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On separability criteria for continuous Bitopological spaces

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Abstract: In this paper, we give characterizations of separation criteria for bitopological spaces via *ij*-continuity. We show that if a bitopological space is a separation axiom space, then that separation axiom space exhibits both topological and heredity properties. For instance, let (X, τ_1, τ_2) be a T_0 space then, the property of T_0 is topological and hereditary. Similarly, when (X, τ_1, τ_2) is a T_1 space then the property of T_1 is topological and hereditary. Next, we show that separation axiom T_0 implies separation axiom T_1 which also implies separation axiom T_2 and the converse is true.

Keywords: Bitopological space; Continuous function; ij-Continuity; Separation axiom.

MSC: 26A03; 54C30.

1. Introduction

Studies have been conducted by different authors on continuity and its aspects. Many results have so far been obtained. Most of these results have been successfully obtained by use of separation criteria. This can be done by choosing a topological space that one may wish to use in testing a property of either topological or bitopological space. Separation criteria involve axioms that guide the separation of topological and bitopological spaces. Therefore, separation axioms are restrictions that are often made depending on the kind of topological or bitopological spaces that we intended to consider. Separation axioms involve the use of spaces which distinguish disjoint sets and distinct points. These separation axioms are also called Tychonoff separation axioms.

Fora [1] states that spaces that can be topologically distinguished are said to be separable spaces. Abu-Donia and El-Tantawy [2] conducted a study to show some classes of sets of a bitopological space (X, τ_1, τ_2) which included the infra topologies and supra topologies. These classes introduce new bitopological properties and new types of continuous functions between bitopological spaces. They did some work which indicated that bitopological separation properties are preserved under some types of continuous functions. The properties $T_{\frac{1}{2}}$, T_b, αT_b, T_d, αT_d are exhibited by topological spaces and finally extended to bitopological spaces. Patil and Nagashree [3] effected some study to show new separation axioms in binary soft topological spaces alongside their properties and characterizations. They have also introduced the notions of binary separation axioms as binary T_0, binary T_1, binary T_2 spaces.

According to [3, Theorem 3.4], every $n - T_0^*$ is binary soft $n - T_0$. This result implies that if we have (x_1, y_1) and (x_1, y_2) as two distinct points of a binary soft $n - T_0^*$ space (U_1, U_2, τ_b, E) then there exists at least one binary soft open set (F, E) or (G, E) such that $(x_1, y_2) \in (F, E)$, $(x_2, y_2) \in (F, E)\prime$ or $(x_2, y_2) \in (G, E)$, $(x_1, y_1) \in (G, E)$. So it implies that $(x_1, y_1) \in (F, E)$, (x_2, y_2) does not exist in (F, E) or $(x_2, y_2) \in (G, E)$, (x_1, y_1) does not exist (G, E). In our study we have considered $T_{2\frac{1}{2}}$-spaces as binary soft bitopological space.

According to [3, Theorem 3.11], a binary soft topological space (U_1, U_2, τ_b, E) is a binary soft $n - T_1^*$ space if for any pair of distinct points (x_1, y_1), $(x_2, y_2) \in U_1 \times U_2$. There exists (F, E), $(G, E)\tau_b$ such that $(x_1, y_1) \in (F, E)'$ and $(x_2, y_2) \in (G, E)$, $(x_1, y_1) \in (G, E)'$.

Binary soft property is also seen to be hereditary to some separation axioms as observed in the next result. In [3, Theorem 3.20], it was proved that the property of binary soft $n - T_2$ is hereditary. It illustrates that when (x_1, y_1) $(x_2, y_2) A \times B$ is a pair of distinct points. Then (x_1, y_1), $(x_2, y_2) \in U_1 \times U_2$ are distinct. The fact that (U_1, U_2, τ_b, E) is binary soft $n - T_2$ there exists disjoint binary soft open sets (F, E) and (G, E) in (U_1, U_2, τ_b, E) such that $(x_1, y_1) \in (F, E)$ and $(x_2, y_2) \in (G, E)$. Therefore, we have disjoint binary soft open sets ${}^Y(F, E)$ and ${}^Y(G, E)$ in (Y, τ_{bY}, E) such that $(x_1, y_1) \in^Y (F, E)$ and also $(x_2, y_2) \in^Y (G, E)$.

Selvanayaki and Rajesh [4] introduced another type of separation axioms in their research of quasi $T_{\frac{1}{2}^*}$ space. A space (X, τ_1, τ_2) is said to be quasi T_0 if for any two distinct points x and y of X there exists $A \in QO(X, \tau_1, \tau_2)$ such that $x \in X$ where y does not exists in A or $y \in A$ where x does not exist in A. A space (X, τ_1, τ_2) is said to be T_1 if for any distinct points x, y of X there exists $A, B \in QO(X, \tau_1, \tau_2)$ such that x A, y does not exist in A and $y \in B$, x does not exist in B.

A space (X, τ_1, τ_2) is said to be quasi T_2 if for any two distinct points x, y of X there exists two disjoint sets $A, B \in QO(X, \tau_1, \tau_2) = QGC(X, \tau_1, \tau_2)$. Therefore, it is true that $T_2 \to T_1 \to T_{\frac{1}{2}} \to T_0$. A point $x \in X$ can be said to be a limit point of a subset A of a bitopological space (X, τ_1, τ_2). In [4, Theorem 2.3] it was proved that every quasi $T_{\frac{1}{2}^*}$ is quasi T_0.

In our study we are aiming at establishing separation axioms that can be used in aspects of continuity through ij notion. For a bitopological space (x, δ, τ) can be referred to as T_0 space if for all $x, y \in X$ with $x \neq y$ then there exists in $\delta \cup \tau$ such that $x \in U$ whereas y is not a members of U on the other hand when x is not a cardinality of U then $y \in U$. The result is indicated in below: According to [5, Theorem 2.76], a bitopological space (X, δ, τ) is called T_1 space if for all $x, y \in X$ with $x \neq y$ then there exists $U \in \delta$ and $V \in \tau$ such that $x \in U$ where y does not exists in V and x does not exists in U and $y \in U$. In [5, Theorem 2.7], it was proved that a bitopological space (X, δ, τ) is called a T_2 space if for all $x, y \in X$ with $x \neq y$ then there exists $U \in \delta$, $V \in \tau$ such that $x \in U$ where $y \in V$ and $U \cap V = \phi$.

Rupaya and Hossan [5] have shown some of the results of heredity property exhibited by some separation axioms as given below: In [5, Theorem 3.1], it was proved that if (X, τ_1, τ_2) is a bitopological space then T_0 is considered to have hereditary property. This result illustrates that (X, δ, τ) is a T_0 space and $A \subset X$ shows that (A, δ, τ) is also T_0 space. Having $x, y \in A$ with $x \neq y$ and $x, y \in X$ with $x \neq y$. Hence if (X, δ, τ) is T_0 space then there exists $U \in \delta \cup \tau$ such that $x \in U$ where y does not exist in U or x also does not exist in U but $y \in U$ so $U \in \delta \cup \tau$. This imply that $U \in \delta$ or $U \in \tau$ then $U \cap A \in \delta_A$ or $U \cap A \in \tau_A$. Therefore, $U \cap A \in \delta \cup \tau$. Then $x, y \in A$ and $x \in U \cap A$ where y does not exist in $U \cap A$ or x does not exist in $U \cap A$ where $y \in U \cap A$. Hence (A, δ_A, τ_A) is also T_0. Separation axioms also act even on the bitopological spaces that exhibit the homeomorphic property. Homeomorphic image of a particular separation axioms is still that axiom. This has been shown by Rupaya and Hossain [5] in the study of properties of separation axioms in bitopological spaces.

Some result shows that a function $f : (X, \delta, \tau) \to (Y, W, Z)$ be a homeomorphism and (X, δ, τ) is T_0 space. Then (Y, W, Z) can be shown to be a T_0 space. This work shows that by letting $f(X, \delta, \tau) \to (Y, W, Z)$ to be a homeomorphism and (X, δ, τ) is T_1 space. Then (Y, W, Z) is also T_1 space. Hence $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since f is onto then there exists $x_1, x_2 \in X$ with $f(x_1) = y_1$ and $f(x_2 = y_2$. A function f is one with $y_1 \neq y_2 \to f(x_1) \neq f(x_2)$ this also implies that $x_1 \neq x_2$. Then $x_1, x_2 \in X$ with $x_1 \neq x_2$. Again since (X, δ, τ) is T_1 space then there exists $U \in \delta$ and $V \in \tau$ such that $x_1 \in U$ and x_2 does not exists in U and x_1 does not in V and $x_2 \in V$. Since f is open then $f(U) \in W$ and $f(V) \in Z$. In summary, the notion of bitopological spaces was first introduced by Kelly [6] in 1963. Kelly stated that a bitopological space is endowed by two independent topological structures say τ_1 and τ_2. So a bitopological space that is equipped by two topologies is denoted by (X, τ_1, τ_2). A non empty set X is equipped by two structures τ_1 and τ_2. By Uryshon's lemma, the results show that existence of quasi-metrics is related to the existence of real-valued functions. Does other properties of bitopological spaces exhibit two topologies a bitopological space?.

In 1984, Ivanov [7] conducted a study on the structures of bitopological spaces. Results obtained show that some properties of a bitopological space as a set X on which there are given two structures τ_1 and τ_2 which are defined on the same set as (X, τ_1, τ_2). The result was obtained by the use of T_0 separation axioms as research methodology. However, from the open question that asks whether all the properties of bitopological spaces inherited by these structures. In 1988, Coy [8] carried out a study on some properties of bitopological spaces such as normality, separability and compactness. Results indicate that they are inherited by a topology in a topological space and as well can be extended to structures in bitopological spaces by the use of conditions of normality and compactness. Does continuity property apply in all spaces?. We consider this question in our study and seek to determine if aspects of continuity can apply to all types of topological spaces. In 2005, Abu-Donia and El-Tantaway [2] gave out generalized separation axioms in bitopological spaces where it was proved that some of the introduced bitopological separation properties are preserved under some types of continuous functions. The extension of semi-continuity, pre-continuity, α-continuity have also been extended to bitopological space. Does it follow that aspects of continuity can be expressed bilaterally as ij-continuity in bitopological spaces and N-topological spaces? We consider this question and try to characterize ij-continuity.

In 2006, Noiri [9] defines semi-open sets and semi-continuity in bitopological spaces. A subset A of X is said to be semi-open if it is $(1,2)$-semi open and $(2,1)$-semiopen. It is also continuous if it is ij-continuous. Is the continuity determined by all the separation axioms?. We also focus on this question and try to find out if other separation axioms apart from T_0 apply. In 2007, Orihuela [10] gave a brief account of topological open problems in the area of renormings of Banach spaces. From question four in open problems, let X be a weakly Cech analytic Banach space where every norm open set is a countable union of sets which are differences of closed sets for the weak topology. Does it follow that the identity map $Id : (X, \sigma(X, X^*)) \rightarrow (X, \|\|)$ is σ-continuous?. We also consider this question and seek to determine if it holds true when X is a bitopological space. In 2010, Kohli [11] in the study of between strong and almost continuity gave an account that strong continuity and weak continuity lie strictly between strong continuity of Lavine and strong continuity of Singal. If X is endowed with a partition topology then, every continuous function $f : X \rightarrow Y$ is perfectly continuous and hence completely continuous. Is the inverse of a real value function f continuous?. This question form basis of our research to determine if inverse functions of n-topologies are also continuous. In 2018, Caldas [12] carried out a study on some topological concepts in bitopological spaces among the concepts given account on are normality, compactness separability among others. From question two in open problems, let function f be continuous if it maps spaces (X, τ_1, τ_2) to (Y, τ_1, τ_2). Does most aspects of continuity hold?. We consider this question and try to determine characterization of aspects of continuity.

Birman [13] conducted a study in continuity of bitopological spaces in 2018. Results obtained by using criterion for continuity as research methodology show that most properties including continuity can be induced in a bitopological space (X, τ_1, τ_2). The open question is that can these properties be extended to n-topological spaces $A_1 \cup A_2 \cup A_3 \cup ... \cup A_n$?. We also look into possibility of extending properties of bitopological spaces to n-topologies. Currently in 2019, Rupaya [5] introduced some concepts of separation axioms in bitopological spaces which satisfy topological and hereditary properties. Since a topological space does not imply bitopological space and vice versa. Open question is that do these properties hold in general for both topological and bitopological space?.

In this paper we characterize separability criteria for bitopological spaces.

2. Preliminaries

In this section, we outline the basic concepts which are useful in the sequel.

Definition 1. [14, Definition 1.1.1] Let X be a nonempty set. A collection of subsets of X denoted by τ is said to be a topology on X if and only if the following properties are satisfied:

(i). X and $\emptyset$ belong to τ.
(ii). Any arbitrary union of members of τ belongs to τ.
(iii). Any finite intersection of any two members of τ belongs to τ.

The ordered pair (X, τ) is called a topological space.

Definition 2. [15, Definition 2.1] Let X be a nonempty set and τ_1, τ_2 be topologies on X. Then (X, τ_1, τ_2) is said to be a bitopological space.

Definition 3. [16, Definition 1.7] Let (X, τ) be a topological space and (Y, τ_X) be a subspace of X which is equipped with a topology induced or inherited from that of X.

Definition 4. [17, Definition 2.4] Let (X, τ_1, τ_2, E) be a soft bitopological space over X and Y be a nonempty subset of X. Then $\tau_{1Y} = \{(Y, F, E) : (F, E) \in \tau_1\}$ and $\tau_{2Y} = \{(Y, G, E) \in \tau_2\}$ are said to be relatively topologies on Y. Therefore, $\{Y, \tau_{1Y}, \tau_{2Y}, E\}$ is called a relatively soft bitopological space of (X, τ_1, τ_2).

Example 1. [18, Example 1.1.2] Let $X = \{a, b, c\}$. Let the collection of subsets of X is denoted by topology τ. Then $\tau = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{X\}, \emptyset\}$. Then τ is a topology on X if it satisfies conditions (i), (ii) and (iii) in Definition 1.

For condition (i) both X and ϕ are in τ.

For condition (ii) $\{a, c\} \cup \{b, c\} = \{a, b, c\} = X$, which is a member of τ.

For condition (iii) $\{a\} \cap \{a, c\} = \{a\}$, which is a member of τ.

Definition 5. [19, Definition 2.4] A mapping $f : (X, \tau_1, \tau_2) \to (X_1, \tau_3, \tau_4)$ is called P-continuous (respectively open, P-closed) if the induced mapping $f : (X, \tau_1) \to (X_1, \tau_3)$ and $f : (X, \tau_2) \to (X_1, \tau_4)$ are continuous (respectively open, closed).

Definition 6. [20, Definition 4.1] Let X and Y be N-topological space. A function $f : X \to Y$ is said to be N^*-continuous on X if the inverse image of every $N\tau$-open set in Y is a $N\tau$-open set in X.

Definition 7. [5, Definition 2.5] A bitopological space (X, τ_1, τ_2) is called T_0 space if $\forall\ x, y \in X$ with $x \neq y$. Then there exists $U \in \tau_1 \cup \tau_2$ such that $x \in U$, y does not exists in U or x does not exists in Y, $y \in U$.

Example 2. [14, Example 4.1.2] Consider $X = \{a, b, c, e, f\}$, $Y = \{b, c, e\}$ and $\tau = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\}$ and $Y = \{b, c, e\}$. Therefore, (Y, τ) is a subspace of (X, τ). Since topology τ on Y is induced from X then all properties of a topological space (X, τ) has are also in a subspace (Y, τ). This subspace is denoted as $\tau_Y = Y \mid \tau_X$.

Definition 8. [6, Definition 1.2] A bitopological space (X, τ_1, τ_2) is a space that is endowed with two independent topologies say τ_1 and τ_2 denoted as (X, τ_1, τ_2).

Example 3. [21, Example 3.2] Let $X = \{a, b, \}, \tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$ and $\tau_\beta = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b\}\}$ then $\{a, c\}$ is connected.

Definition 9. [5, Definition 2.6] A bitopological space (X, τ_1, τ_2) is called T_1 space if $\forall\ x, y \in X$ with $x \neq y$. Then there exists $U \in \tau_1$ and $V \in \tau_2$ such that $x \in U$, y does not exists in U and x does not exists in U and $y \in U$.

Example 4. [19, Example 1] Consider $X = \{a, b, c\}$ with topologies $P = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$ and $Q = \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, x\}$ defined on X. Therefore, P-closed subsets of X are $\emptyset, \{a, c\}, \{a, b\}, \{a\}$ and X. Q-closed subsets of X are $\emptyset, \{b, c\}, \{a, c\}, \{c\}, \{a\}$ and X. It follows that (X, P, Q) does satisfy the condition P-normal. Hence (X, P, Q) is P-normal.

Definition 10. [22, Definition 3.19] A bitopological space (X, τ_1, τ_2) is said to be:

(i). Pairwise S-closed if every pairwise regular, pairwise closed cover of (X, τ_1, τ_2) has a finite subcover.
(ii). Pairwise S-closed if every pairwise countable cover of (X, τ_1, τ_2) by pairwise regular closed sets with respect to τ_1 and τ_2 has a finite subcover.
(iii). Pairwise S-Lindelof if every pairwise cover of (X, τ_1, τ_2) by pairwise regular closed sets with respect to τ_1 and τ_2 has a finite pairwise countable subcover.
(iv). Nearly pairwise compact if every pairwise regular open cover of (X, τ_1, τ_2) has a finite subcover.
(v). Pairwise countably nearly pairwise compact if every pairwise countable cover of (X, τ_1, τ_2) by pairwise regular open sets with respect to τ_1 and τ_2 has a finite subcover.
(vi). Nearly pairwise Lindelof if every pairwise regular open cover of (X, τ_1, τ_2) has a pairwise countable subcover.

Definition 11. [23, Definition 3.14] Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called $(1,2) - \gamma$ semi-continuous if the inverse image of each σ_1-open set in Y is $(1,2) - \gamma$ -semi-open set in X.

Remark 1. [24, Remark 5] Each of ij-irresoluteness ij-almost s-continuity and i-continuity is independent of one another and ij-almost s-continuity is not a generalization of i-continuity. Each of ij-quasi irresoluteness ij-semi-continuity and ij-almost continuity is independent of one another.

Definition 12. [25, Definition 2.1] A function $f : X \to Y$ from a topological space X to A topological space Y is said to be:

(i). Strongly continuous if $f(\overline{A}) \subset A$ for all $A \subset X$.
(ii). Perfectly continuous if $f^{-1}(V)$ is clopen in X for every open set $V \subset Y$.
(iii). σ-perfectly continuous if for each σ-open set V in Y, $f^{-1}(V)$ is a clopen set.

Definition 13. [26, Definition 1.4.1] A function f is continuous at some point $x \in X$ if and only if for any neighborhood V of $f(X)$ hence there is a neighborhood U of x such that $f(U) \subseteq V$.

Definition 14. [11, Definition 3.2] Let (X, τ_1, τ_2) be a topological space and N be a subset of X and p a point in N. Then N is said to be a neighborhood of the point p if there exists an open set U such that $p \in U \subseteq N$.

Example 5. [27, Example 2.3] Let $X = \{1,2,3\}$, $P = \{\emptyset, X, \{1,2\}, \{3\}\}$ and $Q = \{\emptyset, X, \{1\}, \{2,3\}\}$. It is easy to check that (X, P, Q) is pairwise T_0 but not weak pairwise T_1 considering the points 1, 2.

Definition 15. [28, Definition 2] The intersection (resp. union) of all (i, j) -δ-closed (resp. (i, j)-δ-open sets X containing (resp. contained in) $A \subset X$ is called the $(i,j) - \delta$-closure (resp. $(i,j) - \delta$-interior of an A.

Definition 16. [29, Definition 3] A subset of a bitopological space (X, τ_1, τ_2) is said to be (i, j)-δ-b-open. If $A \subset jCl(iInt - \delta(A)) \cup iInt(jCl_\delta(A))$, where $i \neq ji, j = 1,2$. The complement of an $(i,j) - \delta - b$-open set is called an $(i,j) - \delta - b$-closed set.

Example 6. [30, Example 7] Suppose that intersection (resp.union) for all $(i,j) - \delta - b$-closed $(resp.(i,j) - \delta - b$-open) sets of X containing (resp. contained in) $A \subset X$ is called the $(i,j) - \delta - b$-closure $(resp.(i,j) - \delta - b$-interior) of A and is denoted by $(i,j) - bCl_\delta(A)(resp.(i,j) - bInt_\delta(A))$ The intersection of all $(i,j) - b - \delta$-open sets of X containing A is called $(i,j) - b - \delta$-kernel of A and is denoted by $(i,j) - bKer_\delta(A)$.

Example 7. [19, Example 2] Consider $X = \{a,b,c,d\}$ with topologies $P = \{\emptyset, \{a,b\}, X\}$ and $Q = \{\emptyset, \{a\}, \{b,c,d\}, X\}$ defined on X. Observe that P-closed subsets of X are $\emptyset, \{c,d\}$ and X. Q-closed subsets of X are $\emptyset, \{b,c,d\}, \{a\}$ and X. Hence (X, P, Q) is $P1$-normal as we can check since the only pairwise closed sets of X are $\emptyset$ and X. However, (X, P, Q) is not P-normal since the P-closed set $A = \{c,d\}$ and Q-closed set $B = \{a\}$ satisfy $A \cap B = \emptyset$ but do not exists the Q-open set U and P-open set V such that $A \subseteq U, B \subseteq V$ and $U \cup V = \emptyset$.

Definition 17. [31, Definition 4.2] A space (X, τ_1, τ_2) is said to be pairwise R_0 if for every T_i-open set G, $x \in G$ implies $T_i - cl\{x\} \subset G$ i,j = 1, 2, $i \neq j$.

Remark 2. [24, Remark 1] Every ij-regular open set is ji semiregular.

Definition 18. [32, Definition 4.3] A space (X, τ_1, τ_2) is said to be pairwise R_0 if for every T_i-open set G, $x \in G$ implies that $T_j - \delta cl\{x\} \subset G$, i, j = 1, 2, $i \neq j$.

Example 8. [33, Example 3.12] Let $X = \{a,b,c\}$, $\tau_1 = \{X, \emptyset, \{a\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}, \{a,b\}, \{a,c\}\}$. Clearly, $(X, \tau_1) = (X, \tau_2) = (X, \tau_1, \tau_2) = \tau_2$. Thus (X, τ_1, τ_2) is a a quasi-$b - T_{\frac{1}{2}}$ space that is not quasi-$b - T_1$.

Definition 19. [19, Definition 3.6] A space (X, τ_1, τ_2) is called:

(i). Pairwise pre-T_0 (resp. pairwise pre-T_1) if for any pair of distinct points x and y in X, there exists a T_i-preopen set which contains one of them but not the other $i = 1$ or 2 resp. there exists T_i-preopen set U and T_j-preopen set V such that $x \in U, y \in V$ and $U \cap V = \emptyset i, j = 1, 2, i \neq j$.
(ii). A space (X, τ_1, τ_2) is said to be pairwise pre-T_2 if for any pair of distinct points x and y in X, there exists τ_i-preopen set U and τ_j-preopen set V such that $x \in U, y \in V$ and $U \cap V \neq \emptyset$ i, j= 1, 2, where $i \neq j$.

Example 9. [34, Example 2] Let $X = \{a, b, c, d, e\}, \tau_1 = \{\emptyset, X\}, \tau_2 = \{\emptyset, X, \{b, e\}\}$, $Y = \{a, b, c, d\}, \sigma_1 = \{\emptyset, Y, \{c\}\}$ and $\sigma_2 = \{\emptyset, Y, \{b, d\}\}$. Suppose $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \tau_2)$ is defined by $f(a) = f(c) = f(d), f(b) = f(c), f(b) = f(e) = b$. Then the map f is 12-preweak continuous but not 12-preweak semicontinuous since $f^{-1}(\{c\}) = \{a, c, d\}$ which is not 1-open set in the subspace $2 - cl(1 - int(2 - cl(f^{-1}(2 - cl(\{c\})))))$.

Definition 20. [35, Definition 3.19] A bitopological space (X, τ_1, τ_2) is said to be:

(i). Pairwise S-closed if every pairwise regular, pairwise closed cover of (X, τ_1, τ_2) has a finite subcover.
(ii). Pairwise S-closed if every pairwise countable cover of (X, τ_1, τ_2) by pairwise regular closed sets with respect to τ_1 and τ_2 has a finite subcover.
(iii). Pairwise S-Lindelöf if every pairwise cover of (X, τ_1, τ_2) by pairwise regular closed sets with respect to τ_1 and τ_2 has a finite pairwise countable subcover.
(iv). Nearly pairwise compact if every pairwise regular open cover of (X, τ_1, τ_2) has a finite subcover.
(v). Pairwise countably nearly pairwise compact if every pairwise countable cover of (X, τ_1, τ_2) by pairwise regular open sets with respect to τ_1 and τ_2 has a finite subcover.
(vi). Nearly pairwise Lindelöf if every pairwise regular open cover of (X, τ_1, τ_2) has a pairwise countable subcover.

Definition 21. [36, Definition 3.1] Let (X, τ_1, τ_2) be a bitopological space. Then, we say that a subset A of X is τ_1 semi open with respect to τ_2 if and only if there exists a τ_2 open set O such that $O \subset A \subset \tau_2$ closure O.

Definition 22. [37, Definition 2.4] The quasi b-closure of a subset A of (X, τ_1, τ_2) is defined to be $qbCl(A) = \cap F : F \in QBC(X, \tau_1, \tau_2), A \subseteq F$. A subset A of (X, τ_1, τ_2) is quasi-b-generalized closed (simply, quasi-bg-closed) if $qbCl(A) \subseteq U$.

Definition 23. [38, Definition 2.3] A bitopological space (X, τ_1, τ_2) is said to be i-Lindelöf if both the topological space (X, τ_i) is Lindeloöf. X is called Lindelöf if it is both 1-Lindelöf and 2-Lindelöf. Equivalently, (X, τ_1, τ_2) is Lindelöf if every i-open cover of X has a countable subcover for each $i = 1, 2$.

Definition 24. [17, Definition 5] The union of two soft sets of (F, A) and (G, B) over the common universe U is the soft set (H, C), where $C = A \cup B$ and for all $e \in E$. This can be denoted as $(F, A) \cup (G, B) = (H, C)$.

Example 10. [4, Definition 3.1] A space (X, τ_1, τ_2) is quasi-$b - T_0$ if for every two distinct points x, y of X, there exists $A \in QBO(X, \tau_1, \tau_2)$ such that $x \in A$ and y not in A or x not in A and $y \in A$.

Definition 25. [3, Definition 3.2] A binary soft topological space (U_1, U_2, τ_b, E) is said to be binary soft $n - T_0$ if for any pair (x_1, y_1), $(x_1, y_2) \in U_1 \times U_2$ of distinct points that is, $x_1 \neq x_2$ and $y_1 \neq y_2$ there exists at least one binary soft open set (F, E) or (G, E) such that $(x_1, y_1) \in (F, E)$, (x_2, y_2) does not exists in (G, E) or $(x_2, y_2) \in (G, E)$ and (x_1, y_1) does not exists in (G, E).

Definition 26. [33, Definition 3.4] A space (X, τ_1, τ_2) is quasi-$b - T_2$ if for every two distinct points X, y of X, there exists disjoint sets $A, B \in QBO(X, \tau_1, \tau_2)$ such that $x \in A$ and $y \in B$.

Definition 27. [17, Definition 3] The intersection (H, C) of two sets (F, A) and (G, B) over a common universe U. Denoted as $(F, A) \cap (G, B)$, is defined as $C = A \cap B$. Also $H(e) = F(e) \cap G(e)$ for all $e \in C$.

Definition 28. [39, Definition 4.5] A space X is normal on a subset Y. Then if every two disjoint closed subsets F and G of X satisfying $F = \overline{F \cap Y}$ and $G = \overline{G \cap Y}$ can be separated in X by disjoint open sets. Suppose that we have a separable space X then the following conditions are equivalent:

(i). X is said to be normal on every dense countable subset.
(ii). Any two separable disjoint subspaces if X can be represented by disjoint open set.

Remark 3. [40, Definition 2.1] A bitopological space (X, P, Q) is weak pairwise T_0 if and only if each pair of distinct points, there is a set which is either p-open or q-open containing one but not the other.

Definition 29. [4, Definition 3.1] Let (X, τ_1, τ_2) be a topological space and $A \subset X$. Then A is said to be weakly quasi separated from set B if there exists a quasi open set G such that $A \subset G$ and $G \cap B = \emptyset$ or $A \cap qcl(B) = \emptyset$.

Definition 30. [41, Definition 2.2] Let U_1, U_2 be two initial universe sets with powers $P(U_1)$ and $P(U_2)$ respectively and E be a set of parameters. A pair (F, E) is said to be a binary soft set over U_1 and U_2 where F is defined as $F : E \to P(U_1) \times P(U_2)$, $F(e) = (X, Y)$ for each $e \in E$ such that $X \subset U_1$, $Y \subset U_2$.

3. Main results

Proposition 1. *Let (X, τ_1, τ_2) be a T_0 space then the property of T_0 is hereditary and topological.*

Proof. We prove that T_0 has hereditary property. Let (X, τ_1, τ_2) be a T_0 space and let $D \subseteq X$. Then we show that a bitopological subspace $(D, \tau_{D1}, \tau_{D2})$ is also a T_0 space. Since $(D, s\tau_{D1}, \tau_{D2})$ has induced properties from (X, τ_1, τ_2) then it implies that $a, b \in D$ with $a \neq b$, then $a, b \in X$ with $a \neq b$ as in Definition 7. Since (X, τ_1, τ_2) is a T_0 space then $\exists U \in \tau_1 \cup \tau_2$ such that $a \in U$, b does not exists in U or a is not a member of U but $b \in U$. Then it follows that $U \in \tau_1 \cup \tau_2$ such that $U \in \tau_1$ or $U \in \tau_2$. Therefore, $U \cap D \in \tau_{D1}$ or $U \cap D \in \tau_{D2}$ and so $U \cap D \in \tau_{D1} \cap \tau_{D2}$. Again since $a, b \in D$ then $a \in U \cap D$, b does not exists in $U \cap D$ or a does not exists in $U \cap D$, and $b \in U \cap D$. Hence $(D, \tau_{D1}, \tau_{D2})$ is also a T_0 space. Secondly, we prove that T_0 has topological property. Let $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_3, \tau_4)$ be a homeomorphism and let (X, τ_1, τ_2) be a T_0 space we therefore show that (Y, τ_3, τ_4) is also a T_0 space. By Definition, $\chi : (X, \tau_1) \to (Y, \tau_3)$ and $\chi : (X, \tau_2) \to (Y, \tau_4)$ are continuous (open, closed, homeomorphism respectively). Let $b_1, b_2 \in Y$ with $b_1 \neq b_2$, since χ is an onto function then $\exists a_1, a_2 \in X$ with $\chi(a_1) = \chi(b_1)$ and $\chi(a_2) = b_2$ as Definition 5. Since χ is an injective function with $b_1 \neq b_2$ therefore it implies that $\chi(a_1) \neq \chi(a_2)$ hence $a_1 \neq a_2$. Since (X, τ_1, τ_2) is T_0 space and $a_1, a_2 \in X$ where $a_1 \neq a_2$ then it implies that there exists $U \in \tau_1 \cup \tau_2$ such that $a_1 \in U$, a_1 does not exists in U or a_1 does not exists in U, $a_2 \in U$ or $a_1 \in U$, a_2 does not exists in U. Then $U \in \tau_1 \cup \tau_2$ follows that $\chi(U) \in \chi(\tau_1 \cup \tau_2)$ since χ is open and continuous. By Tychonoff separation axiom, it implies that $\chi(U) \in \chi(\tau_1) \cup \chi(\tau_2) \in \tau_3 \cup \tau_4$. Also $a_1 \in U$ which implies that $\chi(a_1) \in \chi(U)$ or $b_1 \in \chi(U)$ and a_2 does not exists in U which imply that $\chi(a_2)$ does not exists in $\chi(U)$ or b_2 does not exists $\chi(U)$. For any $b_1, b_2 \in Y$ with $b_1 \neq b_2$, $\chi(U) \in \tau_3 \cup \tau_4$ is obtained such that $b_1 \in \chi(U)$, b_2 does not exists in $\chi(U)$. Therefore, (Y, τ_3, τ_4) is a T_0 space. Every homeomorphic image of T_0 space implies that it is topological property. □

Proposition 2. *Let (X, τ_1, τ_2) be a T_1 space then the property of T_1 is topological and hereditary.*

Proof. Suppose that T_1 has hereditary property then (X, τ_1, τ_2) is also T_1 space. Let $D \subseteq X$ and hence $(D, \tau_{D1}, \tau_{D2})$ is also T_1 space. Let $a, b \in D$ with $a \neq b$ and $a, b \in X$ with $a \neq b$. Since (X, τ_1, τ_2) is a T_1 space then $\exists U \in \tau_1 \cup \tau_2$ such that $a \in U$, b does not exists in U and a does not exists in V but $b \in U$. By Proposition 1, we have $U \in \tau_1 \cup \tau_2$. Then $U \in \tau_1$ or $U \in \tau_2$ with $U \cap D \in \tau_{D1}$ or $U \cap D \in \tau_{D2}$ also $U \cap D \in \tau_{D1} \cap \tau_{D2}$. Since $a, b \in D$ hence $a \in U \cap D$, b does not exists in $U \cap D$ or a does not exists in $U \cap D$, $b \in U \cap D$. Therefore, $(D, \tau_{D1}, \tau_{D2})$ is also a T_1 space. Secondly, we show that T_1 space has a topological property. Let $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_3, \tau_4)$ be a homeomorphism and (X, τ_1, τ_2) be T_0 space. Then by hypothesis (Y, τ_3, τ_4) is a T_1 space. Let $b_1, b_2 \in Y$ where $b_1 \neq b_2$. Suppose that χ is surjective then it follows that there exists $a_1, a_2 \in X$ with $\chi(a_1) = \chi(b_1)$ and $\chi(a_2) = b_2$. Hence χ is also one to one function with $b_1 \neq b_2$ this implies that $\chi(a_1) \neq \chi(a_2)$ hence $a_1 \neq a_2$. Since (X, τ_1, τ_2) is a T_1 space and $a_1, a_2 \in X$, with $a_1 \neq a_2$. Then $\exists U \in \tau_1 \cup \tau_2$ such that $a_1 \in U$, a_1 does not exists in U or a_1 does not exists in U, $a_2 \in U$. Since $a_1 \in U$, a_2 does not exists in U then it implies that $U \in \tau_1 \cup \tau_2$. Therefore, $\chi(U) \in \chi(\tau_1 \cup \tau_2)$. By Tychonoff Theorem, χ is open and continuous then $\chi(U) \in \chi(\tau_1) \cup \chi(\tau_2) \in \tau_3 \cup \tau_4$. Similarly, $a_1 \in U$ hence it follows that $\chi(a_1) \in \chi(U)$ also $b_1 \in \chi(U)$ and a_2 does not exists in U which implies that $\chi(a_2)$ does not exists in $\chi(U)$, and so b_2 does not exists $\chi(U)$. Then for any $b_1, b_2 \in Y$ with $b_1 \neq b_2$ and $\chi(U) \in \tau_3 \cup \tau_4$ is obtained such that $b_1 \in \chi(U)$, b_2 does not exists in $\chi(U)$. Therefore, (Y, τ_3, τ_4) is also a T_1 space. Hence χ is continuous (open, closed and homeomorphism) if and only if the maps $\chi : (X, \tau_1) \to (Y, \tau_3)$ and $\chi : (X, \tau_2) \to (Y, \tau_4)$ are continuous (open, closed and homeomorphism respectively). By hypothesis, every homeomorphism image of T_1 space imply T_0 space. Therefore, a T_1 space is a topological property. □

Proposition 3. *Suppose that (X, τ_1, τ_2) is a T_2 space then the property of T_2 is topological and hereditary.*

Proof. Let (X, τ_1, τ_2) and (Y, τ_1, τ_2) be two bitopological spaces. If (X, τ_1, τ_2) is a T_2 space then it exhibits topological properties. Let $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_3, \tau_4)$ be a homeomorphism and (X, τ_1, τ_2) is also a T_2 space. Then we show that (Y, τ_3, τ_4) is also T_2 space. By Definition 27, let $b_1, b_2 \in Y$ with $y_1 \neq y_2$. Since all elements in Y are images of elements in X then χ is a surjective function. Then there exists $a_1, a_2 \in X$ with $\chi(a_1) = b_1$ and $\chi(a_2) = b_2$. Again since χ is an injective function then $b_1 \neq b_2$. This implies that $\chi(a_1) \neq \chi(a_2)$, and $a_1 \neq a_2$. Therefore, $a_1, a_2 \in X$ with $a_1 \neq a_2$. Consequently, since (X, τ_1, τ_2) is a T_2 space then it shows that $\exists U \in \tau_1$ and $V \in \tau_2$ such that $a_1 \in U$, $a_2 \in V$ then $U \cap V \neq \emptyset$. Suppose that χ is open then $\chi(U) \in \tau_3$ and $\chi(V) \in \tau_4$. Therefore, $\chi(U) \cap \chi(V) \neq \emptyset$ then there exists $c \in X$ such that $c \in \chi(U) \cap \chi(V)$. This shows that $c \in \chi(U)$ and $c \in \chi(V)$ then $\exists p_1 \in U$ and $p_2 \in V$ such that $c = \chi(p_1)$ and $c \in \chi(p_2)$ with $\chi(p_1) = \chi(p_2)$ and $p_1 = p_2$ since χ is a one to one function and so $p_1 \in U$ and $p_1 \in V$. hence $p_1 \in U \cap V \neq \emptyset$ which is by contradiction. Suppose that $U \cap V = \emptyset$ which implies that $\chi(U) \cap \chi(V) = \emptyset$. Therefore, for any $b_1, b_2 \in Y$ with $b_1 \neq b_2$ hence $\chi(U) = \tau_3$ and $\chi(V) \in c$ is obtained such that $b_1 \in \chi(U)$, $b_2 \in \chi(V)$ and so $\chi(U) \cap \chi(V) \neq \emptyset$. Hence (Y, τ_3, τ_4) is a T_2 space. So it implies that every homeomorphism image of a T_2 is a T_2 space. Then T_2 is a topological property. Let (X, τ_1, τ_2) be a T_2 space then it has hereditary property. Let (X, τ_1, τ_2) also be T_2 space. Since $D \subseteq X$, we prove that $(D, \tau_{D1}, \tau_{D2})$ is also T_2 space. Let $a, b \in D$ with $a \neq b$ and $a, b \in X$ and also $a \neq b$. By Definition 9, it follows that $\exists U \in \tau_1 \cup \tau_2$ such that $a \in U$, b does not exists in U and also a does not exists in U but $b \in U$. So $U \in \tau_1 \cup \tau_2$, it implies that $U \in \tau_1$ or $U \in \tau_2$ where $U \cap D \in \tau_{D1}$ or $U \cap D \in \tau_{D2}$. By Tychonoff theorem, $U \cap D \in \tau_{D1} \cap \tau_{D2}$. Again since $a, b \in D$ then $a \in U \cap D$, b does not exists in $U \cap D$ or a does not exists in $U \cap D$, $b \in U \cap D$. Therefore, $(D, \tau_{D1}, \tau_{D2})$ is also a T_2 space and has a topological property. □

Proposition 4. *Let (X, τ_1, τ_2) be a $T_{\frac{5}{2}}$ space then the property of $T_{\frac{5}{2}}$ is topological and hereditary.*

Proof. Let $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_3, \tau_4)$. By hypothesis, T_1 space imply T_2 space which also implies $T_{\frac{5}{2}}$ space. Therefore, we prove that $T_{\frac{5}{2}}$ space has hereditary property. Let (X, τ_1, τ_2) be a $T_{\frac{5}{2}}$ space and let $K \subseteq X$. Then $(K, \tau_{K1}, \tau_{K2})$ is a $T_{\frac{5}{2}}$ space since it is bitopological subspace of (X, τ_1, τ_2). Then it implies that $(K, \tau_{K1}, \tau_{K2})$ is a $T_{\frac{5}{2}}$ space. Let $m, n \in K$ with $m \neq n$ then if (X, τ_1, τ_2) is a $T_{\frac{5}{2}}$ space then $\exists A \in \tau_1$ and $B \in \tau_2$ such that $m \in A$, $n \in B$ such that the intersection of A and B is empty that is, $A \cap B = \emptyset$. Hence $A \in \tau_1$, $B \in \tau_2$ then it follows that $A \cap K \in \tau_{K1}$ and $B \cap K \in \tau_{K2}$. Therefore, $m, n \in K$ then $m \in A \cap K$, $n \in B \cap K$. So $(A \cap K) \cap (B \cap K) = (A \cap K) \cap K = \emptyset \cap K = \emptyset$, hence $(K, \tau_{K1}, \tau_{K2})$ is $T_{\frac{5}{2}}$ space. Then (X, τ_1, τ_2) is also a $T_{\frac{5}{2}}$ space so it also has a topological property. By hypothesis $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_3, \tau_4)$ and χ is a homeomorphic function then it follows that (Y, τ_3, τ_4) is also a $T_{\frac{5}{2}}$ space. Therefore, $n_1, n_2 \in Y$ with $n_1 \neq n_2$. Since χ is onto function then it implies that $\exists m_1, m_2 \in X$ with $\chi(m_1) = \chi(n_1)$ and $\chi(m_2) = n_2$. Suppose χ is injective with $n_1 \neq n_2$ then it implies that $\chi(m_1) \neq \chi(m_2)$ and so $m_1 \neq m_2$. Hence (X, τ_1, τ_2) is $T_{\frac{5}{2}}$ space then $m_1, m_2 \in X$, with $m_1 \neq m_2$ and $\exists A \in \tau_1 \cup \tau_2$ such that $m_1 \in A$, m_1 does not exists in A or m_1 does not exists in A, $a_2 \in A$. Similarly, $m_1 \in A$, m_2 does not exists in U therefore it implies that $A \in \tau_1 \cup \tau_2$ such that $\chi(A) \in \chi(\tau_1 \cup \tau_2)$. By condition for separation axioms, $\chi(A) \in \chi(\tau_1) \cup \chi(\tau_2) \in \tau_3 \cup \tau_4$, $m_1 \in A$ such that $\chi(m_1) \in \chi(A)$ hence $n_1 \in \chi(A)$ and m_2 does not exists in A and $\chi(m_2)$ does not exists in $\chi(A)$, this implies that n_2 does not exists $\chi(A)$ for any $n_1, n_2 \in Y$ with $n_1 \neq n_2$, $\chi(A) \in \tau_3 \cup \tau_4$ is obtained such that $n_1 \in \chi(A)$, n_2 does not exists in $\chi(A)$. Therefore, (Y, τ_3, τ_4) is also $T_{\frac{5}{2}}$ space. Each homeomorphic image of $T_{\frac{5}{2}}$ space is also $T_{\frac{5}{2}}$ space and hence it has topological property. □

Lemma 1. *Let (X, τ_1, τ_2) be a normal space then the property of normality imply topological and hereditary.*

Proof. Let (X, τ_1, τ_2) be a normal bitopological space. Then there exist two disjoint closed sets x and y with $x \neq y$ and two disjoint open sets say U and V such that $x \subset U$ and $y \subset V$. By Definition, two disjoint closed sets $x, y \in X$ this therefore implies that $x \in U$, y does not exists in U and x does not exists in V but $y \in V$. Since normal bitopological space implies T_2 space then we have $x, y \in X$ with $x \neq y$ then $\exists U \in \tau_1 \cup \tau_2$ such that $x \in U$, y does not exists in U and also x does not exists in V but $y \in V$ therefore normal spaces have topological property. Secondly, we prove that normality imply hereditary property. By Proposition 2, $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_3, \tau_4)$ and χ is homeomorphism if and only if it is a bijective function. Let $A \subseteq X$ and let (X, τ_1, τ_2) be a normal space then this implies that A is also normal. By conditions for normality, (X, τ_1, τ_2) is a normal space then $(A, \tau_{A1}, \tau_{A2})$ is also normal since disjoints closed sets $x, y \in A$ and $x, y \in X$ with $x \neq y$. Therefore, considering disjoint open sets U and V we have $U \in \tau_1$ and $V \in \tau_2$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Therefore, $U \in \tau_1$ and $V \in \tau_2$ then it implies that $U \cap A \in \tau_{A1}$ and $V \cap A \in \tau_{A2}$ hence it follows that $x \in \cap AU$, $y \in V \cap A$. Then it implies that $(U \cap A) \cap (V \cap A) \cap A = \emptyset \cap A = \emptyset$. By hypothesis, $(A, \tau_{A1}, \tau_{A2})$ is normal and has topological property induced from (X, τ_1, τ_2). □

Proposition 5. *A bitopological space (X, τ_1, τ_2) is $ij - \pi_\lambda - T_\lambda$ if and only if it is $ij - \pi_\lambda$-symmetric.*

Proof. Assume that (X, τ_1, τ_2) is $ij - \pi_\lambda - T_\lambda$. Let $x \in ij - Cl\pi_\lambda(\{y\})$ and U to be any $ij - \pi_\lambda$-open set such that $y \in U$ and $x \in U$. Therefore, this implies that every $ij - \pi_\lambda$-open set that is containing y also contains x with $x \neq y$ hence $y \in ij - Cl\pi_\lambda(\{x\})$. Conversely, let U be $ij - \pi_\lambda$-open in X hence it implies that $x \in U$, y does not exist in U, and x does not exists in $ij - Cl\pi_\lambda(\{y\})$. Then if y does not exist in $ij - Cl\pi_\lambda(\{x\})$ then $ij - Cl\pi_\lambda(\{x\}) \subseteq U$. Hence it implies that both U and $ij - Cl\pi_\lambda(\{y\})$ are disjoint open sets where $x \in U$ and y does not exists in U or $y \in ij - Cl\pi_\lambda(\{x\})$ and x does not exist in $ij - Cl\pi_\lambda(\{x\})$. By Proposition 1, (X, τ_1, τ_2) is $ij - \pi_\lambda - T_0$ since it has topological property. Therefore, (X, τ_1, τ_2) is $ij - \pi_\lambda - T_\lambda$. □

Proposition 6. *A bitopological space (X, τ_1, τ_2) is $ij - \pi_\lambda - T_0$ and $ij - \pi_\lambda - T_1$ if and only if it is $ij - \pi_\lambda - T_\lambda$ symmetric.*

Proof. Let (X, τ_1, τ_2) be $ij - \pi_\lambda - T_\lambda$ symmetric and let $x, y \in X$ with $x \neq y$. Since (X, τ_1, τ_2) $ij - \pi_\lambda - T_\lambda$ symmetric we need prove that it is also $ij - \pi_\lambda - T_0$. Let x and y be two disjoint closed sets in X. Then U and $ij - \pi_\lambda(\{y\})$ be any two disjoint open sets. By Definition 9, two disjoint closed sets x and y are both members of open sets either U or $ij - \pi_\lambda(\{y\})$. Hence suppose that each $ij - \pi_\lambda$-open set contains x and y then $y \in U$ and $x \in U$. Since U is a member of $ij - \pi_\lambda$-open set then it implies that $x \in U$ and y does not exists in U. By Tychonoff theorem, it follows that $ij - \pi_\lambda(\{x\}) \subseteq U$ hence y does not exists U. Hence it implies that y does not exists in $ij - \pi_\lambda(\{x\})$ thus by assumption x does not exists in $ij - \pi_\lambda(\{x\})$. Since $ij - \pi_\lambda(\{x\}) \subseteq U$ then (X, τ_1, τ_2) is $ij - \pi_\lambda - T_0$. Now, it implies that (X, τ_1, τ_2) is $ij - \pi_\lambda - T_\lambda$ symmetric. Therefore, every $ij - \pi_\lambda - T_\lambda$ symmetric imply $ij - \pi_\lambda - T_1$. Since (X, τ_1, τ_2) is $ij - \pi_\lambda - T_0$ we assume without any loss of generality that $x \in K \subset X\{y\}$ for $ij - \pi_\lambda$-open set K where x does not exists in $ij - Cl\pi_\lambda(\{y\})$ and y does not exists in $ij - Cl\pi(\{x\})$. Therefore, $X \setminus ij - Cl\pi(\{x\})$ is an $ij - \pi_\lambda$-open set containing y but not x. Hence (X, τ_1, τ_2) is $ij - \pi_\lambda - T_1$. □

Lemma 2. *A bitopological space (X, τ_1, τ_2) is $ij - \pi_\lambda - T_0$ if and if for any $x, y \in X$ and $ij - Cl\pi_\lambda(\{x\}) = ij - Cl\pi_\lambda(\{y\})$ then implies that $ij - Cl\pi_\lambda(\{x\}) \cap ij - Cl\pi_\lambda(\{y\}) = \emptyset$.*

Proof. Let (X, τ_1, τ_2) is $ij - \pi_\lambda - T_0$. Then we have disjoint closed sets x and y where $x \in X$ such that $ij - Cl\pi_\lambda(\{y\}) \neq ij - Cl\pi_\lambda(\{x\})$. Therefore, there exists $x \in ij - Cl\pi_\lambda(\{x\})$ such that x does not exists in $ij - Cl\pi_\lambda(\{y\})$ this implies that $y \in ij - Cl\pi_\lambda(\{y\})$ and n does not exists in $ij - Cl\pi_\lambda(\{x\})$. Since x is not a member of $ij - Cl\pi_\lambda(\{y\})$ therefore there exists $V \in ij - B\lambda O(X, x)$ such that y does not exists in V. However, $x \in ij - Cl\pi_\lambda(\{x\})$ hence $x \in V$. Therefore, this follows that x is not a member of $ij - Cl\pi_\lambda(\{y\})$. Then it implies that $x \in X \setminus ij - Cl\pi_\lambda(\{y\}) \in ij - B\lambda O(X)$. Since (X, τ_1, τ_2) is $ij - \pi_\lambda - T_0$ then $ij - Cl\pi_\lambda(\{x\}) \subset X \setminus ij - Cl\pi_\lambda(\{y\})$. By Proposition 6, we have $ij - Cl\pi_\lambda(\{x\}) \cap ij - \pi Cl_\lambda(\{y\}) = \emptyset$. Conversely, let $V \in ij - B\lambda O(X, x)$. We show that $ij - Cl\pi_\lambda(\{x\}) \subset V$. Let y not to be an element of V then it follows that $y \in X \setminus V$ hence $y = x$ and x does not exists in $ij - Cl\pi_\lambda(\{y\})$. This shows that $ij - Cl\pi_\lambda(\{y\}) \neq ij - Cl\pi_\lambda(\{x\})$. By assumption $ij - Cl\pi_\lambda(\{y\}) \cap ij - Cl\pi_\lambda(\{x\}) = \emptyset$. By hypothesis, y does not exists in $ij - Cl\pi_\lambda(\{x\})$ and so $ij - Cl\pi_\lambda(\{x\}) \subseteq V$. Therefore, (X, τ_1, τ_2) is $ij - \pi_\lambda - T_0$. □

Theorem 1. *A bitopological space (X, τ_1, τ_2) is $ij - \pi_\lambda - T_1$ if and only if for any points x and y in X, $ij - Ker\pi_\lambda(\{x\}) = ij - Ker\pi_\lambda(\{y\})$ implying that $ij - Ker\pi_\lambda(\{x\}) \cap ij - Ker\pi_\lambda(\{y\}) = \emptyset$.*

Proof. Let (X, τ_1, τ_2) be $ij - \pi_\lambda - T_1$. Then let disjoint closed sets $x, y \in X$. By hypothesis, $ij - Ker\pi_\lambda(\{x\}) \neq ij - Ker\pi_\lambda(\{y\})$ then it follows that $ij - \pi Cl_\lambda(\{y\}) \neq ij - \pi Cl_\lambda(\{x\})$. Suppose that $z \in ij - Ker\pi_\lambda(\{x\}) \cap ij - Ker\pi_\lambda(\{y\})$ then it implies that $z \in ij - Ker\pi_\lambda(\{x\})$. Therefore, it follows that $z \in ij - Cl\pi_\lambda(\{z\})$. Thus by Lemma 2, we have $ij - Cl\pi_\lambda(\{x\}) = ij - Cl\pi_\lambda(\{z\})$ this is by contraction. Hence $ij - Cl\pi_\lambda(\{y\}) = ij - Cl\pi_\lambda(\{z\}) = ij - Cl\pi_\lambda(\{x\})$. Therefore, $ij - Ker\pi_\lambda(\{x\}) \cap ij - \pi Ker_\lambda(\{y\}) = \emptyset$. Conversely, let (X, τ_1, τ_2) be a bitopological space such that disjoint closed points x and y are members of X. Then it implies that $ij - Ker\pi_\lambda(\{x\}) \neq ij - Ker\pi_\lambda(\{y\})$ hence $ij - Ker\pi_\lambda(\{x\}) \cap ij - Ker\pi_\lambda(\{y\}) = \emptyset$. Therefore, $ij - Ker\pi_\lambda(\{x\}) \neq ij - Cl\pi_\lambda(\{y\})$ then $\cap ij - Ker\pi_\lambda(\{y\}) = \emptyset$. By assumption we have $z \in ij - Ker\pi_\lambda(\{x\})$ and $x \in ij - Ker\pi_\lambda(\{y\})$ therefore $ij - Ker\pi_\lambda(\{x\}) \cap ij - Ker\pi_\lambda(\{z\}) = \emptyset$. Therefore, $ij - \pi Ker_\lambda(\{x\}) = ij - \pi Ker_\lambda(\{z\})$. Thus it follows that $z \in ij - Cl\pi_\lambda(\{x\}) = ij - Cl\pi_\lambda(\{y\})$. Then $ij - Ker\pi_\lambda(\{x\}) = ij - Ker\pi_\lambda(\{z\}) = ij - Ker\pi_\lambda(\{y\})$ this is by contradiction. Therefore, $ij - Ker\pi_\lambda(\{x\}) \neq ij - Ker\pi_\lambda(\{y\})$. Then this implies that $ij - Ker\pi_\lambda(\{x\}) \cap ij - Ker\pi_\lambda(\{y\}) = \emptyset$. Therefore, (X, τ_1, τ_2) is $ij - \pi_\lambda - T_1$. □

Proposition 7. *For bitopological space (X, τ_1, τ_2) the following are equivalent:*

(i). (X, τ_1, τ_2) *is* $ij - \pi_\lambda - T_1$.
(ii). *For each $x, y \in X$ then U is $ij - \pi_\lambda - T_1$-open then $x \in U$ if and only if $y \in U$. $x \in U$ and $y \in V$.*
(iii). *If $x, y \in X$ such that $ij - \pi - Cl_\lambda(\{x\}) \neq ij - \pi - Cl_\lambda(\{y\})$, then there exists closed sets F_1 and F_2 whereby $x \in F_1$, y does not exists in F_1, $y \in F_2$ x does not exists in F_2 and $X = F_1 \cup F_2$.*

Proof. Proving for $(i) \Rightarrow (ii)$. Let x, y be two closed disjoint sets in X. By Theorem 1, $ij - Cl\pi_\lambda(\{x\} = ij - Cl\pi_\lambda(\{y\}$ or $ij - Cl\pi_\lambda(\{x\} \neq ij - Cl\pi_\lambda(\{y\}$. Let U be $ij - \pi_\lambda$-open therefore $x \in U$ such that $y \in ij - Cl\pi_\lambda(\{x\} \subset U$. Since $x \in U$ then $x \in ij - \pi - Cl_\lambda(\{y\} \subset U$. Therefore, $ij - Cl\pi_\lambda(\{x\} \neq ij - Cl\pi_\lambda(\{y\}$. This implies that there exists disjoint $ij - \pi_\lambda$-open sets U and V such that $x \in ij - Cl\pi_\lambda(\{x\} \subset U$ and $y \in ij - Cl\pi_\lambda(\{y\} \subset V$.

For $(ii). \Rightarrow (iii)$. Let $x, y \in X$ such that $ij - Cl\pi_\lambda(\{x\} \neq ij - Cl\pi_\lambda(\{y\}$. Since x is not a member of $ij - Cl\pi_\lambda(\{y\}$ and y is not a member of $ij - Cl\pi_\lambda(\{x\}$ then x does not belongs to $ij - Cl\pi_\lambda(\{y\}$. Therefore, there exists $ij - \pi_\lambda$-open set A such that $x \in A$, y does not exists in A. By (ii) there exists disjoints $ij - \pi_\lambda$-open sets U and V such that $x \in U$, $y \in V$. Then it implies that $F_1 = X \setminus V$ and $F_2 = X \setminus U$ are $ij - \pi_\lambda$-closed sets such that $x \in F_1$, y does not exists in F_1, and $y \in F_2$, x does not exists in F_2 hence it follows that $X = F_1 \cap F_2$.

For $(iii) \Rightarrow (i)$. We need to show that (X, τ_1, τ_2) is $ij - \pi_\lambda - T_0$ space. Let U be an $ij - \pi_\lambda$-open set such that $x \in U$. Then $ij - \pi - Cl_\lambda(\{x\} \subset U$ this implies that $y \in ij - Cl\pi_\lambda(\{x\} \cap (X \setminus U)$. By (i) it implies that $ij - Cl\pi_\lambda(\{x\} \neq ij - Cl\pi_\lambda(\{y\} = ij - Cl\pi_\lambda(\{y\})$ then $y \in U$. By (iii), there exists $ij - \pi_\lambda$-closed sets F_1 and F_2 such that $x \in F_1$, y does not exists F_1 but $y \in F_2$, x does not exists in F_2 such that $X = F_1 \cup F_2$. Hence $y \in F_2 \setminus F_1 \setminus X \setminus F_1 \in ij - B\lambda O(X)$ and x does not exists in $X \setminus F_1$ this is by contradiction. Therefore, (X, τ_1, τ_2) is $ij - \pi_\lambda - T_0$ space. Let $p, q \in X$ such that $ij - Cl\pi_\lambda(\{p\} \neq ij - Cl\pi_\lambda(\{q\}$, then there are $ij - \pi_\lambda$-closed sets H_1 and H_2 such that $p \in H_1$, $q \in H_1$, q is not a member of H_1 and $q \in H_2$ while $p \in H_2$ and $X = H_1 \cap H_2$. Therefore, $p \in H_1 \setminus H_2$ and $q \in H_2 \setminus H_1$. Then $H_1 \setminus H_2$ and $H_2 \setminus H_1$ are disjoint $ij - \pi_\lambda$-open sets. Hence $ij - Cl\pi_\lambda(\{p\}) \subset H_1 \setminus H_2$ and $ij - Cl\pi_\lambda(\{q\}) \subset H_2 \setminus H_1$. □

Theorem 2. *Every normal $ij - \pi_\lambda - T_2$ bitopological space (X, τ_1, τ_2) is Hausdorff space.*

Proof. Let (X, τ_1, τ_2) be a normal bitopological space then we have disjoint closed sets x and y with $x \neq y$. Let U and V be disjoint open sets such that $x \subset U$ and $y \subset V$. By Definition 24, if there exist two disjoint closed sets $x, y \in X$ then it implies that $x \in U$, y is not a member of U and x does not exists in V but $y \in V$. By hypothesis, normal bitopological spaces are also T_2 spaces. Since $x, y \in X$ with $x \neq y$, then $\exists U \in \tau_1 \cup \tau_2$ such that $x \in U$, y does not exists in U and x does not exists in V but $y \in V$. By Lemma 1, if (X, τ_1, τ_2) is a normal space then $(A, \tau_{A1}, \tau_{A2})$ is also normal since $A \subseteq X$. Then there exists open disjoint sets U and V

where $U \in \tau_1$ and $V \in \tau_2$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Consequently, by conditions for normality it implies that $U \in \tau_1$ and $V \in \tau_2$ then $U \cap A \in \tau_{A1}$ therefore, $V \cap A \in \tau_{A2}$, and $x \in \cap AU$, $y \in V \cap A$. Then $(U \cap A) \cap (V \cap A) \cap A = \emptyset \cap A = \emptyset$. Since $(A, \tau_{A1}, \tau_{A2})$ is a bitopological subspace then it implies that it has a topological property. Let T_2 space be Hausdorff space then we have two closed sets x and y with $x \neq y$. By hypothesis, we have two disjoint open sets U and V such that $x \in U$, y does not exists in V and x does not exists in V but $y \in V$ Then it implies that (X, τ_1, τ_2) is $ij - \pi_\lambda - T_2$. Therefore, (X, τ_1, τ_2) is a Hausdorff space and every normal $ij - \pi_\lambda - T_2$ space is also Hausdorff space. □

Corollary 1. *The property of $ij - \pi_\lambda - T_2$ in bitopological space (X, τ_1, τ_2) is hereditary.*

Proof. By hypothesis, (X, τ_1, τ_2) is $ij - \pi_\lambda - T_2$. Let X be any set and $A \subset X$ then $(A, \tau_{A1}, \tau_{A2})$ is also T_2 space. Since $(A, \tau_{A1}, \tau_{A2})$ is a bitopological subspace therefore it inherits properties from (X, τ_1, τ_2). Let $m, n \in A$ with $m \neq n$ and $m, n \in X$ with $m \neq n$. Then it follows that we have disjoint open sets U and V. Therefore, it implies that there exists $U \in \tau_1 \cup \tau_2$ such that $m \in U$, n does not exists in V and m does not exists in V but $m \in V$. By Theorem 2, we have $U \in \tau_1 \cup \tau_2$. This implies that $U \in \tau_1$ or $U \in \tau_2$ such that $U \cap A \in \tau_{A1}$ or $U \cap A \in \tau_{A2}$. Hence by separation axioms it suffices that $U \cap A \in \tau_{A1} \cap \tau_{A2}$. Then $m, n \in A$ and $m \in U \cap A$, where n is not a member of $U \cap A$ or m is not a member of $U \cap A$, $n \in U \cap A$. Hence $(A, \tau_{A1}, \tau_{A2})$ implies T_2 space. Since $(A, \tau_{A1}, \tau_{A2})$ is a T_2 space then it implies that it is also $ij - \pi_\lambda - T_2$. Therefore, $(A, \tau_{A1}, \tau_{A2})$ is also $ij - \pi_\lambda - T_2$ space and so it is hereditary. □

Corollary 2. *The property of $ij - \pi_\lambda - T_2$ in bitopological space (X, τ_1, τ_2) is topological.*

Proof. Let $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_1, \tau_2)$. By hypothesis, (X, τ_1, τ_2) is a T_2 space. Let $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_3, \tau_4)$ be a homeomorphism. We have disjoint open sets $y_1, y_2 \in Y$ with $y_1 \neq y_2$. By Definition 11, all elements in Y are images of elements in X and so χ is a bijective function. Therefore, it implies that $\exists x_1, x_2 \in X$ with $\chi(x_1) = y_1$ and $\chi(x_2) = y_2$. Suppose χ is a one to one function with $y_1 \neq y_2$ therefore this implies that $\chi(x_1) \neq \chi(x_2)$, then it implies that $x_1 \neq x_2$ hence $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since (X, τ_1, τ_2) is a T_2 space then it implies that $\exists U \in \tau_1$ and $V \in \tau_2$ such that $x_1 \in U$, $x_2 \in V$ and so $U \cap V \neq \emptyset$. Since χ is open hence $\chi(U) \in \tau_3$ and $\chi(V) \in \tau_4$. By Tychonoff theorem, $\chi(U) \cap \chi(V) \neq \emptyset$ suppose that there exists $c \in X$ therefore $c \in \chi(U) \cap \chi(V)$. This shows that $c \in \chi(U)$ and $c \in \chi(V)$ then $\exists p_1 \in U$ and $p_2 \in V$ such that $c = \chi(p_1)$ and $c \in \chi(p_2)$ with $\chi(p_1 = \chi(p_2)$. Suppose that $p_1 = p_2$ then it implies that $p_1 \in U$ and $p_1 \in V$ so $p_1 \in U \cap V \neq \emptyset$. by contradiction if $U \cap V = \emptyset$ then $\chi(U) \cap \chi(V) = \emptyset$. For any $y_1, y_2 \in Y$ with $y_1 \neq y_2$ then $\chi(U) = \tau_3$ and $\chi(V) \in c$ is obtained such that $y_1 \in \chi(U)$, $y_2 \in \chi(V)$ and $\chi(U) \cap \chi(V) \neq \emptyset$. Therefore, (Y, τ_3, τ_4) is a T_2 space. Hence (X, τ_1, τ_2) and (Y, τ_3, τ_4) are $ij - \pi_\lambda - T_2$ spaces and so they are topological. □

Corollary 3. *The property of $ij - \pi_\lambda - T_{\frac{5}{2}}$ in bitopological space (X, τ_1, τ_2) is both topological and heredity.*

Proof. Let (X, τ_1, τ_2) be a $T_{\frac{5}{2}}$ space and let $M \subseteq X$. By hypothesis, it implies that a bitopological subspace $(M, \tau_{M1}, \tau_{M2})$ is also a $T_{\frac{5}{2}}$ space. By Proposition 4, we have $x \in M$ with $x \neq y$. Since (X, τ_1, τ_2) is a $T_{\frac{5}{2}}$ space it implies that there exists $A \in \tau_1$ and $B \in \tau_2$ such that $x \in A$, $y \in B$ and $A \cap B = \emptyset$ hence it implies that $A \in \tau_1$ and $B \in \tau_2$. By separation axioms technique, $A \cap M \in \tau_{M1}$ and $B \cap M \in \tau_{M2}$ therefore $x, y \in M$ hence $x \in A \cap M$ and $y \in B \cap M$. Then it follows that $(A \cap M) \cap (B \cap M) = (A \cap M) \cap M = \emptyset \cap M = \emptyset$. Therefore, $(M, \tau_{M1}, \tau_{M2})$ is $ij - \pi_\lambda - T_{\frac{5}{2}}$. Since (X, τ_1, τ_2) is an $ij - \pi_\lambda - T_{\frac{5}{2}}$ space then it has topological property. Let $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_3, \tau_4)$ and χ is homeomorphic. Since (X, τ_1, τ_2) is $ij - \pi_\lambda - T_{\frac{5}{2}}$ then it implies that (Y, τ_3, τ_4) is also an $ij - \pi_\lambda - T_{\frac{5}{2}}$ space. By hypothesis, $y_1, y_2 \in Y$ with $n_1 \neq y_2$. Suppose that χ is onto function then there exists $x_1, x_2 \in X$ such that $\chi(x_1) = \chi(y_1)$ and $\chi(y_2) = x_2$. Also if χ is an injective function with $y_1 \neq y_2$ it follows that $\chi(x_1) \neq \chi(x_2)$ and $x_1 \neq x_2$. Therefore, since (X, τ_1, τ_2) is $ij - \pi_\lambda - T_{\frac{5}{2}}$ space then it implies that $x_1, x_2 \in X$, with $x_1 \neq x_2$ and so $\exists A \in \tau_1 \cup \tau_2$ hence $x_1 \in A$, x_1 does not exists in A or x_1 does not exists in A, $x_2 \in A$ similarly, $x_1 \in A$, x_2 does not exists in A hence $A \in \tau_1 \cup \tau_2$. Then $\chi(A) \in \chi(\tau_1 \cup \tau_2)$ since χ is open we have $\chi(A) \in \chi(\tau_1) \cup \chi(\tau_2) \in \tau_3 \cup \tau_4$. Since $x_1 \in A$ then it implies that $\chi(x_1) \in \chi(A)$ so $y_1 \in \chi(A)$ and x_2 does not exists in A which implies that $\chi(x_2)$ does not exists in $\chi(A)$ and y_2 does not exists $\chi(A)$. For any $y_1, y_2 \in Y$ with $y_1 \neq y_2$, $\chi(A) \in \tau_3 \cup \tau_4$ is obtained such that $y_1 \in \chi(A)$ and y_2 does not exists in $\chi(A)$. Hence (Y, τ_3, τ_4) is a $ij - \pi_\lambda - T_{\frac{5}{2}}$ space. Therefore, $ij - \pi_\lambda - T_{\frac{5}{2}}$ space is both topological and hereditary. □

Theorem 3. *A bitopological space (X, τ_1, τ_2) is pairwise λT_0 if and only if there is either closed distinct point of X either $\tau_1 - \eta$ or $\tau_2 - \eta$.*

Proof. Let $x, y \in X$ be two distinct points in X. By hypothesis there exist two disjoint open sets U and V. Then U is a $\tau_1 - \eta$-open set containing x but not y. Similarly, V is a $\tau_2 - \eta$-open set that contains y but not x. Thus by Definition 17, $y \in \tau_1 - \eta cl\{y\} \subset X - U$ and so it implies that x does not belongs to $\tau_1 - \eta cl\{y\}$. By Proposition 7, it follows that $\tau_1 - \eta cl\{x\} \neq \tau_1 - \eta cl\{y\}$ as x, y are distinct points in X. Therefore $\tau_1 - \eta cl\{x\} \neq \tau_1 - \eta cl\{y\}$ and $\tau_2 - \eta cl\{x\} \neq \tau_2 - \eta cl\{y\}$. Suppose that p is a point of X such that $p \in \tau_1 - \eta cl\{y\}$ then p does not belongs to $\tau_1 - \eta cl\{x\}$. This implies that y does not belongs to $\tau_1 - \eta cl\{x\}$. Hence if $y \in \tau_1 - \eta cl\{x\}$ then $\tau_1 - \eta cl\{y\} \subset \tau_1 - \eta cl\{x\}$ and $p \in \tau_1 - \eta cl\{y\} \subset \tau_1 - \eta cl\{x\}$. By contradiction p does not belongs to $\tau_1 - \eta cl\{x\}$ and so $p \in \tau_1 - \eta cl\{y\} \subset \tau_1 - \eta cl\{x\}$. This contradicts the fact that p is not a member of $\tau_1 - \eta cl\{x\}$ hence y also does not belongs to $\tau_1 - \eta cl\{x\}$. Therefore, $U = X - \tau_1 - \eta cl\{x\}$ is a $\tau_1 - \eta$-open set containing y but not x. This implies that $\tau_2 - \eta cl\{x\} \neq \tau_2 - \eta cl\{y\}$. Hence $\tau_1 - \eta$ and $\tau_2 - \eta$ closures are distinct points of X. □

Theorem 4. *A bitopological space (X, τ_1, τ_2) is pairwise λT_0 if either (X, τ_1) or (X, τ_2) is λT_0.*

Proof. Let (X, τ_1, τ_2) be a bitopological space. Then (X, τ_1, τ_2) is pairwise λT_0 if and only if either (X, τ_1) or (X, τ_2) is λT_0. By hypothesis, there are two disjoint closed sets x and y which are members of X. Similarly, there are two disjoint open sets U and V. By Theorem 3, U is a $\tau_1 - \eta$-open set containing x but not y hence $y \in \tau_1 - \eta cl\{y\} \subset X - U$ and x does not belongs to $\tau_1 - \eta cl\{y\}$. Since topological spaces imply bitopological spaces we have $\tau_1 - \eta cl\{x\} \neq \tau_1 - \eta cl\{y\}\tau_1 - \eta$ and $\tau_2 - \eta$ closures are distinct points. Conversely, this need not to be true in general. Let $X = \{a, b, c\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{b, c\}\}$ and $\tau_2 = \{X, \emptyset, \{c\}, \{a, b\}\}$. Therefore, it shows that a bitopological space (X, τ_1, τ_2) is pairwise λT_0 when neither (X, τ_1) nor is (X, τ_2) is λT_0. □

Theorem 5. *A bitopological space (X, τ_1, τ_2) is $ij - \lambda T_0$ if it is normal.*

Proof. Let (X, τ_1, τ_2) be a normal space. By hypothesis, it follows that there exists two disjoint closed sets a and b where $a \neq b$. Similarly, this implies that there exists two disjoint open sets M and N such that $a \subset M$ and $b \subset N$. Then $a, b \in X$ and so $a \in M$ where a does not exists in N but $b \in N$ for all $a, b \in X$ with $a \neq b$. By conditions for normality, there exists $M \in \tau_1 \cup \tau_2$ such that $a \in M$, b does not exists in U also a does not exists in N but $b \in N$. This shows that (X, τ_1, τ_2) is a normal space. Since (X, τ_1, τ_2) is λT_0 then it implies that it is $ij - \lambda T_0$-normal. Thus, by Definition 18, M is a $\tau_1 - \eta$-open set containing a but does not contain b. Therefore, it implies that $b \in \tau_1 - \eta cl\{b\} \subset X - M$ and so a does not belongs to $\tau_1 - \eta cl\{b\}$. By Tychonoff theorem, we have $\tau_1 - \eta cl\{a\} \neq \tau_1 - \eta cl\{b\}$. Since a, b are two distinct points of X then neither $\tau_1 - \eta cl\{a\} \neq \tau_1 - \eta cl\{b\}$ nor $\tau_2 - \eta cl\{a\} \neq \tau_2 - \eta cl\{b\}$. Let c to be any point of X such that $c \in \tau_1 - \eta cl\{b\}$, c does not belongs to $\tau_1 - \eta cl\{a\}$ and b does not belong to $\tau_1 - \eta cl\{a\}$. If $b \in \tau_1 - \eta cl\{a\}$ then $\tau_1 - \eta cl\{b\} \subset \tau_1 - \eta cl\{a\}$ and $c \in \tau_1 - \eta cl\{b\} \subset \tau_1 - \eta cl\{a\}$. By contradiction, since c does not belongs to $\tau_1 - \eta cl\{a\}$ then it implies that b also does not exists in $\tau_1 - \eta cl\{a\}$ thus $M = X - \tau_1 - \eta cl\{a\}$ is a $\tau_1 - \eta$-open set that contains b but not x. Hence it implies that $\tau_2 - \eta cl\{a\} \neq \tau_2 - \eta cl\{b\}$. Therefore, (X, τ_1, τ_2) is $ij - \lambda T_0$ and it implies that is a normal space. □

Corollary 4. *Every $ij - \pi_\lambda - \lambda T_2$ is $ij - \pi_\lambda - \lambda T_1$ and $ij - \pi_\lambda - \lambda T_0$.*

Proof. Let (X, τ_1, τ_2) be $ij - \pi_\lambda - \lambda T_2$. By hypothesis, (X, τ_1, τ_2) is pairwise λT_0. Let G be any $T_i - \pi_\lambda$-open set and hence $x \in G$ such that each point $y \in X - G$, $T_j - \pi Cl\{y\}$. Then it implies that there exists $T_i - \pi_\lambda$ open set U_y and any $T_j - \pi_\lambda$-open set V_y such that $x \in U_y$, $y \in V_y$. Therefore, by hypothesis we have $U_y \cap V_y = \emptyset$. If $A = \bigcup\{V_y : y \in X - G\}$ then $X - G \subset A$ and x does not exists in A. Therefore, $T_j - \pi_\lambda$ openness of A implies that $T_j - \pi Cl\{x\} \subset X - A \subset G$. Hence X is λT_0 and (X, τ_1, τ_2) is $ij - \pi\lambda - T_0$. By hypothesis, (X, τ_1, τ_2) is $ij - \pi_\lambda - \lambda T_0$ and there exists closed disjoint sets x and y with $x \neq y$ and $x \in ij - \pi Cl_\lambda(\{y\})$. By assumption

y does not exists in $ij - \pi Cl_{\lambda}(\{x\})$ then $ij - \pi Cl_{\lambda}(\{x\}) \subseteq U$. Thus this implies that (X, τ_1, τ_2) is $ij - \pi_{\lambda} - \lambda T_0$. Since (X, τ_1, τ_2) is $ij - \pi_{\lambda} - \lambda T_0$ then it is $ij - \pi_{\lambda} - \lambda T_2$. Therefore, $ij - \pi_{\lambda} - \lambda T_2$ imply $ij - \pi_{\lambda} - \lambda T_1$ which also imply $ij - \pi_{\lambda} - \lambda T_0$. □

Proposition 8. *A space (X, τ_1, τ_2) is pairwise $\Omega - T_1$ if and only if (X, τ_1) and (X, τ_2) are $\Omega - T_1$.*

Proof. Let (X, τ_1, τ_2) be pairwise $\Omega - T_1$ space. Let x and y be a pair of distinct points of set X. By Definition 19, there exists a τ_i-preopen set which is containing x but does not contain y. Since x and y are closed sets in X then it implies that $T_i - \Omega Cl\{x\} \neq T_j - \Omega Cl\{y\}$, $i, j = 1, 2$ with $i \neq j$. Therefore, there exists a $T_i - \Omega$-open set U and a $T_j - \Omega$-open set V such that $x \in V$, $y \in U$ and $U \cap V = \emptyset$. Therefore, it follows that $i = 1$ or 2 then there exist τ_i-preopen set U and τ_j-preopen set V. This therefore implies that $T_i - \Omega Cl\{x\} \subset V$ and $y \in U$ implies $T_j - \Omega\{y\} \subset U$, with $U \cap V = \emptyset$ and $i, j = 1, 2$ where $i \neq j$. Therefore, (X, τ_1) and (X, τ_2) are $\Omega - T_1$ and hence (X, τ_1, τ_2) is pairwise $\Omega - T_1$. □

Theorem 6. *Every bisemiopen subset of a pairwise p-regular space is pairwise p-regular.*

Proof. Let (X, τ_1, τ_2) be pairwise p-regular space and let $Y \subset X$ be a bisemiopen set. We show that subspace $(Y, \tau_{1Y}, \tau_{2Y})$ is pairwise p-regular. Let F be any τ_{iY}-closed set and so x does not belongs to F. Then there exists a τ_i-closed set A such that $F = A \cap Y$. Since (X, τ_1, τ_2) is a pairwise p-regular space and $x \in A$ then there exists $U \in PO(X, \tau_j)$ and $V \in PO(X, \tau_i)$ hence we have $A \subset U$, $x \in V$ and $U \cap V = \emptyset$ where $i, j = 1, 2$ with $i \neq j$. Now, it implies that we have $L = U \cap Y$ and $M = V \cap Y$. By separation axioms, it follows that $L \in PO(X, \tau_i)$, $M \in PO(Y, \tau_{iY})$. Hence $F \subset L$, $x \in M$ and so it implies that the intersection of L and M is emptyset that is, $L \cap M = \emptyset$. Therefore, $(Y, \tau_{1Y}, \tau_{2Y})$ is pairwise p-regular space. □

Theorem 7. *Every quasi $T_{\frac{7}{2}}$ space is quasi T_0.*

Proof. Let (X, τ_1, τ_2) be a quasi $T_{\frac{7}{2}}$ space. We have $x, y \in X$ with $x \neq y$. Hence it follows that $qKer(\{x\}) \cap qKer(\{y\}) = \emptyset$ since $\{x\}$ and $\{y\}$ are disjoint open sets with $\{x\} \neq \{y\}$. If $qKer(\{x\}) \cap qKer(\{y\}) = \emptyset$ then it implies that $T_{\frac{7}{2}}$ is a quasi space. Since $\{x\} \neq \{y\}$ then by separation axioms it suffices that $qKer(\{x\}) \neq qKer(\{y\})$. Therefore, it implies that (X, τ_1, τ_2) is quasi T_0. Since (X, τ_1, τ_2) is quasi T_0 then it is also quasi $T_{\frac{7}{2}}$. □

Theorem 8. *A bitopological space (X, τ_1, τ_2) is quasi $T_{\frac{7}{2}}$ if and only if $(qcl(\{y\})) = \cap\{y\} \bigcup (qcl(\{y\})) \cap \{x\}$ is degenerate.*

Proof. Let X be quasi $T_{\frac{7}{2}}$. Therefore, it shows that we have any of the two cases either x is weakly quasi separated from y or y weakly quasi separated from x. Suppose that x is weakly quasi separated from y then it implies that we have $\{x\} \cap qcl\{y\} = \emptyset$ and $\{y\} \cap qcl(\{x\})$ is called a degenerated set. Similarly, if y is weakly quasi separated from x then $\{y\} \cap qcl(\{x\}) = \emptyset$ and $\{x\} \cap qcl(\{y\})$ is also a degenerated set. By Definition 29, it suffices that $(qcl(\{x\}) \cap \{y\}) \bigcup (qcl(\{x\}) \cap \{y\})$ is a degenerate set. Hence $(qcl(\{x\}) \cap \{y\}) \bigcup (qcl(\{x\}) \cap \{y\})$ is a degenerate set. By separation axioms, it implies that it is either an empty or singleton set. Then suppose that it is a singleton then its value is either $\{x\}$ or $\{y\}$. Therefore, if it is $\{x\}$ then y is weakly quasi separated from x. Also if it is $\{y\}$ it is then x is weakly separated from y. Hence (X, τ_1, τ_2) is quasi $T_{\frac{7}{2}}$. □

4. Conclusion

In this paper, we have given necessary conditions and characterized separation criteria for bitopological spaces via ij-continuity. We have shown that if a bitopological space is a separation axiom space, then that separation axiom space exhibits both topological and heredity properties. For instance, let (X, τ_1, τ_2) be a T_0 space then, the property of T_0 is topological and hereditary. Similarly, when (X, τ_1, τ_2) is a T_1 space then the property of T_1 is topological and hereditary. Lastly, we have shown that separation axiom T_0 implies separation axiom T_1 which also implies separation axiom T_2 and the converse is true.

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