + \frac{1}{\gamma^4} (|u_{yy}(t)|^2 + |v_{yy}(t)|^2) \\
 & + \frac{1}{\gamma^2} \widehat{M} (|u_y(t)|^2 + |v_y(t)|^2) + a_1(t) (|u_y(t)|^2 + |v_y(t)|^2) + 2 \int_0^t (|u_t(s)|^2 + |v_t(s)|^2) ds \\
 & \leq |u_1|^2 + |v_1|^2 + \frac{1}{\gamma^4} (|u_{0yy}|^2 + |v_{0yy}|^2) + \frac{1}{\gamma^2} \widehat{M} (|u_{0y}|^2 + |v_{0y}|^2) \\
 & + a_1(0) (|u_{0y}|^2 + |v_{0y}|^2) + \int_0^t |a'_1(s)| (|u_y(s)|^2 + |v_y(s)|^2) ds + 2\alpha_0 \int_0^t (|u_t(s)|^2 + |v_t(s)|^2) ds \\
 & + 2 \int_0^t |a_3(s)| |u_y(s)| |u_t(s)| ds + 2 \int_0^t |a_3(s)| |v_y(s)| |v_t(s)| ds.
 \end{aligned} \tag{33}$$

Taking the convergence on the initial conditions and the estimates

$$\begin{cases} 2|a_3(s)| |u_y(s)| |u_t(s)| \leq |a_3(s)|^2 |u_y(s)|^2 + |u_t(s)|^2, \\ 2|a_3(s)| |v_y(s)| |v_t(s)| \leq |a_3(s)|^2 |v_y(s)|^2 + |v_t(s)|^2, \end{cases} \tag{34}$$

we arrive at

$$\begin{aligned}
 & |u_t(t)|^2 + |v_t(t)|^2 + \frac{1}{\gamma^4} (|u_{yy}(t)|^2 + |v_{yy}(t)|^2) \\
 & + \frac{1}{\gamma^2} \widehat{M} (|u_y(t)|^2 + |v_y(t)|^2) + a_1(t) (|u_y(t)|^2 + |v_y(t)|^2) + \int_0^t (|u_t(s)|^2 + |v_t(s)|^2) ds \\
 & \leq k_0 + \int_0^t |a'_1(s)| (|u_y(s)|^2 + |v_y(s)|^2) ds \\
 & + 2\alpha_0 \int_0^t (|u_t(s)|^2 + |v_t(s)|^2) ds + \int_0^t |a_3(s)|^2 (|u_y(s)|^2 + |v_y(s)|^2) ds.
 \end{aligned} \tag{35}$$

Now, using the hypotheses (16)-(19), we conclude that there exist positive constants C_0 , C_1 and C_3 such that

$$a_1(t) \leq C_0; \quad |a'_1(t, y)| \leq C_1 \quad \text{and} \quad |a_3(t, y)| \leq C_3. \tag{36}$$

Using (36) in (35) we obtain

$$\begin{aligned} &|u_t(t)|^2 + |v_t(t)|^2 + \frac{1}{\gamma^4}(|u_{yy}(t)|^2 + |v_{yy}(t)|^2) \\ &+ \frac{1}{\gamma^2} \widehat{M}(|u_y(t)|^2 + |v_y(t)|^2) + C_0(|u_y(t)|^2 + |v_y(t)|^2) + \int_0^t (|u_t(s)|^2 + |v_t(s)|^2) ds \\ &\leq k_0 + 2\alpha_0 \int_0^t (|u_t(s)|^2 + |v_t(s)|^2) ds + C_4 \int_0^t (|u_y(s)|^2 + |v_y(s)|^2) ds, \end{aligned} \tag{37}$$

where $C_4 = C_1 + C_3$.

We define $k_1 = \min\{1, C_0\}$ and $k_2 = \max\{\frac{\alpha_0}{k_1} + \frac{C_4}{k_1}\}$ and set

$$X(t) = |u_t(s)|^2 + |v_t(s)|^2 + |u_y(s)|^2 + |v_y(s)|^2.$$

It follows from (37) that

$$X(t) \leq \frac{k_0}{k_1} + k_2 \int_0^t X(s) ds. \tag{38}$$

Employing Gronwall’s Lemma in (38) we obtain

$$\frac{k_0}{k_1} + k_2 \int_0^t X(s) ds \leq \frac{k_0}{k_1} e^{k_2 T}; \quad 0 < T < \infty, \tag{39}$$

therefore

$$\begin{aligned} &|u_t(t)|^2 + |v_t(t)|^2 + \frac{1}{\gamma^4}(|u_{yy}(t)|^2 + |v_{yy}(t)|^2) + \frac{1}{\gamma^2} \widehat{M}(|u_y(t)|^2 + |v_y(t)|^2) \\ &+ C_0(|u_y(t)|^2 + |v_y(t)|^2) + \int_0^t (|u_t(s)|^2 + |v_t(s)|^2) ds \leq \frac{k_0}{k_1} e^{k_2 T} \end{aligned} \tag{40}$$

for all $t \in (0, T)$, $0 < T < \infty$.

3.2. The second priori estimate

Let $w_j = -\sqrt{\lambda_j} u_t^m(t)$ in (24) and $w_j = -\sqrt{\lambda_j} v_t^m(t)$ in (25) we obtain the following identities

$$\begin{aligned} &-(u_{tt}(t), u_{tyy}(t)) - \frac{1}{\gamma^4} (u_{yyyy}(t), u_{tyy}(t)) + \frac{1}{\gamma^2} M(u(t), v(t)) (u_{yy}(t), u_{tyy}(t)) - (u_t(t), u_{tyy}(t)) \\ &- (F(u(t)), u_{tyy}(t)) - (a_1(t)u_{yy}(t), u_{tyy}(t)) - (a_2(t)u_{ty}(t), u_{tyy}(t)) - (a_3(t)u_y(t), u_{tyy}(t)) = 0, \end{aligned} \tag{41}$$

$$\begin{aligned} &-(v_{tt}(t), v_{tyy}(t)) - \frac{1}{\gamma^4} (v_{yyyy}(t), v_{tyy}(t)) + \frac{1}{\gamma^2} M(u(t), v(t)) (v_{yy}(t), v_{tyy}(t)) - (v_t(t), v_{tyy}(t)) \\ &- (F(v(t)), v_{tyy}(t)) - (a_1(t)v_{yy}(t), v_{tyy}(t)) - (a_2(t)v_{ty}(t), v_{tyy}(t)) - (a_3(t)v_y(t), v_{tyy}(t)) = 0. \end{aligned} \tag{42}$$

It’s easy to see that,

$$-(u_{tt}(t), u_{tyy}(t)) = \frac{1}{2} \frac{d}{dt} |u_{ty}(t)|^2, \tag{43}$$

$$-(u_{yyyy}(t), u_{tyy}(t)) = \frac{1}{2\gamma^2} \frac{d}{dt} |u_{yyy}(t)|^2, \tag{44}$$

$$-(u_t(t), u_{tyy}(t)) = |u_{ty}(t)|^2, \tag{45}$$

$$-(a_1(y, t)u_{yy}(t), u_{tyy}(t)) = -\frac{1}{2} \frac{d}{dt} [a_1(y, t)u_{yy}^2(t)] + \frac{1}{2} \int_0^1 a_1'(y, t)u_{yy}^2(t) dy, \tag{46}$$

$$-(a_3(y, t)u_y(t), u_{tyy}(t)) = \left(\frac{2\gamma'(t) - \gamma''(t)}{\gamma(t)} u_y(t), u_{ty}(t) \right) + (a_3(y, t)u_{yy}(t), u_{ty}(t)), \tag{47}$$

$$-(a_2(y, t)u_{ty}(t), u_{tyy}(t)) = -\frac{1}{2} \int_0^1 \frac{\gamma'(t)}{\gamma(t)} u_{ty}^2(t) dy, \tag{48}$$

$$\frac{1}{\gamma^2} M(u(t), v(t))(u_{yy}(t), u_{tyy}(t)) = \frac{1}{2\gamma^2} \frac{d}{dt} \left[M(u(t), v(t)) |u_{yy}(t)|^2 \right] - \frac{1}{2\gamma^2} \left[\frac{d}{dt} M(u(t), v(t)) \right] |u_{yy}(t)|^2 \tag{49}$$

and

$$-a'_1(y, t) = \frac{\alpha'(t) + y\gamma'(t)}{\gamma^3(t)} [2\gamma(t)(\alpha''(t) + \gamma''(t)) - 2\gamma'(t)(\alpha'(t) + \gamma'(t))]. \tag{50}$$

We omit the demonstration of equation (42) because it is identical to the previous demonstration.

Using the identities (43)-(50) and the hypothesis (20) in (41) and (42) we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |u_{ty}(t)|^2 + \frac{1}{2\gamma^4} \frac{d}{dt} |u_{yyy}(t)|^2 + |u_{ty}(t)|^2 + \frac{1}{2\gamma^2} \frac{d}{dt} \left[M(u(t), v(t)) |u_{yy}(t)|^2 \right] \\ & - \frac{1}{2\gamma^2} \left[\frac{d}{dt} M(u(t), v(t)) \right] |u_{yy}(t)|^2 - (F(u(t)), u_{tyy}) - \frac{1}{2} \frac{d}{dt} [a_1(y, t)u_{yy}^2(t)] \\ & + \frac{1}{2} \int_0^1 a'_1(y, t)u_{yy}^2(t) dy - \frac{1}{2} \int_0^1 \frac{\gamma'(t)}{\gamma(t)} u_{ty}^2(t) dy + \left(\frac{2\gamma'(t) - \gamma''(t)}{\gamma(t)} u_y(t), u_{ty}(t) \right) \\ & + (a_3(y, t)u_{yy}, u_{ty}) = 0, \end{aligned} \tag{51}$$

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |v_{ty}(t)|^2 + \frac{1}{2\gamma^4} \frac{d}{dt} |v_{yyy}(t)|^2 + |v_{ty}(t)|^2 + \frac{1}{2\gamma^2} \frac{d}{dt} \left[M(u(t), v(t)) |v_{yy}(t)|^2 \right] \\ & - \frac{1}{2\gamma^2} \left[\frac{d}{dt} M(u(t), v(t)) \right] |v_{yy}(t)|^2 - (F(v(t)), v_{tyy}) - \frac{1}{2} \frac{d}{dt} [a_1(y, t)v_{yy}^2(t)] \\ & + \frac{1}{2} \int_0^1 a'_1(y, t)v_{yy}^2(t) dy - \frac{1}{2} \int_0^1 \frac{\gamma'(t)}{\gamma(t)} v_{ty}^2(t) dy + \left(\frac{2\gamma'(t) - \gamma''(t)}{\gamma(t)} v_y(t), v_{ty}(t) \right) \\ & + (a_3(y, t)v_{yy}, v_{ty}) = 0. \end{aligned} \tag{52}$$

Summing up (51) with (52) and integrating the result over (0, t) we infer

$$\begin{aligned} & |u_{ty}(t)|^2 + |v_{ty}(t)|^2 + \frac{1}{\gamma^4} |u_{yyy}(t)|^2 + \frac{1}{\gamma^4} |v_{yyy}(t)|^2 + \frac{2}{\gamma^2} M(u(t), v(t)) |u_{yy}(t)|^2 \\ & + \frac{2}{\gamma^2} M(u(t), v(t)) |v_{yy}(t)|^2 + 2 \int_0^t [|u_{ty}(s)|^2 + |v_{ty}(s)|^2] ds \\ & = \int_0^1 a_1(y, t)(u_{yy}^2(t) + v_{yy}^2(t)) dy + \int_0^1 a_1(y, 0)(u_{0yy}^2(y) + v_{0yy}^2(y)) dy \\ & \times \int_0^t \frac{\gamma'(s)}{\gamma(s)} (u_{ty}^2(s) + v_{ty}^2(s)) ds - 2 \int_0^t (\gamma(s)u_y(s), u_{ty}(s)) ds \\ & - 2 \int_0^t (\gamma(s)v_y(s), v_{ty}(s)) ds + 2 \int_0^t (a_3(y, s)u_{yy}(s), u_{ty}(s)) ds \\ & + 2 \int_0^t (a_3(y, s)v_{yy}(s), v_{ty}(s)) ds + \frac{1}{\gamma^2} \int_0^t \left[\frac{d}{dt} M(u(s), v(s)) \right] \left(|u_{yy}(s)|^2 + |v_{yy}(s)|^2 \right) ds \\ & \times \int_0^t \int_0^1 a'_1(y, s) \left(u_{yy}^2(y, s) + v_{yy}^2(y, s) \right) dy ds + 2 \int_0^t \int_0^1 (F(u(y, s)), u_{tyy}(y, s)) dy ds \\ & \times \int_0^t \int_0^1 (F(v(y, s)), v_{tyy}(y, s)) dy ds. \end{aligned} \tag{53}$$

Next, we will analyse the terms on the right-hand side of (53). Using the hypothesis (16) we obtain

$$|a_1(y, t)| \leq C_{01}, \quad |a_1(y, 0)| \leq C_{02}, \quad |a'_1(y, t)| \leq C_{11}, \quad |a_2(y, t)| \leq C_{21}, \quad |a_3(y, t)| \leq C_{31}. \tag{54}$$

Estimate for $I_1 = \int_0^t (\gamma(s)u_y(s), u_{ty}(s)) ds$:

$$|I_1| \leq C_1 + \frac{C_\gamma}{2} \int_0^t |u_{ty}(s)|^2 ds. \tag{55}$$

Estimate for $I_2 = \int_0^t (\gamma(s)v_y(s), v_{ty}(s)) ds$:

$$|I_2| \leq C_1 + \frac{C_\gamma}{2} \int_0^t |v_{ty}(s)|^2 ds. \tag{56}$$

Estimate for $I_3 = \frac{1}{\gamma^2} \int_0^t \left[\frac{d}{dt} M(u(s), v(s)) \right] \left(|u_{yy}(s)|^2 + |v_{yy}(s)|^2 \right) ds$:

$$\begin{aligned} |I_3| &= \left| \frac{1}{\gamma^2} \int_0^t M'(\lambda(s)) \left[2(u_y(s), u_{ty}(s)) + 2(v_y(s), v_{ty}(s)) \right] \left[|u_{yy}(s)|^2 + |v_{yy}(s)|^2 \right] \right| \\ &\leq \frac{2}{\gamma^2} \int_0^t |M'(\lambda(s))| |(u_y(s), u_{ty}(s))| |u_{yy}(s)|^2 + |v_{yy}(s)|^2 ds \\ &\quad + \frac{2}{\gamma^2} \int_0^t |M'(\lambda(s))| |(v_y(s), v_{ty}(s))| |u_{yy}(s)|^2 + |v_{yy}(s)|^2 ds. \end{aligned}$$

By using (20) and (30), we conclude that $\{u_{yy}, v_{yy}\} \in L^\infty(0, T; L^2(0, 1))$ and $|M'(\lambda(s))|$ is bounded for $0 \leq s \leq C$, where $\lambda(s) = |u_y(t)|^2 + |v_y(t)|^2$ and C is a positive constant. Therefore,

$$\begin{aligned} |I_3| &\leq \frac{2C}{\gamma^2} \int_0^t |(u_y(s), u_{ty}(s))| ds + \frac{2C}{\gamma^2} \int_0^t |(v_y(s), v_{ty}(s))| ds \\ &\leq \frac{C}{\gamma^2} \int_0^t |u_y(s)|^2 ds + \frac{C}{\gamma^2} \int_0^t |u_{ty}(s)|^2 ds + \frac{C}{\gamma^2} \int_0^t |v_y(s)|^2 ds + \frac{C}{\gamma^2} \int_0^t |v_{ty}(s)|^2 ds. \end{aligned} \tag{57}$$

Now, from the hypothesis (20), we have

$$|F_\mu(u(y, t))| \leq C.$$

Estimate for $I_4 = \int_0^t (F(u(y, s)), u_{tyy}(y, s)) ds$: Note that

$$\int_0^t (F(u(y, s)), u_{tyy}(y, s)) ds = (F(u(y, t))u_{yy}) - \int_0^t (F_\mu(u(y, s))u_t(y, s), u_{yy}(y, s)) ds.$$

We have

$$\begin{aligned} |I_4| &\leq \left| (F(u(y, t))u_{yy}) \right| + \left| \int_0^t (F_\mu(u(y, s))u_t(y, s), u_{yy}(y, s)) ds \right| \\ &\leq \left| (F(u(y, t))u_{yy}) \right| + \int_0^t \int_0^1 C |u_t(y, s)| |u_{yy}(y, s)| dy ds \\ &\leq \left| (F(u(y, t))u_{yy}) \right| + \frac{C}{2} \int_0^t (|u_t(y, s)|^2 + |u_{yy}(y, s)|^2) ds. \end{aligned} \tag{58}$$

Proceeding in a similar way, we have the estimate for $I_5 = \int_0^t (F(v(y, s)), v_{tyy}(y, s)) ds$:

$$|I_5| \leq \left| (F(v(y, t))v_{yy}) \right| + \frac{C}{2} \int_0^t (|v_t(y, s)|^2 + |v_{yy}(y, s)|^2) ds. \tag{59}$$

Now, combining (55)-(59) in (53) we infer

$$\begin{aligned}
 & |u_{ty}(t)|^2 + |v_{ty}(t)|^2 + \frac{1}{\gamma^4}|u_{yyy}(t)|^2 + \frac{1}{\gamma^4}|v_{yyy}(t)|^2 + 2 \int_0^t [|u_{ty}(s)|^2 + |v_{ty}(s)|^2] ds \\
 & + \frac{2}{\gamma^2}M(u(t), v(t)) \left(|u_{yy}(t)|^2 + |v_{yy}(t)|^2 \right) \\
 & \leq C_{01}|u_{yy}(s)|^2 + C_{01}|v_{yy}(s)|^2 + C_{01}|u_{0yy}|^2 + C_{01}|v_{0yy}|^2 + \frac{\gamma'(t)}{\gamma(t)} \int_0^t \left(|u_{ty}(s)|^2 + |v_{ty}(s)|^2 \right) ds \\
 & + 2|\gamma(t)| \int_0^t |u_y(s)|^2 ds + 2|\gamma(t)| \int_0^t |u_{ty}(s)|^2 ds + 2|\gamma(t)| \int_0^t |v_y(s)|^2 ds + 2|\gamma(t)| \int_0^t |v_{ty}(s)|^2 ds \\
 & + C_{31} \int_0^t |u_{yy}(s)|^2 ds + C_{31} \int_0^t |u_{ty}(s)|^2 ds + 2C_{31} \int_0^t |v_{yy}(s)|^2 ds + 2C_{31} \int_0^t |v_{ty}(s)|^2 ds \\
 & + \frac{C}{\gamma^2} \int_0^t |u_y(s)|^2 ds + \frac{C}{\gamma^2} \int_0^t |u_{ty}(s)|^2 ds + \frac{C}{\gamma^2} \int_0^t |v_y(s)|^2 ds + \frac{C}{\gamma^2} \int_0^t |v_{ty}(s)|^2 ds \\
 & + C_{11} \int_0^t |u_{yy}(s)|^2 ds + |(F(u(y, t))u_{yy})| + \frac{C}{2} \int_0^t (|u_t(y, s)|^2 + |u_{yy}(y, s)|^2) ds \\
 & + |(F(v(y, t))v_{yy})| + \frac{C}{2} \int_0^t (|v_t(y, s)|^2 + |v_{yy}(y, s)|^2) ds.
 \end{aligned} \tag{60}$$

Using the first estimate, the hypothesis (16) and the convergence on the initial condition in (60), we obtain

$$\begin{aligned}
 & \mu_0 \left\{ |u_{ty}(t)|^2 + |v_{ty}(t)|^2 + |u_{yyy}(t)|^2 + |v_{yyy}(t)|^2 + |u_{yy}(t)|^2 + |v_{yy}(t)|^2 + \int_0^t [|u_{ty}(s)|^2 + |v_{ty}(s)|^2] ds \right\} \\
 & \leq 4C_{01}C + \frac{\gamma'(t)}{\gamma(t)} \int_0^t \left(|u_{ty}(s)|^2 + |v_{ty}(s)|^2 \right) ds + 6\gamma(t)C^2T + 2\gamma(t) \int_0^t \left(|u_{ty}(s)|^2 + |v_{ty}(s)|^2 \right) ds \\
 & + C_{31} \int_0^t |u_{yy}(s)|^2 ds + C_{31} \int_0^t |u_{ty}(s)|^2 ds + 2C_{31} \int_0^t |v_{yy}(s)|^2 ds + 2C_{31} \int_0^t |v_{ty}(s)|^2 ds + 2\frac{C}{\gamma^2}C^2T \\
 & + \frac{C}{\gamma^2} \int_0^t |u_{ty}(s)|^2 ds + \frac{C}{\gamma^2} \int_0^t |v_{ty}(s)|^2 ds + C_{11} \int_0^t \left(|u_{yy}(s)|^2 + |v_{yy}(s)|^2 \right) ds + CC^2T \\
 & + \frac{C}{2} \int_0^t \left(|u_{yy}(s)|^2 + |v_{yy}(s)|^2 \right) ds + |(F(u(y, t))u_{yy})| + |(F(v(y, t))v_{yy})|,
 \end{aligned} \tag{61}$$

where $\mu_0 = \{1, \frac{1}{2\gamma^4}, \frac{2}{\gamma^2}, \frac{m_0}{\gamma^2}\}$.

Again, using the hypothesis (16) in (60) and making

$$G(t) = |u_{ty}(t)|^2 + |v_{ty}(t)|^2 + |u_{yyy}(t)|^2 + |v_{yyy}(t)|^2 + |u_{yy}(t)|^2 + |v_{yy}(t)|^2,$$

we have

$$\mu_0G(t) + \mu_0 \int_0^t [|u_{ty}(s)|^2 + |v_{ty}(s)|^2] ds \leq C + k_1 \int_0^t [|u_{ty}(s)|^2 + |v_{ty}(s)|^2] ds + k_2 \int_0^t [|u_{yy}(s)|^2 + |v_{yy}(s)|^2] ds, \tag{62}$$

where $k_1 = \left(\frac{\gamma'}{\gamma} + 2\gamma + C_{31} + 2C_{31} + \frac{C}{\gamma^2} + \frac{C}{\gamma^2}\right)$ and $k_2 = \left(C_{31} + 2C_{31} + C_{11} + \frac{C}{2}\right)$. Now we get

$$\begin{aligned}
 G(t) + \int_0^t [|u_{ty}(s)|^2 + |v_{ty}(s)|^2] ds & \leq k_0 + k_0 \int_0^t \left(|u_{ty}(s)|^2 + |v_{ty}(s)|^2 + |u_{yy}(s)|^2 + |v_{yy}(s)|^2 \right) ds \\
 & \leq k_0 + k_0 \int_0^t G(s) ds.
 \end{aligned} \tag{63}$$

Employing Gronwall’s Lemma, from (63) we obtain the second estimate

$$|u_{ty}(t)|^2 + |v_{ty}(t)|^2 + |u_{yyy}(t)|^2 + |v_{yyy}(t)|^2 + |u_{yy}(t)|^2 + |v_{yy}(t)|^2 \leq k_0 e^{k_0 T} \quad \forall t \in [0, T]. \tag{64}$$

3.3. Passage to the Limit

From the estimates (40) and (64), imply that there exists a subsequence of (u^m, v^m) , which we still denoted by (u^m, v^m) and function $(u(t), v(t))$ such that

$$(u^m, v^m) \overset{*}{\rightharpoonup} (u, v) \text{ weak star in } [L^\infty(0, T; H_0^2(0, 1) \cap H^4(0, 1))]^2, \tag{65}$$

$$(u_y^m, v_y^m) \overset{*}{\rightharpoonup} (u_y, v_y) \text{ weak star in } [L^\infty(0, T; H_0^1(0, 1))]^2, \tag{66}$$

$$(u_t^m, v_t^m) \overset{*}{\rightharpoonup} (u_t, v_t) \text{ weak star in } [L^\infty(0, T; L^2(0, 1))]^2, \tag{67}$$

$$(u_{ty}^m, v_{ty}^m) \overset{*}{\rightharpoonup} (u_{ty}, v_{ty}) \text{ weak star in } [L^\infty(0, T; L^2(0, 1))]^2, \tag{68}$$

$$(u_{yyy}^m, v_{yyy}^m) \overset{*}{\rightharpoonup} (u_{yyy}, v_{yyy}) \text{ weak star in } [L^\infty(0, T; L^2(0, 1))]^2, \tag{69}$$

$$(u_{yy}^m, v_{yy}^m) \overset{*}{\rightharpoonup} (u_{yy}, v_{yy}) \text{ weak star in } [L^\infty(0, T; L^2(0, 1))]^2. \tag{70}$$

Analysis of the nonlinear terms

Since $M(u(x, y), v(x, t))$ is in $W_{Loc}^{1,\infty}(\mathbb{R})$ as a function of t and (u_y^m, v_y^m) is bounded in $L^\infty(0, T; H_0^1((0, 1)))$, we obtain

$$M(u^m(t), v^m(t)) \longrightarrow M(u(x, y), v(x, t)), \tag{71}$$

and

$$\begin{cases} M(u^m(t), v^m(t)) (u_{yy}^m, w_j) \longrightarrow M(u(x, y), v(x, t)) (u_{yy}, w_j), \\ M(u^m(t), v^m(t)) (v_{yy}^m, w_j) \longrightarrow M(u(x, y), v(x, t)) (v_{yy}, w_j). \end{cases} \tag{72}$$

On the other hand, we observe that $F \in W_{Loc}^{1,\infty}(\mathbb{R})$ and (u_y^m, v_y^m) is bounded in $L^\infty(0, T; H_0^1((0, 1)) \hookrightarrow L^2(0, T; L^2(0, 1)))$, therefore

$$(F(u^m(t), F(v^m(t))) \text{ is bounded in } L^2(0, T; L^2(0, 1)). \tag{73}$$

Then, by compactness arguments, it follow that

$$\begin{cases} F(u^m(t) \longrightarrow F(u), & \text{a.e. in } Q, \\ F(v^m(t) \longrightarrow F(v), & \text{a.e. in } Q, \text{ as } m \rightarrow \infty. \end{cases} \tag{74}$$

From (73)-(74) and Lions [5], Chapter 1, Lemma 1.3, we conclude that

$$\begin{cases} F(u^m(t) \longrightarrow F(u) \text{ weakly in } L^2(0, T; L^2(0, 1)), \\ F(v^m(t) \longrightarrow F(v) \text{ weakly in } L^2(0, T; L^2(0, 1)). \end{cases} \tag{75}$$

The convergence (65)-(70), (72) and (75) are sufficient to pass to limit in the approximated system (24)-(27) in order to obtain

$$\begin{cases} u_{tt}^m + \frac{1}{\gamma^4} u_{yyy}^m - \frac{1}{\gamma^2} M(u, v) u_{yy}^m + u_t^m + F(u^m) + a_1 u_{yy}^m + a_2 u_{ty}^m + a_3 u_y^m = 0, \\ v_{tt}^m + \frac{1}{\gamma^4} v_{yyy}^m - \frac{1}{\gamma^2} M(u, v) v_{yy}^m + v_t^m + F(v^m) + a_1 v_{yy}^m + a_2 v_{ty}^m + a_3 v_y^m = 0, \end{cases} \tag{76}$$

in $L^2(0, T; L^2(0, 1))$. The uniqueness follows by using standard methods. To verify the initial conditions we use the usual argument, as in Lions [5]. \square

Now we present the principal result of this work.

Theorem 2. *Let us take $(u_0, v_0) \in (H_0^2(I_0) \cap H^4(I_0))^2$, $(u_1, v_1) \in (H_0^2(I_0))^2$ and suppose that the assumptions (15)-(22) holds. Then there exists a unique strong solution (u, v) of the initial boundary value problem (1) in the sense $L^2(0, \infty; L^2(I_t))$, satisfying (23).*

Proof. To show existence in non-cylindrical domain we return to our original problem by using the change variable given in (3). Let (u, v) the solution obtained from Theorem 1 and (u, v) defined by (5), then (u, v) belongs to the class

$$\begin{cases} (u, v) \in [L^\infty(0, \infty; H_0^2(I_t) \cap H^4(I_t))]^2, \\ (u_t, v_t) \in [L^2(\infty; L^2(I_t))]^2, \\ (u_{tt}, v_{tt}) \in [L^2(\infty; L^2(I_t))]^2, \end{cases}$$

where $I_t =]\alpha(t), \beta(t)[$ for any $t \geq 0$. Then from (15)-(17), it is easy to see that (u, v) satisfies first and second Equations in (1) in the sense $L^2(0, \infty; L^2(I_t))$. The uniqueness follows from the uniqueness of Theorem 1. \square

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