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On a generalized class of bi-univalent functions defined by subordination and q-derivative operator

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Abstract: In this paper, the q-derivative operator and the principle of subordination were employed to define a subclass $\mathcal{B}_q(\tau,\lambda,\phi)$ of analytic and bi-univalent functions in the open unit disk $\mathcal{U}$. For functions $f(z) \in \mathcal{B}_q(\tau,\lambda,\phi)$, we obtained early coefficient bounds and some Fekete-Szegö estimates for real and complex parameters.

Keywords: Analytic function; Bi-univalent function; Subordination; Fekete-Szegö problem; Ma-Minda function; Carathéodory function; q-differentiation.

MSC: 30C45; 30C50.

1. Introduction

Let $\mathcal{U} = \{z : z \in \mathbb{C}, |z| < 1\}$ be a unit disk and let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathcal{U}), \tag{1}$$

normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S} \subset \mathcal{A}$ be the class of analytic and univalent functions in $\mathcal{U}$.

Let $\mathcal{W}$ denote the class of functions

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \quad (z \in \mathcal{U}),$$

such that $w(0) = 0$ and $|w(z)| < 1$. The class $\mathcal{W}$ is known as the class of Schwarz functions.

By [1], let $j(z), J(z) \in \mathcal{A}$, then $j(z) \prec J(z)$, $z \in \mathcal{U}$, if $\exists w(z)$ analytic in $\mathcal{U}$, such that $w(0) = 0$, $|w(z)| < 1$ and $j(z) = J(w(z))$. If the function $J(z)$ is univalent in $\mathcal{U}$, then $j(z) \prec J(z) \implies j(0) = J(0)$ and $j(\mathcal{U}) \subset J(\mathcal{U})$.

Let $\mathcal{P}$ denote the class of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \mathcal{U}), \tag{2}$$

which are analytic in $\mathcal{U}$ such that $\mathcal{R}e(p(z)) > 0$ and $p(0) = 1$. It is known that functions in classes $\mathcal{P}$ and $\mathcal{W}$ are related such that

$$p(z) = \frac{1+w(z)}{1-w(z)} \Longleftrightarrow w(z) = \frac{p(z)-1}{p(z)+1}. \tag{3}$$

In [2], Ma and Minda defined a function $\phi \in \mathcal{P}$ $(z \in \mathcal{U})$ such that $\phi(0) = 1$, $\phi'(0) > 0$ and $\phi(\mathcal{U})$ is starlike with respect to 1 and symmetric with respect to the real axis. Such function ϕ can be expressed as

$$\phi(z) = 1 + \beta_1 z + \beta_2 z^2 + \cdots \quad (z \in \mathcal{U},\ \beta_1 > 0). \tag{4}$$

Fekete and Szegö [3] investigated the coefficient functional

$$g_\rho(f) = |a_3 - \rho a_2^2|,$$

which arose from the disproof of Littlewood-Parley conjecture (see [1]) that says modulus of coefficients of odd univalent functions are less than 1. This functional has been investigated by many researchers, see for instance [4,5].

Historically, Lewin [6] introduced a subclass of $\mathcal{A}$ called the class of *bi-univalent* functions and established that $|a_2| \leq 1.51$ for all bi-univalent functions. Also, the Koebe 1/4 theorem (see [1]) states that the range of every function $f \in \mathcal{S}$ contains the disk $D = \{\omega : |\omega| < 0.25\} \subseteq f(\mathcal{U})$. This implies that $\forall f \in \mathcal{S}$ has an inverse function f^{-1} such that

$$f^{-1}(f(z)) = z \quad (z \in \mathcal{U}),$$

and

$$f(f^{-1}(\omega)) = \omega \quad (\omega : |\omega| < r_0(f);\ r_0(f) \geq 0.25),$$

where $f^{-1}(\omega)$ is expressed as

$$F(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots . \tag{5}$$

Thus, a function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathcal{U}$ if both $f(z)$ and $F(\omega)$ are univalent in $\mathcal{U}$. Let $\mathcal{B}$ denote the class of analytic and bi-univalent functions in $\mathcal{U}$.

Some functions $f \in \mathcal{B}$ includes $f(z) = z$, $f(z) = z/(1-z)$, $f(z) = -\log(1-z)$ and $f(z) = \frac{1}{2}\log[(1+z)/(1-z)]$. Observe that some familiar functions $f \in \mathcal{S}$ such as the *Koebe function* $K(z) = z/(1-z)^2$, its rotation function $K_\sigma(z) = z/(1-e^{i\sigma}z)^2$, $f(z) = z - z^2/2$ and $f(z) = z/(1-z^2)$ are nonmembers of $\mathcal{B}$. See [4,5,7–11] for more details.

Jackson [12] (see also [8,13,14]) introduced the concept of q-derivative operator. For functions $f \in \mathcal{A}$, the q-derivative of f can be defined by

$$\mathcal{D}_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} \qquad (z \neq 0,\ 0 < q < 1), \tag{6}$$

where $\mathcal{D}_q f(0) = f'(0)$ and $\mathcal{D}_q f(z)z = \mathcal{D}_q(\mathcal{D}_q f(z))$. From (1) and (6) we get

$$\left.\begin{aligned} \mathcal{D}_q f(z) &= 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \\ \mathcal{D}_q f(z)z &= \sum_{n=2}^{\infty} [n]_q [n-1]_q a_n z^{n-2} \end{aligned}\right\} \tag{7}$$

where $[n]_q = \frac{1-q^n}{1-q}$, $[n-1]_q = \frac{1-q^{n-1}}{1-q}$, $\lim_{q\uparrow 1}[n]_q = n$ and $\lim_{q\uparrow 1}[n-1]_q = n-1$.

For instance, if α is a constant, then for the function $f(z) = \alpha z^n$,

$$\mathcal{D}_q f(z) = \mathcal{D}_q(\alpha z^n) = \frac{1-q^n}{1-q}\alpha z^{n-1} = [n]_q \alpha z^{n-1},$$

and note that

$$\lim_{q\uparrow 1} \mathcal{D}_q f(z) = \lim_{q\uparrow 1} [n]_q \alpha z^{n-1} = n\alpha z^{n-1} =: f'(z),$$

where $f'(z)$ is the classical derivative.

In this study, the q-derivative operator and the subordination principle are used to define and generalize a subclass of bi-univalent functions. Afterwards, some coefficient bounds and some Fekete-Szegö estimates were investigated. Some of our results generalised that of Srivastava and Bansal in [10] and some new results are added.

Definition 1. Let $0 < q < 1$, $\tau \in \mathbb{C} \setminus \{0\}$, $0 \leq \lambda \leq 1$ and ϕ is defined in (4). A function $f \in \mathcal{B}$ is said to be in the class $\mathcal{B}_q(\tau, \lambda, \phi)$ if the subordination conditions

$$1 + \frac{1}{\tau}[\mathcal{D}_q f(z) + \lambda z \mathcal{D}_q f(z) z - 1] \prec \phi(z) \qquad (z \in \mathcal{U}), \tag{8}$$

and

$$1 + \frac{1}{\tau}[\mathcal{D}_q F(\omega) + \lambda \omega \mathcal{D}_q^2 F(\omega) - 1] \prec \phi(\omega) \quad (\omega \in \mathcal{U}), \tag{9}$$

where $F(\omega) = f^{-1}(\omega)$ are satisfied.

Remark 1. Let $q \uparrow 1$ in (8) and (9), then $\mathcal{B}_q(\tau, \lambda, \phi)$ becomes the class $\mathcal{B}(\tau, \lambda, \phi)$ investigated by Srivastava and Bansal [10].

2. Preliminary Lemmas

To establish our results, we shall need the following lemmas. Let $p(z)$ be as defined in (2).

Lemma 2 ([1]). *If $p(z) \in \mathcal{P}$, then $|p_n| \leq 2$ $(n \in \mathbb{N})$. The result is sharp for the well-known Möbius function.*

Lemma 3 ([15,16]). *If $p(z) \in \mathcal{P}$, then $2p_2 = p_1^2 + (4 - p_1^2)x$ for some x and $|x| \leq 1$.*

3. Main Results

Unless otherwise mentioned in what follows, we assume throughout this work that $0 < q < 1$, $\tau \in \mathbb{C} \setminus \{0\}$, $0 \leq \lambda \leq 1$, ϕ is as defined in (4) and $f \in \mathcal{B}$, hence our results are as follows:

Theorem 4. *Let $f \in \mathcal{B}_q(\tau, \lambda, \phi)$, then*

$$|a_2| \leq \frac{\beta_1^{3/2}|\tau|}{\sqrt{\left|\beta_1^2 \tau [3]_q (1 + [2]_q \lambda) + [2]_q^2 (1 + [1]_q \lambda)^2 (\beta_1 - \beta_2)\right|}}, \tag{10}$$

$$|a_3| \leq \frac{\beta_1^2 |\tau|^2}{[2]_q^2 (1 + [1]_q \lambda)^2} + \frac{\beta_1 |\tau|}{[3]_q (1 + [2]_q \lambda)}, \tag{11}$$

where $\beta_1 > 0$ and β_n $(n \in \mathbb{N})$ are coefficients of $\phi(z)$ in (4).

Proof. Let $f(z) \in \mathcal{B}$ and $F(\omega) = f^{-1}(\omega)$, then there exists the analytic functions $u(z), v(\omega) \in \mathcal{W}$, $z, \omega \in \mathcal{U}$ such that $u(0) = 0 = v(0)$, $|u(z)| < 1$, $|v(\omega)| < 1$ so that they satisfy the subordination conditions:

$$1 + \frac{1}{\tau}[\mathcal{D}_q f(z) + \lambda z \mathcal{D}_q f(z) z - 1] = \phi(u(z)) \quad (z \in \mathcal{U}), \tag{12}$$

and

$$1 + \frac{1}{\tau}[\mathcal{D}_q F(\omega) + \lambda \omega \mathcal{D}_q^2 F(\omega) - 1] = \phi(v(\omega)) \quad (\omega \in \mathcal{U}). \tag{13}$$

By substituting (7) into LHS of (12) we respectively get

$$1 + \frac{1}{\tau}[\mathcal{D}_q f(z) + \lambda z \mathcal{D}_q f(z) z - 1] = 1 + \frac{[2]_q (1 + [1]_q \lambda) a_2}{\tau} z + \frac{[3]_q (1 + [2]_q \lambda) a_3}{\tau} z^2 + \cdots, \tag{14}$$

and following the same process for $F(\omega)$ in (5) gives

$$1 + \frac{1}{\tau}[\mathcal{D}_q F(\omega) + \lambda \omega \mathcal{D}_q^2 F(\omega) - 1] = 1 - \frac{[2]_q (1 + [1]_q \lambda) a_2}{\tau} \omega + \frac{[3]_q (1 + [2]_q \lambda)(2a_2^2 - a_3)}{\tau} \omega^2 + \cdots. \tag{15}$$

Now to expand

$$\phi(u(z)), \tag{16}$$

and

$$\phi(v(\omega)), \tag{17}$$

in series form, let $\delta_1(z) = 1 + b_1 z + b_2 z^2 + \dots$, $\delta_2(\omega) = 1 + c_1\omega + c_2\omega^2 + \cdots \in \mathcal{P}$, then by (3),

$$\delta_1(z) = \frac{1+u(z)}{1-u(z)} \Longrightarrow u(z) = \frac{\delta_1(z)-1}{\delta_1(z)+1} = \frac{1}{2}\left[b_1 z + \left(b_2 - \frac{b_1^2}{2}\right)z^2 + \left(\frac{b_1^3}{2^2} - b_1 b_2 + b_3\right)z^3 + \cdots\right], \tag{18}$$

and following the same process

$$\delta_2(\omega) = \frac{1+v(\omega)}{1-v(\omega)} \Longrightarrow v(\omega) = \frac{\delta_2(\omega)-1}{\delta_2(\omega)+1} = \frac{1}{2}\left[c_1\omega + \left(c_2 - \frac{c_1^2}{2}\right)\omega^2 + \left(\frac{c_1^3}{2^2} - c_1 c_2 + c_3\right)\omega^3 + \cdots\right]. \tag{19}$$

Substituting (18) into (16) as expressed by (4) we get

$$\begin{aligned}\phi(u(z)) = 1 &+ \frac{1}{2}\beta_1 b_1 z + \frac{1}{2}\left[\beta_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{2}\beta_2 b_1^2\right]z^2 \\ &+ \frac{1}{2}\left[\beta_1\left(\frac{b_1^3}{2^2} - b_1 b_2 + b_3\right) + \beta_2 b_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}\beta_3 b_1^3\right]z^3 + \cdots,\end{aligned} \tag{20}$$

and substituting (19) into (17) as expressed by (4) we get

$$\begin{aligned}\phi(v(\omega)) = 1 &+ \frac{1}{2}\beta_1 c_1\omega + \frac{1}{2}\left[\beta_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{2}\beta_2 c_1^2\right]\omega^2 \\ &+ \frac{1}{2}\left[\beta_1\left(\frac{c_1^3}{2^2} - c_1 c_2 + c_3\right) + \beta_2 c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}\beta_3 c_1^3\right]\omega^3 + \cdots.\end{aligned} \tag{21}$$

Now comparing the coefficients in (14) and (20) we get

$$\frac{[2]_q(1+[1]_q\lambda)a_2}{\tau} = \frac{\beta_1 b_1}{2}, \tag{22}$$

$$\frac{[3]_q(1+[2]_q\lambda)a_3}{\tau} = \frac{1}{2}\left[\beta_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{2}\beta_2 b_1^2\right], \tag{23}$$

and comparing the coefficients in (15) and (21) gives

$$-\frac{[2]_q(1+\lambda[1]_q)a_2}{\tau} = \frac{\beta_1 c_1}{2}, \tag{24}$$

$$\frac{[3]_q(1+[2]_q\lambda)(2a_2^2 - a_3)}{\tau} = \frac{1}{2}\left[\beta_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{2}\beta_2 c_1^2\right]. \tag{25}$$

Now adding (22) and (24) and simplifying we get

$$b_1 = -c_1 \quad \text{and} \quad b_1^2 = c_1^2. \tag{26}$$

Also from (22) and (24) we get

$$8[2]_q^2(1+[1]_q\lambda)^2 a_2^2 = \tau^2\beta_1^2(b_1^2 + c_1^2), \tag{27}$$

and adding (23) and (25) and using (26) we get

$$4[3]_q(1+[2]_q\lambda)a_2^2 = \tau\beta_1(b_2+c_2) - \tau b_1^2(\beta_1 - \beta_2). \tag{28}$$

From (27) and using (26) we get

$$b_1^2 = \frac{4[2]_q^2(1+[1]_q\lambda)^2 a_2^2}{\tau^2\beta_1^2}. \tag{29}$$

So that by substituting for b_1^2 in (28) we get

$$a_2^2 = \frac{\tau^2\beta_1^3(b_2+c_2)}{4\{\tau\beta_1^2[3]_q(1+[2]_q\lambda)+[2]_q^2(1+[1]_q\lambda)^2(\beta_1-\beta_2)\}}, \tag{30}$$

and applying Lemma 2 gives (10).

Again by subtracting (23) from (25), using (26) and simplifying we get

$$a_3 = a_2^2 + \frac{\tau\beta_1(b_2-c_2)}{4[3]_q(1+[2]_q\lambda)}. \tag{31}$$

Thus, from (27), using (26) and simplifying we get

$$a_3 = \frac{\tau^2\beta_1^2 b_1^2}{4[2]_q^2(1+[1]_q\lambda)^2} + \frac{\tau\beta_1(b_2-c_2)}{4[3]_q(1+[2]_q\lambda)}, \tag{32}$$

and applying Lemma 2 gives (11). □

Let $q \uparrow 1$, then Theorem 4 becomes

Corollary 5. *Let $f(z) \in \mathcal{B}_q(\tau,\lambda,\phi)$, then as $q \uparrow 1$,*

$$|a_2| \le \frac{|\tau|\beta_1^{3/2}}{\sqrt{|\tau[3]_q\beta_1^2 + [2]_q^2(\beta_1-\beta_2)|}},$$
$$|a_3| \le \frac{|\tau|^2\beta_1^2}{[2]_q^2} + \frac{|\tau|\beta_1}{[3]_q}.$$

which is the result of Srivastava and Bansal [10].

Theorem 6 (Fekete-Szegö Estimate, $\varrho \in \mathbb{R}$). *If $f \in \mathcal{B}_q(\tau,\lambda,\phi)$ and $\varrho \in \mathbb{R}$, then*

$$|a_3 - \varrho a_2^2| \le \begin{cases} \frac{|\tau|\beta_1}{[3]_q(1+[2]_q\lambda)} & \text{for } 0 \le |h(\varrho)| \le \frac{1}{[3]_q(1+[2]_q\lambda)}; \\ |\tau|\beta_1|h(\rho)| & \text{for } |h(\varrho)| \ge \frac{1}{[3]_q(1+[2]_q\lambda)}, \end{cases}$$

where

$$h(\varrho) = \frac{\tau\beta_1^2(1-\varrho)}{\{\tau\beta_1^2[3]_q(1+[2]_q\lambda)+[2]_q^2(1+[1]_q\lambda)^2(\beta_1-\beta_2)\}}. \tag{33}$$

Proof. From (30) and (31),

$$\begin{aligned} |a_3-\varrho a_2^2| &= \left|\frac{\tau\beta_1(b_2-c_2)}{4[3]_q(1+[2]_q\lambda)} + (1-\varrho)a_2^2\right| \\ &= \left|\frac{\tau\beta_1}{4}\left\{\frac{(b_2-c_2)}{[3]_q(1+[2]_q\lambda)} + (b_2+c_2)h(\varrho)\right\}\right|, \end{aligned}$$

where $h(\varrho)$ is given in (33), so that by applying triangle inequality, (4), Lemma 2 and simplifying complete the proof. □

Theorem 7 (Fekete-Szegö Estimate, $\rho \in \mathbb{C}$). *If $f \in \mathcal{B}_q(\tau, \lambda, \phi)$ and $\rho \in \mathbb{C}$, then*

$$|a_3 - \rho a_2^2| \leq \begin{cases} \frac{|\tau|\beta_1}{[3]_q(1+[2]_q\lambda)} & \text{for } |1-\rho| \in [0,\xi); \\ \frac{\beta_1^2|\tau|^2}{[2]_q^2(1+[1]_q\lambda)^2}|1-\rho| & \text{for } |1-\rho| \in [\xi, \infty), \end{cases} \tag{34}$$

where

$$\xi = \frac{[2]_q^2(1+[1]_q\lambda)^2}{|\tau|\beta_1[3]_q(1+[2]_q\lambda)}.$$

Proof. From (27) and (31) and using (26),

$$a_3 - \rho a_2^2 = (1-\rho)\frac{\beta_1^2 b_1^2 \tau^2}{4[2]_q^2(1+[1]_q\lambda)^2} + \frac{\beta_1\tau(b_2-c_2)}{4[3]_q(1+[2]_q\lambda)}. \tag{35}$$

From Lemma 3 and (26)

$$b_2 - c_2 = \frac{1}{2}(4-b_1^2)(x-y), \tag{36}$$

for some $x, y, |x| \leq 1, |y| \leq 1$ and $|b_1| \in [0,2]$. Thus using (36) in (35) simplifies to

$$a_3 - \rho a_2^2 = (1-\rho)\frac{\beta_1^2 b_1^2 \tau^2}{4[2]_q^2(1+[1]_q\lambda)^2} + \frac{\beta_1\tau(4-b_1^2)}{8[3]_q(1+[2]_q\lambda)}(x-y).$$

For $\delta(z) = 1 + b_1 z + b_2 z^2 + \cdots \in \mathcal{P}$, $|b_1| \leq 2$ by Lemma 2. Letting $b = b_1$, we may assume without any restriction that $b \in [0,2]$. Now using triangle inequality, letting $X = |x| \leq 1$ and $Y = |y| \leq 1$, then we get

$$|a_3 - \rho a_2^2| \leq |1-\rho|\frac{\beta_1^2 b^2|\tau|^2}{4[2]_q^2(1+[1]_q\lambda)^2} + \frac{\beta_1|\tau|(4-b^2)}{8[3]_q(1+[2]_q\lambda)}(X+Y) = H(X,Y).$$

For $X, Y \in [0,1]$;

$$\max\{H(X,Y)\} = H(1,1) = \frac{\beta_1^2|\tau|^2}{4[2]_q^2(1+[1]_q\lambda)^2}\left\{|1-\rho| - \frac{[2]_q^2(1+[1]_q\lambda)^2}{\beta_1|\tau|[3]_q(1+[2]_q\lambda)}\right\}b^2 + \frac{\beta_1|\tau|}{[3]_q(1+[2]_q\lambda)} = G(b).$$

For $b \in [0,2]$;

$$G'(b) = \frac{\beta_1^2|\tau|^2}{2[2]_q^2(1+[1]_q\lambda)^2}\left\{|1-\rho| - \frac{[2]_q^2(1+[1]_q\lambda)^2}{\beta_1|\tau|[3]_q(1+[2]_q\lambda)}\right\}b, \tag{37}$$

which implies that there is a critical point at $G'(b) = 0$, that is at $b = 0$. Hence for

$$G'(b) < 0; \ |1-\rho| \in \left[0, \frac{[2]_q^2(1+[1]_q\lambda)^2}{\beta_1|\tau|[3]_q(1+[2]_q\lambda)}\right),$$

thus, $G(b)$ is strictly a decreasing function of $|1-\rho|$, therefore from (3),

$$\max\{G(b) : b \in [0,2]\} = G(0) = \frac{\beta_1|\tau|}{[3]_q(1+[2]_q\lambda)}.$$

Also for

$$G'(b) \geq 0; \ |1-\rho| \in \left[\frac{[2]_q^2(1+[1]_q\lambda)^2}{\beta_1|\tau|[3]_q(1+[2]_q\lambda)}, 0\right),$$

thus, $G(b)$ is an increasing function of $|1-\rho|$, therefore from (3),

$$\max\{G(b) : b \in [0,2]\} = G(2) = \frac{|1-\rho|\beta_1^2|\tau|^2}{[2]_q^2(1+[1]_q\lambda)^2}.$$

So that by putting the results together leads to (34). □

4. Conclusion

In this work, we were able to establish the first two coefficient bounds and also solve the Fekete-Szegö problem for the class $\mathcal{B}_q(\tau, \lambda, \phi)$ of analytic and bi-univalent functions in $\mathcal{U}$. The results in the first theorem generalized that of Srivastava and Bansal [10].

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