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Article

Limit cycles of a planar differential system via averaging theory

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Abstract: In this article, we consider the limit cycles of a class of planar polynomial differential systems of the form

$$\dot{x} = -y + \varepsilon(1 + \sin^n \theta)xP(x, y)$$
$$\dot{y} = x + \varepsilon(1 + \cos^m \theta)yQ(x, y),$$

where $P(x, y)$ and $Q(x, y)$ are polynomials of degree n_1 and n_2 respectively and ε is a small parameter. We obtain the maximum number of limit cycles that bifurcate from the periodic orbits of a linear center $\dot{x} = -y, \dot{y} = x$, by using the averaging theory of first order.

Keywords: Mathieu-Duffing type; Averaging theory; Planar differential system; Limit cycle.

MSC: 34C07; 34C05; 34C40.

1. Introduction

One of the main problems in the theory of ordinary differential equations is the study of their limit cycles, their existence, their number and their stability, these properties of limit cycles were studied extensively by mathematicians and physicists, and more recently also by chemists, biologists, economists, etc. A limit cycle of a differential equation is a periodic orbit in the set of all isolated periodic orbits of the differential equation.

The second part of the 16th Hilbert's problem [1] is related to the least upper bound on the number of limit cycles of polynomial vector fields having a fixed degree. The study of differential equations or planar differential systems has been considered by several authors. In [2] the authors studied the limit cycles for a variant of a generalized Riccati equation. Mathieu, in [3] considered the second order differential equation

$$\ddot{x} + b(1 + \cos t)x = 0,$$

where b is a real constant. It is called Mathieu equation, which is the simplest mathematical model of an excited system on a parameter. We also recall the Ermakov-Pinney equation which is the Mathieu-Duffing type equation

$$\ddot{x} + b(1 + \cos t)x - x^{\beta} = 0,$$

where β is an integer and $b > 0$. The last two equations modeled the dynamics of a system with harmonic parametric excitation and a nonlinear term corresponding to a restoring force, see [4,5].

There are several methods exist to study the number of limit cycles that bifurcate from the periodic orbits such as the integrating factor, the abelian integral method, the Poincaré-Melnikov integral method, Poincaré return map and averaging theory. The study of limit cycles for differential equations or planar differential systems by applying the averaging method has been considered by several authors see for instance [6–8].

In [9], the authors studied the limit cycles of the second-order differential equations

$$\ddot{x} + \varepsilon(1 + \cos^m \theta)Q(x, y) + x = 0,$$

where $Q(x,y)$ is an arbitrary polynomial of degree n, and for each integer non-negative m.

In this paper, our goal is to study the maximum number of limit cycles of a differential planar system bifurcating from the periodic orbits of the linear center

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x, \end{cases}$$

given by

$$\begin{cases} \dot{x} = -y + \varepsilon(1 + \sin^n \theta)xP(x,y) \\ \dot{y} = x + \varepsilon(1 + \cos^m \theta)yQ(x,y), \end{cases} \tag{1}$$

where $|\varepsilon| > 0$ is a small parameter, m, n are non-negative integers, $P(x,y)$ and $Q(x,y)$ are polynomials of degree n_1 and n_2 respectively. Our main result is the following theorem .

Theorem 1. *For all polynomials $P(x,y)$ and $Q(x,y)$ have degree n_1 and n_2 respectively, n and m are non-negative integers, then for $|\varepsilon| > 0$ sufficiently small, the maximum number of limit cycles of the differential systems (1) bifurcating from the periodic orbits of the linear center $\dot{x} = -y, \dot{y} = x$ using averaging theory of first order*

(1) If m odd and n odd

$$\max\{n_1, n_2\},$$

(2) If m even and n even

$$\max\left\{\left[\frac{n_1}{2}\right], \left[\frac{n_2}{2}\right]\right\},$$

(3) If m odd and n even

$$\max\left\{n_2, n_2 + \left[\frac{n_1}{2}\right] - \left[\frac{n_2}{2}\right]\right\},$$

(4) If m even and n odd

$$\max\left\{n_1, n_1 + \left[\frac{n_2}{2}\right] - \left[\frac{n_1}{2}\right]\right\},$$

where $[.]$ denotes the integer part function.

The statements of Theorem 1 is proved in §3. In §2 we recall the averaging theory of first order.

2. The averaging theory of first order

The averaging theory of first for studying periodic orbits was developed in [10,11].

Theorem 2. *We Consider the differential system*

$$\dot{x}(t) = \varepsilon H(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{2}$$

where $H : \mathbb{R} \times D \to \mathbb{R}^n, R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. We define $h : D \to \mathbb{R}^n$ as

$$h(z) = \frac{1}{T}\int_0^T H(s,z)ds, \tag{3}$$

and assume that

(i) H and R are locally Lipschitz with respect to x,

(ii) For $a \in D$ with $h(a) = 0$, there exists a neighborhood V of a such that $h(z) \neq 0$ for all $z \in \bar{V} \setminus \{a\}$ and $d_B(h, V, 0) \neq 0$.

Then for $|\varepsilon| > 0$ sufficiently small there exists an isolated T-periodic solution $\varphi(\cdot,\varepsilon)$ of system (2) such that $\varphi(\cdot,\varepsilon) \to a$ as $\varepsilon \to 0$.

Here we will need some facts from the proof of Theorem 2. Hypothesis (i) assures the existence and uniqueness of the solution of each initial value problem on the interval $[0, T]$. Hence, for each $z \in D$, it is possible to denote by $x(\cdot, z, \varepsilon)$ the solution of (2) with the initial value $x(0, z, \varepsilon) = z$.

We consider also the function $\zeta : D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ defined by

$$\zeta(z, \varepsilon) = \int_0^T \left[\varepsilon H(t, x(t, z, \varepsilon)) + \varepsilon^2 R(t, x(t, z, \varepsilon), \varepsilon \right] dt. \tag{4}$$

From (2) it follows for every $z \in D$ that

$$\zeta(z, \varepsilon) = x(T, z, \varepsilon) - x(0, z, \varepsilon). \tag{5}$$

The function ζ can be written in the form

$$\zeta(z, \varepsilon) = \varepsilon h(z) + O(\varepsilon^2), \tag{6}$$

where h is given by (3), then for $|\varepsilon| > 0$ sufficiently small satisfies that $z_\varepsilon = x(0, \varepsilon)$ tends to be an isolated zero of $\zeta(., \varepsilon)$ when $\varepsilon \to 0$. Of course, due to (5) the function ζ is a displacement function for system (2), and its fixed points are initial conditions for the T-periodic solution of system (2).

For additional information on the averaging theory, see the books [12,13].

Theorem 3 (Discartes Theorem). *Consider the real polynomial $p(x) = a_{i_1} x^{i_1} + a_{i_2} x^{i_2} + ... + a_{i_k} x^{i_k}$ with $0 \le i_1 < i_2 < ... < i_k$ and $a_{i_j} \neq 0$ real constants for $j \in \{1, 2, ..., k\}$. When $a_{i_j} a_{i_{j+1}} < 0$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of sign. If the number of variations of signs is m, then $p(x)$ has at most m positive real roots. Moreover, it is always possible to choose the coefficients of $p(x)$ in such a way that $p(x)$ has exactly $k - 1$ positive real roots.*

3. Proof Theorem 1

We need the first order averaging theory in to prove of Theorem 1. In order to apply first order averaging method we write system (1), in polar coordinates (r, θ) where $x = r\cos(\theta), y = rsin(\theta)$ $r > 0$. If we take

$$\begin{cases} P(x, y) = \sum\limits_{i+j=0}^{n_1} a_{ij} x^i y^j, \\ Q(x, y) = \sum\limits_{i+j=0}^{n_2} b_{ij} x^i y^j, \end{cases}$$

system (1) can be written as follows

$$\begin{cases} \dot{r} = \varepsilon \left(\sum\limits_{i+j=0}^{n_1} a_{ij} \left(\cos^{i+2}\theta \sin^j\theta + \cos^{i+2}\theta \sin^{j+n}\theta \right) r^{i+j+1} \right. \\ \qquad \left. + \sum\limits_{i+j=0}^{n_2} b_{ij} \left(\cos^i\theta \sin^{j+2}\theta + \cos^{i+m}\theta \sin^{j+2}\theta \right) r^{i+j+1} \right), \\ \dot{\theta} = 1 + \varepsilon \left(\sum\limits_{i+j=0}^{n_2} b_{ij} \left(\cos^{i+1}\theta \sin^{j+1}\theta + \cos^{i+m+1}\theta \sin^{j+1}\theta \right) r^{i+j} \right. \\ \qquad \left. - \sum\limits_{i+j=0}^{n_1} a_{ij} \left(\cos^{i+1}\theta \sin^{j+1}\theta + \cos^{i+1}\theta \sin^{j+n+1}\theta \right) r^{i+j} \right). \end{cases}$$

Taking θ as the new independent variable system, (1) can be written as

$$\begin{aligned} \frac{dr}{d\theta} =& \varepsilon \left(\sum_{i+j=0}^{n_1} a_{ij} \left(\cos^{i+2}\theta \sin^j\theta + \cos^{i+2}\theta \sin^{j+n}\theta \right) r^{i+j+1} \right. \\ & \left. + \sum_{i+j=0}^{n_2} b_{ij} \left(\cos^i\theta \sin^{j+2}\theta + \cos^{i+m}\theta \sin^{j+2}\theta \right) r^{i+j+1} \right) + O(\varepsilon^2) \end{aligned}$$

$$=\varepsilon F(r,\theta)+O(\varepsilon^2),$$

where

$$\begin{aligned}
F(r,\theta) &= F_1(r,\theta)+F_2(r,\theta),\\
F_1(r,\theta) &= \sum_{i+j=0}^{n_1} a_{ij}\left(\cos^{i+2}\theta\sin^j\theta+\cos^{i+2}\theta\sin^{j+n}\theta\right)r^{i+j+1},\\
F_2(r,\theta) &= \sum_{i+j=0}^{n_2} b_{ij}\left(\cos^{i}\theta\sin^{j+2}\theta+\cos^{i+m}\theta\sin^{j+2}\theta\right)r^{i+j+1}.
\end{aligned}$$

Let F_{10} be the averaging equation of first order associated with system (1), using the notation introduced in Theorem 2 we compute F_{10} by integrating F_1 with respect to θ,

$$F_{10}(r)=\frac{1}{2\pi}\int_0^{2\pi}F(r,\theta)d\theta=\frac{1}{2\pi}\int_0^{2\pi}F_1(r,\theta)d\theta+\frac{1}{2\pi}\int_0^{2\pi}F_2(r,\theta)d\theta. \tag{7}$$

In order to calculate the exact expression of F_{10} we use the following formulas

$$\int_0^{2\pi}\sin^p\theta\cos^{2q}\theta d\theta = \frac{(2q-1)!!}{(2q+p)(2q+p-2)..(p+2)}\int_0^{2\pi}\sin^p\theta d\theta,$$

$$\int_0^{2\pi}\cos^p\theta\sin^{2q}\theta d\theta = \frac{(2q-1)!!}{(2q+p)(2q+p-2)..(p+2)}\int_0^{2\pi}\cos^p\theta d\theta.$$

These formulas are applicable for arbitrary real p and arbitrary positive integer q, except for the following negative even integers $p=-2,-4,...,-2n$.

If p is a natural number and $q=0$ we have

$$\begin{aligned}
&\int_0^{2\pi}\sin^{2l}\theta d\theta=\frac{(2l-1)!!}{2^l l!}2\pi,\\
&\int_0^{2\pi}\sin^{2l+1}\theta d\theta=0,\\
&\int_0^{2\pi}\cos^{2l}\theta d\theta=\frac{(2l-1)!!}{2^l l!}2\pi,\\
&\int_0^{2\pi}\cos^{2l+1}\theta d\theta=0.
\end{aligned}$$

We have also

$$\begin{aligned}
\int_0^{2\pi}\sin^p\theta\cos^{2q+1}\theta d\theta &= 0,\\
\int_0^{2\pi}\cos^p\theta\sin^{2q+1}\theta d\theta &= 0.
\end{aligned}$$

These last formulas are applicable for arbitrary real p and non-negative integer q, except the following negative odd integers $p=-1,-3,...,-(2n+1)$. For more details of these integrals and other, see [14].

Now we determine $\frac{1}{2\pi}\int_0^{2\pi}F_1(r,\theta)d\theta$, in the following cases

(1) If n odd and n_1 even

$$\begin{aligned}
f_1(r) &= \frac{1}{2\pi}\int_0^{2\pi}F_1(r,\theta)d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{i+j=0}^{n_1}\left[a_{i,j}\left(\sin^j\theta+\sin^{j+n}\theta\right)\cos^{i+2}\theta\right]r^{i+j+1}d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{2q+j=2}^{n_1+2}\left[a_{2q-2,j}(\sin^j\theta+\sin^{j+n}\theta)\cos^{2q}\theta\right]r^{2q+j-1}d\theta\\
&= \frac{1}{2\pi}\left[\sum_{2q+2l+1=3}^{n_1+1}a_{2q-2,2l+1}r^{2q+2l}\int_0^{2\pi}\sin^{2l+n+1}\theta\cos^{2q}\theta d\theta\right.\\
&\quad\left.+\sum_{2l+2q=2}^{n_1+2}a_{2q-2,2l}r^{2q+2l-1}\int_0^{2\pi}\sin^{2l}\theta\cos^{2q}\theta d\theta\right]\\
&= \sum_{l+q=1}^{n_1/2}a_{2q-2,2l+1}r^{2l+2q}\frac{(2q-1)!!}{(2q+2l+1)(2q+2l-1)...(2l+3)}\frac{(2l+n)!!}{2^{\frac{2l+n+1}{2}}(\frac{2l+n+1}{2})!}\\
&\quad+\sum_{l+q=1}^{(n_1+2)/2}a_{2q-2,2l}r^{2l+2q-1}\frac{(2q-1)!!}{(2q+2l)(2q+2l-2)...(2l+2)}\frac{(2l-1)!!}{2^l l!}\\
&= \sum_{l+q=1}^{n_1/2}a_{2q-2,2l+1}r^{2q+2l}\frac{(2l+n)!!(2q-1)!!}{2^{\frac{2l+n+1}{2}}(\frac{2l+n+1}{2})!(2q+2l+1)(2q+2l-1)...(2l+3)}\\
&\quad+\sum_{l+q=1}^{(n_1+2)/2}a_{2q-2,2l}r^{2q+2l-1}\frac{(2q-1)!!(2l-1)!!}{2^{l+q}l!(q+l)(q+l-1)...(l+1)}\\
&= \sum_{k=1}^{n_1+1}A_k r^k.
\end{aligned}
$$

(2) If n odd and n_1 odd

$$
\begin{aligned}
f_2(r) &= \frac{1}{2\pi}\int_0^{2\pi}F_1(r,\theta)d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{i+j=0}^{n_1}\left[a_{i,j}(\sin^j\theta+\sin^{j+n}\theta)\cos^{i+2}\theta\right]r^{i+j+1}d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{2q+j=2}^{n_1+2}\left[a_{2q-2,j}(\sin^j\theta+\sin^{j+n}\theta)\cos^{2q}\theta\right]r^{2q+j-1}d\theta\\
&= \frac{1}{2\pi}\left[\sum_{2q+2l+1=3}^{n_1+2}a_{2q-2,2l+1}r^{2q+2l}\int_0^{2\pi}\sin^{2l+n+1}\theta\cos^{2q}\theta d\theta\right.\\
&\quad\left.+\sum_{2l+2q=2}^{n_1+1}a_{2q-2,2l}r^{2q+2l-1}\int_0^{2\pi}\sin^{2l}\theta\cos^{2q}\theta d\theta\right]\\
&= \sum_{l+q=1}^{(n_1+1)/2}a_{2q-2,2l+1}r^{2q+2l}\frac{(2l+n)!!(2q-1)!!}{2^{\frac{2l+n+1}{2}}(\frac{2l+n+1}{2})!(2q+2l+1)(2q+2l-1)...(2l+3)}\\
&\quad+\sum_{l+q=1}^{(n_1+1)/2}a_{2q-2,2l}r^{2q+2l-1}\frac{(2q-1)!!(2l-1)!!}{2^{l+q}l!(q+l)(q+l-1)...(l+1)}\\
&= \sum_{k=1}^{n_1+1}\tilde{A}_k r^k.
\end{aligned}
$$

(3) If n even and n_1 even

$$
\begin{aligned}
f_3(r) &= \frac{1}{2\pi}\int_0^{2\pi}F_1(r,\theta)d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{i+j=0}^{n_1}[a_{i,j}(\sin^j\theta+\sin^{j+n}\theta)\cos^{i+2}\theta]r^{i+j+1}d\theta
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{2q+j=2}^{n_1+2}[a_{2q-2,j}(\sin^j\theta+\sin^{j+n}\theta)\cos^{2q}\theta]r^{2q+j-1}d\theta\\
&= \frac{1}{2\pi}\left[\sum_{2l+2q=2}^{n_1+2}a_{2q-2,2l}r^{2q+2l-1}\int_0^{2\pi}(\sin^{2l}\theta+\sin^{2l+n}\theta)\cos^{2q}\theta d\theta\right]\\
&= \sum_{l+q=1}^{(n_1+2)/2}a_{2q-2,2l}r^{2q+2l-1}\frac{(2q-1)!!}{2^q(q+l)(q+l-1)...(l+1)}\left[\frac{(2l-1)!!}{2^l l!}+\frac{(2l+n-1)!!}{2^{\frac{2l+n}{2}}(\frac{2l+n}{2})!}\right]\\
&= \sum_{\substack{k=1\\ k\ odd}}^{n_1+1}\bar{A}_k r^k.
\end{aligned}
$$

(4) If n even and n_1 odd

$$
\begin{aligned}
f_4(r) &= \frac{1}{2\pi}\int_0^{2\pi}F_1(r,\theta)d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{i+j=0}^{n_1}[a_{i,j}(\sin^j\theta+\sin^{j+n}\theta)\cos^{i+2}\theta]r^{i+j+1}d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{2q+j=2}^{n_1+2}[a_{2q-2,j}(\sin^j\theta+\sin^{j+n}\theta)\cos^{2q}\theta]r^{2q+j-1}d\theta\\
&= \frac{1}{2\pi}\left[\sum_{2l+2q=2}^{n_1+1}a_{2q-2,2l}r^{2q+2l-1}\int_0^{2\pi}(\sin^{2l}\theta+\sin^{2l+n}\theta)\cos^{2q}\theta d\theta\right]\\
&= \sum_{l+q=1}^{(n_1+1)/2}a_{2q-2,2l}r^{2q+2l-1}\frac{(2q-1)!!}{2^q(q+l)(q+l-1)...(l+1)}\left[\frac{(2l-1)!!}{2^l l!}+\frac{(2l+n-1)!!}{2^{\frac{2l+n}{2}}(\frac{2l+n}{2})!}\right]\\
&= \sum_{\substack{k=1\\ k\ impair}}^{n_1}\bar{A}_k r^k.
\end{aligned}
$$

And we determine $\frac{1}{2\pi}\int_0^{2\pi}F_2(r,\theta)d\theta$ in the following cases

(5) If m odd and n_2 even

$$
\begin{aligned}
f_5(r) &= \frac{1}{2\pi}\int_0^{2\pi}F_2(r,\theta)d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{i+j=0}^{n_2}\left[b_{i,j}(\cos^i\theta+\cos^{i+m}\theta)\sin^{j+2}\theta\right]r^{i+j+1}d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{i+2q=2}^{n_2+2}\left[b_{i,2q-2}(\cos^i\theta+\cos^{i+m}\theta)\sin^{2q}\theta\right]r^{i+2q-1}d\theta\\
&= \frac{1}{2\pi}\Bigg[\sum_{2l+1+2q=3}^{n_2+1}b_{2l+1,2q-2}r^{2l+2q}\int_0^{2\pi}\cos^{2l+m+1}\theta\sin^{2q}\theta d\theta\\
&\quad+\sum_{2l+2q=2}^{n_2+2}b_{2l,2q-2}r^{2l+2q-1}\int_0^{2\pi}\cos^{2l}\theta\sin^{2q}\theta d\theta\Bigg]\\
&= \sum_{l+q=1}^{n_2/2}b_{2l+1,2q-2}r^{2l+2q}\frac{(2q-1)!!}{(2q+2l+1)(2q+2l-1)...(2l+3)}\frac{(2l+m)!!}{2^{\frac{2l+m+1}{2}}(\frac{2l+m+1}{2})!}\\
&\quad+\sum_{l+q=1}^{(n_2+2)/2}b_{2l,2q-2}r^{2l+2q-1}\frac{(2q-1)!!}{(2q+2l)(2q+2l-2)...(2l+2)}\frac{(2l-1)!!}{2^l l!}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{l+q=1}^{n_2/2} b_{2l+1,2q-2} r^{2l+2q} \frac{(2l+m)!!(2q-1)!!}{2^{\frac{2l+m+1}{2}}(\frac{2l+m+1}{2})!(2q+2l+1)(2q+2l-1)...(2l+3)} \\
&+ \sum_{l+q=1}^{(n_2+2)/2} b_{2l,2q-2} r^{2l+2q-1} \frac{(2l-1)!!(2q-1)!!}{2^{l+q} l!(q+l)(q+l-1)...(l+1)} \\
&= \sum_{k=1}^{n_2+1} B_k r^k.
\end{aligned}
$$

(6) If m odd and n_2 odd

$$
\begin{aligned}
f_6(r) &= \frac{1}{2\pi}\int_0^{2\pi} F_2(r,\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{i+j=0}^{n_2} \left[b_{i,j}(\cos^i\theta + \cos^{i+m}\theta)\sin^{j+2}\theta\right] r^{i+j+1} d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{i+2q=2}^{n_2+2} \left[b_{i,2q-2}(\cos^i\theta + \cos^{i+m}\theta)\sin^{2q}\theta\right] r^{i+2q-1} d\theta \\
&= \frac{1}{2\pi}\left[\sum_{2l+1+2q=3}^{n_2+2} b_{2l+1,2q-2} r^{2l+2q} \int_0^{2\pi} \cos^{2l+m+1}\theta \sin^{2q}\theta d\theta \right. \\
&\left. + \sum_{2l+2q=2}^{n_2+1} b_{2l,2q-2} r^{2l+2q-1} \int_0^{2\pi} \cos^{2l}\theta\sin^{2q}\theta d\theta\right] \\
&= \sum_{l+q=1}^{(n_2+1)/2} b_{2l+1,2q-2} r^{2l+2q} \frac{(2l+m)!!(2q-1)!!}{2^{\frac{2l+m+1}{2}}(\frac{2l+m+1}{2})!(2q+2l+1)(2q+2l-1)...(2l+3)} \\
&+ \sum_{l+q=1}^{(n_2+1)/2} b_{2l,2q-2} r^{2l+2q-1} \frac{(2l-1)!!(2q-1)!!}{2^{l+q} l!(q+l)(q+l-1)...(l+1)} \\
&= \sum_{k=1}^{n_2+1} \tilde{B}_k r^k.
\end{aligned}
$$

(7) If m even and n_2 even

$$
\begin{aligned}
f_7(r) &= \frac{1}{2\pi}\int_0^{2\pi} F_2(r,\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{i+j=0}^{n_2} \left[b_{i,j}(\cos^i\theta + \cos^{i+m}\theta)\sin^{j+2}\theta\right] r^{i+j+1} d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{i+2q=2}^{n_2+2} \left[b_{i,2q-2}(\cos^i\theta + \cos^{i+m}\theta)\sin^{2q}\theta\right] r^{i+2q-1} d\theta \\
&= \frac{1}{2\pi}\left[\sum_{2l+2q=2}^{n_2+2} b_{2l,2q-2} r^{2l+2q-1} \int_0^{2\pi} (\cos^{2l}\theta + \cos^{2l+m}\theta)\sin^{2q}\theta d\theta\right] \\
&= \sum_{l+q=1}^{(n_2+2)/2} b_{2l,2q-2} r^{2l+2q-1} \frac{(2q-1)!!}{2^q (q+l)(q+l-1)...(l+1)} \left[\frac{(2l-1)!!}{2^l l!} + \frac{(2l+m-1)!!}{2^{\frac{2l+m}{2}}(\frac{2l+m}{2})!}\right] \\
&= \sum_{\substack{k=1 \\ k\ odd}}^{n_2+1} \bar{B}_k r^k.
\end{aligned}
$$

(8) If m even and n_2 odd

$$
f_8(r) = \frac{1}{2\pi}\int_0^{2\pi} F_2(r,\theta)d\theta
$$

$$\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{i+j=0}^{n_2}\left[b_{i,j}(\cos^i\theta+\cos^{i+m}\theta)\sin^{j+2}\theta\right]r^{i+j+1}d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{i+2q=2}^{n_2+2}\left[b_{i,2q-2}(\cos^i\theta+\cos^{i+m}\theta)\sin^{2q}\theta\right]r^{i+2q-1}d\theta\\
&= \frac{1}{2\pi}\left[\sum_{2l+2q=2}^{n_2+1}b_{2l,2q-2}r^{2l+2q-1}\int_0^{2\pi}(\cos^{2l}\theta+\cos^{2l+m}\theta)\sin^{2q}\theta d\theta\right]\\
&= \sum_{l+q=1}^{(n_2+1)/2}b_{2l,2q-2}r^{2l+2q-1}\frac{(2q-1)!!}{2^q(q+l)(q+l-1)...(l+1)}\left[\frac{(2l-1)!!}{2^l l!}+\frac{(2l+m-1)!!}{2^{\frac{2l+m}{2}}(\frac{2l+m}{2})!}\right]\\
&= \sum_{\substack{k=1\\ k\ odd}}^{n_2}\bar{B}_k r^k.
\end{aligned}$$

Going back to the Equation (7), and we distinguish the following cases and subcases

(a) If m odd and n odd

(a.1) n_1 even et n_2 even

$$F_{10}(r)=\sum_{k=1}^{n_1+1}A_k r^k+\sum_{k=1}^{n_2+1}B_k r^k,$$

(a.2) n_1 odd et n_2 odd

$$F_{10}(r)=\sum_{k=1}^{n_1+1}\tilde{A}_k r^k+\sum_{k=1}^{n_2+1}\tilde{B}_k r^k,$$

(a.3) n_1 odd et n_2 even

$$F_{10}(r)=\sum_{k=1}^{n_1+1}\tilde{A}_k r^k+\sum_{k=1}^{n_2+1}B_k r^k,$$

(a.4) n_1 even et n_2 odd

$$F_{10}(r)=\sum_{k=1}^{n_1+1}A_k r^k+\sum_{k=1}^{n_2+1}\tilde{B}_k r^k.$$

We have that F_{10} is the polynomial in the variable r, then by Descartes Theorem F_{10} has most max $\{n_1,n_2\}$ limit cycles, this completes the proof of statement (1) of Theorem 1.

(b) If m even and n even.

(b.1) n_1 even et n_2 even

$$F_{10}(r)=\sum_{\substack{k=1\\ k\ odd}}^{n_1+1}\bar{A}_k r^k+\sum_{\substack{k=1\\ k\ odd}}^{n_2+1}\bar{B}_k r^k,$$

(b.2) n_1 odd et n_2 odd

$$F_{10}(r)=\sum_{\substack{k=1\\ k\ odd}}^{n_1}\bar{A}_k r^k+\sum_{\substack{k=1\\ k\ odd}}^{n_2}\bar{B}_k r^k,$$

(b.3) n_1 odd et n_2 even

$$F_{10}(r)=\sum_{\substack{k=1\\ k\ odd}}^{n_1}\bar{A}_k r^k+\sum_{\substack{k=1\\ k\ odd}}^{n_2+1}\bar{B}_k r^k,$$

(b.4) n_1 even et n_2 odd

$$F_{10}(r)=\sum_{\substack{k=1\\ k\ odd}}^{n_1+1}\bar{A}_k r^k+\sum_{\substack{k=1\\ k\ odd}}^{n_2}\bar{B}_k r^k.$$

We have that F_{10} is the polynomial in the variable r^2, then by Descartes Theorem F_{10} has most max$\{[\frac{n_1}{2}],[\frac{n_2}{2}]\}$ limit cycles, this completes the proof of statement (2) of Theorem 1.

(c) If m odd and n even.

(c.1) n_1 even et n_2 even

$$F_{10}(r)=\sum_{\substack{k=1\\k\ odd}}^{n_1+1}\bar{A}_k r^k+\sum_{k=1}^{n_2+1}B_k r^k,$$

(c.2) n_1 odd et n_2 odd

$$F_{10}(r)=\sum_{\substack{k=1\\k\ odd}}^{n_1}\bar{A}_k r^k+\sum_{k=1}^{n_2+1}\tilde{B}_k r^k,$$

(c.3) n_1 odd et n_2 even

$$F_{10}(r)=\sum_{\substack{k=1\\k\ odd}}^{n_1}\bar{A}_k r^k+\sum_{k=1}^{n_2+1}B_k r^k,$$

(c.4) n_1 even et n_2 odd

$$F_{10}(r)=\sum_{\substack{k=1\\k\ odd}}^{n_1+1}\bar{A}_k r^k+\sum_{k=1}^{n_2+1}\tilde{B}_k r^k.$$

We have that F_{10} is the sum of two polynomials, one in the variable r and the other in r^2, then by Descartes Theorem F_{10} has most $\max\left\{n_2, n_2+\left[\frac{n_1}{2}\right]-\left[\frac{n_2}{2}\right]\right\}$ limit cycles, this completes the proof of statement (3) of Theorem 1.

(d) If n odd and m even.

(d.1) n_1 even et n_2 even

$$F_{10}(r)=\sum_{k=1}^{n_1+1}A_k r^k+\sum_{\substack{k=1\\k\ odd}}^{n_2+1}\bar{B}_k r^k,$$

(d.2) n_1 odd et n_2 odd

$$F_{10}(r)=\sum_{k=1}^{n_1+1}\tilde{A}_k r^k+\sum_{\substack{k=1\\k\ odd}}^{n_2}\bar{B}_k r^k,$$

(d.3) n_1 odd et n_2 even

$$F_{10}(r)=\sum_{k=1}^{n_1+1}\tilde{A}_k r^k+\sum_{\substack{k=1\\k\ odd}}^{n_2+1}\bar{B}_k r^k,$$

(d.4) n_1 even et n_2 odd

$$F_{10}(r)=\sum_{k=1}^{n_1+1}A_k r^k+\sum_{\substack{k=1\\k\ odd}}^{n_2}\bar{B}_k r^k.$$

We have that F_{10} is the sum of two polynomials, one in the variable r and the other in r^2, then by Descartes Theorem F_{10} has most $\max\left\{n_1, n_1+\left[\frac{n_2}{2}\right]-\left[\frac{n_1}{2}\right]\right\}$ limit cycles, this completes the proof of statement (4) of Theorem 1.

4. Example

We consider the system

$$\begin{cases}\dot{x}=-y+\varepsilon(1+\sin^2\theta)x(\frac{1}{84}x^2-\frac{23}{240}),\\ \dot{y}=x+\varepsilon(1+\cos^3\theta)y(\frac{1}{8}x^2y^2-\frac{23}{18}xy^2+\frac{1}{12}y^2-\frac{13}{48}x+\frac{1}{8}).\end{cases}\tag{8}$$

By doing the change of variables $x=r\cos\theta, y=r\sin\theta$ and taking θ as a new independent variable, we get

$$\begin{aligned}\dot{r}\;=\;&\varepsilon\left(\left(\frac{1}{84}r^3\cos^4\theta-\frac{23}{240}r\cos^2\theta\right)\left(1+\sin^2\theta\right)+\left(\frac{1}{8}r^5\sin^4\theta\cos^2\theta-\frac{23}{18}r^4\sin^4\theta\cos\theta\right.\right.\\&\left.\left.+\frac{1}{12}r^3\sin^4\theta-\frac{13}{48}r^2\sin^2\theta\cos\theta+\frac{1}{8}r\sin^2\theta\right)\left(1+\cos^3\theta\right)\right),\end{aligned}$$

$$\dot{\theta} = 1+\varepsilon\left(\left(-\frac{1}{84}r^2\cos^3\theta\sin\theta+\frac{23}{240}\sin\theta\cos\theta\right)\left(1+\sin^2\theta\right)+\left(-\frac{23}{18}r^3\cos^2\theta\sin^3\theta\right.\right.$$
$$\left.\left.+\frac{1}{12}r^2\sin^3\theta\cos\theta+\frac{1}{8}r^4\cos^3\theta\sin^3\theta-\frac{13}{48}r\cos\theta+\frac{1}{8}\cos\theta\sin\theta\right)\right).$$

Taking θ as the new independent variable, we get

$$\frac{dr}{d\theta} = \varepsilon F(r,\theta)+O(\varepsilon^2),$$

where

$$F(r,\theta) = \left(\frac{1}{84}r^3\cos^4\theta-\frac{23}{240}r\cos^2\theta\right)\left(1+\sin^2\theta\right)+\left(\frac{1}{8}r^5\sin^4\theta\cos^2\theta-\frac{23}{18}r^4\sin^4\theta\cos\theta\right.$$
$$\left.+\frac{1}{12}r^3\sin^4\theta-\frac{13}{48}r^2\sin^2\theta\cos\theta+\frac{1}{8}r\sin^2\theta\right)\left(1+\cos^3\theta\right).$$

The function of averaging theory of first order is

$$F_{10} = \frac{1}{768}r\left(6r^4-23r^3+28r^2-13r+2\right),$$

that has exactly four positive zeros which are $r_1=\frac{1}{3}, r_2=\frac{1}{2}, r_3=1$, and $r_4=2$. Which satisfy

$$\frac{dF_{10}(r)}{dr}\Big|_{r=r_1} = -\frac{5}{10368}\neq 0,$$
$$\frac{dF_{10}(r)}{dr}\Big|_{r=r_2} = \frac{1}{2048}\neq 0,$$
$$\frac{dF_{10}(r)}{dr}\Big|_{r=r_3} = -\frac{1}{384}\neq 0,$$
$$\frac{dF_{10}(r)}{dr}\Big|_{r=r_4} = \frac{5}{128}\neq 0,$$

then we conclude that the system (8) has two stable limit cycles for $r_1=\frac{1}{3}$ and $r_3=1$, and two unstable limit cycles for $r_2=\frac{1}{2}$ and $r_4=2$ (see Figure 1).

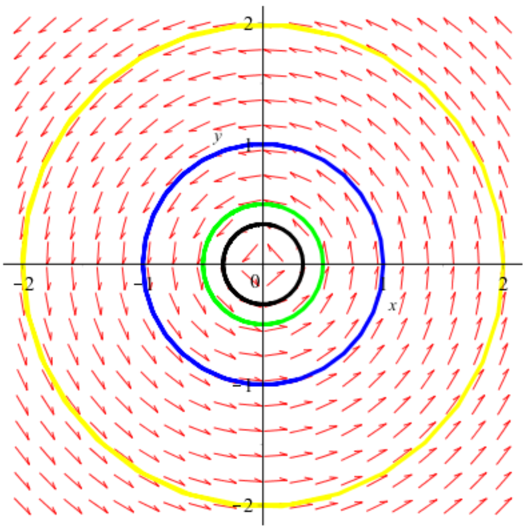

Figure 1. Four limit cycles for $\varepsilon = 10^{-3}$

5. Conclusion

In the present paper, by using the averaging theory of the first order we show that the maximum number of the limit cycles bifurcating from linear center $\dot{x} = -y, \dot{y} = x$, for a generalized planar differential system.

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