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Simpson's type inequalities for exponentially convex functions with applications

Yenny Rangel-Oliveros[1] and Eze R. Nwaeze[2,*]

[1] Facultad de Ciencias Exactas y Naturales, Escuela de Ciencias Físicas y Matemática, Pontificia Universidad Católica del Ecuador, Quito 170143, Ecuador.

[2] Department of Mathematics and Computer Science, Alabama State University, Montgomery, AL 36101, USA.

* Correspondence: enwaeze@alasu.edu

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Abstract: The Simpson's inequality cannot be applied to a function that is twice differentiable but not four times differentiable or have a bounded fourth derivative in the interval under consideration. Loads of articles are bound for twice differentiable convex functions but nothing, to the best of our knowledge, is known yet for twice differentiable exponentially convex and quasi-convex functions. In this paper, we aim to do justice to this query. For this, we prove several Simpson's type inequalities for exponentially convex and exponentially quasi-convex functions. Our findings refine, generalize and complement existing results in the literature. We regain previously known results by taking $\alpha = 0$. In addition, we also show the importance of our results by applying them to some special means of positive real numbers and to the Simpson's quadrature rule. The obtained results can be extended for different kinds of convex functions.

Keywords: Simpson inequalities; Convex functions; Exponentially convex functions; Exponentially quasi-convex functions.

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1. Introduction

Suppose $g : [a_1, a_2] \to \mathbb{R}$ is a four times continuously differentiable mapping on (a_1, a_2) and $||g^{(4)}||_\infty = \sup_{x\in(a_1,a_2)} |g^{(4)}(x)| < \infty$. Then the following inequality

$$\left|\frac{1}{3}\left[\frac{g(a_1)+g(a_2)}{2} + 2g\left(\frac{a_1+a_2}{2}\right)\right] - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} g(x)dx\right| \le \frac{1}{2880}||g^{(4)}||_\infty (a_2-a_1)^4$$

holds, and it is well known in the literature as Simpson's inequality, named after the English mathematician Thomas Simpson. If the mapping g is neither four times differentiable nor is the fourth derivative $g^{(4)}$ bounded on (a_1, a_2), then we cannot apply the classical Simpson quadrature formula. In [1] Dragomir *et al.*, proved some recent developments on Simpson's inequality for which the remainder is expressed in terms of lower derivatives than the fourth. For recent refinements, counterparts, generalizations and new Simpson's type inequalities, see [1–4].

The classical convexity of functions is a fundamental notions in mathematics, they have widely applications in many branches of mathematics and physics. A function $g : I \subset \mathbb{R} \to \mathbb{R}$ is said to be a convex in the classical sense, if

$$g(ta_2 + (1-t)a_1) \le tg(a_2) + (1-t)g(a_1)$$

for all $a_1, a_2 \in I$ and $t \in [0, 1]$. We call g concave if the inequality is reversed. A somewhat generalization of the above definition is given by Awan *et al.*, [5] as follows:

Definition 1 ([5]). A function $g : I \subset \mathbb{R} \to \mathbb{R}$ is called exponentially convex, if

$$g(ta_2 + (1-t)a_1) \leq t\frac{g(a_2)}{e^{\alpha a_2}} + (1-t)\frac{g(a_1)}{e^{\alpha a_1}} \tag{1}$$

for all $a_1, a_2 \in I$, $t \in [0,1]$, and $\alpha \in \mathbb{R}$. If (1) holds in the reversed sense, then g is said to be exponentially concave.

For example, the function $g : \mathbb{R} \to \mathbb{R}$, defined by $g(x) = -x^2$ is a concave function, thus this function is exponentially convex for all $\alpha < 0$. An exponentially convex function on a closed interval is bounded, it also satisfies the Lipschitzian condition on any closed interval $[a_1, a_2] \subset \mathring{I}$ (interior of I). Therefore an exponentially convex function is absolutely continuous on $[a_1, a_2] \subset \mathring{I}$ and continuous on $\mathring{I}$.

Recently, Nie *et al.*, [6] introduced the notion of exponentially quasi-convex as thus:

Definition 2 ([6]). Let $\alpha \in \mathbb{R}$. Then a mapping $g : I \subset \mathbb{R} \to \mathbb{R}$ is said to be exponentially quasi-convex if

$$g(ta_2 + (1-t)a_1) \leq \max\left\{\frac{g(a_2)}{e^{\alpha a_2}}, \frac{g(a_1)}{e^{\alpha a_1}}\right\}$$

for all $a_1, a_2 \in I$, and $t \in [0,1]$.

Remark 1. Note that if $\alpha = 0$, then the classes of exponentially convex and quasi-convex functions reduce to the classes of classical convex and quasi-convex function.

Inspired by the work of Sarikaya *et al.*, [7] and Vivas *et al.*, [8], we aim to establish new Simpson's type results for the class of functions whose derivatives in absolute value at certain powers are exponentially convex and exponentially quasi-convex. By taking $\alpha = 0$, we recapture some already established results in the literature. The main results are framed and justified in §2, followed by applications of our results to some special means in §3, and Simpson's quadrature in §4.

2. Main results

For the proof of our main results, the following lemma will be useful.

Lemma 1 ([7]). *Let $g : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I° such that $g'' \in L_1[a_1, a_2]$, where $a_1, a_2 \in I$ with $a_1 < a_2$. Then the following equality holds:*

$$\begin{aligned}&\frac{1}{6}\left[g(a_1) + 4g\left(\frac{a_1+a_2}{2}\right) + g(a_2)\right] - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} g(x)\,dx \\ &= (a_2-a_1)^2\int_0^1 k(t)g''(ta_2 + (1-t)a_1)\,dt,\end{aligned}$$

where

$$k(t) = \begin{cases} \frac{t}{2}\left(\frac{1}{3} - t\right) & \text{if} \quad t \in \left[0, \frac{1}{2}\right) \\ (1-t)\left(\frac{t}{2} - \frac{1}{3}\right) & \text{if} \quad t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

2.1. Simpson's inequality for exponentially convex

We now give a new refinement of Simpson's inequality for twice differentiable functions:

Theorem 1. *Let $g : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I° such that $g'' \in L_1[a_1, a_2]$, where $a_1, a_2 \in I$ with $a_1 < a_2$. If $|g''|$ is exponentially convex on $[a_1, a_2]$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[g(a_1) + 4g\left(\frac{a_1+a_2}{2}\right) + g(a_2)\right] - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} g(x)dx\right| \leq \frac{(a_2-a_1)^2}{162}\left[\frac{|g''(a_2)|}{e^{\alpha a_2}} + \frac{|g''(a_1)|}{e^{\alpha a_1}}\right].$$

Proof. Using Lemma 1 and since $|g''|$ is exponentially convex, we have

$$\begin{aligned}&\left|\frac{1}{6}\Big[g(a_1)+4g\Big(\frac{a_1+a_2}{2}\Big)+g(a_2)\Big]-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}g(x)dx\right|\\ \leq&(a_2-a_1)^2\int_0^1|k(t)|\,|g''(ta_2+(1-t)a_1)|\,dt\\ \leq&(a_2-a_1)^2\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\Big(t\frac{|g''(a_2)|}{e^{\alpha a_2}}+(1-t)\frac{|g''(a_1)|}{e^{\alpha a_1}}\Big)dt\\ &+(a_2-a_1)^2\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\Big(t\frac{|g''(a_2)|}{e^{\alpha a_2}}+(1-t)\frac{|g''(a_1)|}{e^{\alpha a_1}}\Big)dt\\ =&(a_2-a_1)^2(I_1+I_2),\end{aligned}$$

where

$$\begin{aligned}I_1=&\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left(t\frac{|g''(a_2)|}{e^{\alpha a_2}}+(1-t)\frac{|g''(a_1)|}{e^{\alpha a_1}}\right)dt\\ =&\frac{|g''(a_2)|}{e^{\alpha a_2}}\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|tdt+\frac{|g''(a_1)|}{e^{\alpha a_1}}\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|(1-t)\,dt\\ =&\frac{59}{31104}\frac{|g''(a_2)|}{e^{\alpha a_2}}+\frac{133}{31104}\frac{|g''(a_1)|}{e^{\alpha a_1}},\end{aligned}$$

and

$$\begin{aligned}I_2=&\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\Big(t\frac{|g''(a_2)|}{e^{\alpha a_2}}+(1-t)\frac{|g''(a_1)|}{e^{\alpha a_1}}\Big)dt\\ =&\frac{|g''(a_2)|}{e^{\alpha a_2}}\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|tdt+\frac{|g''(a_1)|}{e^{\alpha a_1}}\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|(1-t)dt\\ =&\frac{133}{31104}\frac{|g''(a_2)|}{e^{\alpha a_2}}+\frac{59}{31104}\frac{|g''(a_1)|}{e^{\alpha a_1}},\end{aligned}$$

which completes the proof. □

Corollary 1. *Let $g:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I° such that $g''\in L_1[a_1,a_2]$, where $a_1,a_2\in I$ with $a_1<a_2$. If $g(a_1)=g\Big(\frac{a_1+a_2}{2}\Big)=g(a_2)$ and $|g''|$ is exponentially convex on $[a_1,a_2]$, then the following inequality holds:*

$$\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2}g(x)dx-g\Big(\frac{a_1+a_2}{2}\Big)\right|\leq\frac{(a_2-a_1)^2}{162}\left[\frac{|g''(a_2)|}{e^{\alpha a_2}}+\frac{|g''(a_1)|}{e^{\alpha a_1}}\right].$$

Remark 2. In Theorem 1, by letting $\alpha=0$ we get [7, Theorem 2.2].

Theorem 2. *Let $g:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I° such that $g''\in L_1[a_1,a_2]$, where $a_1,a_2\in I$ with $a_1<a_2$. If $|g''|^q$ is exponentially convex on $[a_1,a_2]$ and $q\geq1$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{6}\Big[g(a_1)+4g\Big(\frac{a_1+a_2}{2}\Big)+g(a_2)\Big]-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}g(x)dx\right|\\ &\leq(a_2-a_1)^2\left(\frac{1}{162}\right)^{1-\frac{1}{q}}\left[\left(\frac{59}{31104}\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q+\frac{133}{31104}\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\right)^{\frac{1}{q}}\right.\\ &\quad\left.+\left(\frac{133}{31104}\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q+\frac{59}{31104}\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

Proof. Suppose that $q\geq1$. From Lemma 1, we have

$$\begin{aligned}
&\Big|\frac{1}{6}\Big[g(a_1)+4g\Big(\frac{a_1+a_2}{2}\Big)+f(a_2)\Big]-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}g(x)dx\Big|\\
&\leq (a_2-a_1)^2\int_0^1|k(t)|\left|g''(ta_2+(1-t)a_1)\right|dt\\
&=(a_2-a_1)^2\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left|g''(ta_2+(1-t)a_1)\right|dt\\
&\quad+(a_2-a_1)^2\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left|g''(ta_2+(1-t)a_1)\right|dt.
\end{aligned}$$

Using the Hölder's inequality for functions

$$\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|^{1-\frac{1}{q}}$$

and

$$\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|^{\frac{1}{q}}\left|g''(ta_2+(1-t)a_1)\right|,$$

for the first integral, and the functions

$$\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|^{1-\frac{1}{q}}$$

and

$$\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|^{\frac{1}{q}}\left|g''(ta_2+(1-t)a_1)\right|,$$

for the second integral, from the above relation we get the inequality:

$$\begin{aligned}
&\Big|\frac{1}{6}\Big[g(a_1)+4g\Big(\frac{a_1+a_2}{2}\Big)+g(a_2)\Big]-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}g(x)dx\Big|\\
&\leq (a_2-a_1)^2\Big(\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|dt\Big)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left|g''(ta_2+(1-t)a_1)\right|^q dt\right)^{\frac{1}{q}}\\
&\quad+(a_2-a_1)^2\Big(\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|dt\Big)^{1-\frac{1}{q}}\\
&\quad\times\Big(\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left|g''(ta_2+(1-t)a_1)\right|^q dt\Big)^{\frac{1}{q}}.
\end{aligned}$$

Using the fact that

$$\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|dt=\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|dt=\frac{1}{162}$$

and the exponentially convexity of $|g''|^q$, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left|g''(ta_2+(1-t)a_1)\right|^q dt\\
&\leq\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left[t\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q+(1-t)\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\right]dt\\
&=\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|tdt+\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|(1-t)dt\\
&=\frac{59}{31104}\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q+\frac{133}{31104}\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q
\end{aligned}\tag{2}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left|g''(ta_2+(1-t)a_1)\right|^q dt\\
&\leq \int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left[t\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q+(1-t)\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\right]dt\\
&=\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|tdt+\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|(1-t)dt\\
&=\frac{133}{31104}\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q+\frac{59}{31104}\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q.
\end{aligned} \tag{3}$$

From (2) and (3), we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[g(a_1)+4g\Big(\frac{a_1+a_2}{2}\Big)+\Psi(a_2)\right]-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}g(x)dx\right|\\
&\leq (a_2-a_1)^2\Big(\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|dt\Big)^{1-\frac{1}{q}}\left(\frac{59}{31104}\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q+\frac{133}{31104}\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\right)^{\frac{1}{q}}\\
&\quad+(a_2-a_1)^2\Big(\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|dt\Big)^{1-\frac{1}{q}}\left(\frac{133}{31104}\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q+\frac{59}{31104}\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\right)^{\frac{1}{q}}\\
&=(a_2-a_1)^2\left(\frac{1}{162}\right)^{1-\frac{1}{q}}\left[\left(\frac{59}{31104}\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q+\frac{133}{31104}\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\frac{133}{31104}\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q+\frac{59}{31104}\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof. □

Corollary 2. *Let $g: I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I° such that $g''\in L_1[a_1,a_2]$, where $a_1,a_2\in I$ with $a_1<a_2$. If $g(a_1)=g\Big(\frac{a_1+a_2}{2}\Big)=g(a_2)$ and $|g''|^q$ is exponentially convex on $[a_1,a_2]$ and $q\geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2}g(x)dx-g\Big(\frac{a_1+a_2}{2}\Big)\right|\\
&\leq (a_2-a_1)^2\left(\frac{1}{162}\right)^{1-\frac{1}{q}}\left[\left(\frac{59}{31104}\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q+\frac{133}{31104}\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\frac{133}{31104}\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q+\frac{59}{31104}\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Remark 3. By setting $\alpha=0$ in Theorem 2, we recapture [7, Theorem 2.5].

Corollary 3. *Let $g: I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I° such that $g''\in L_1[a_1,a_2]$, where $a_1,a_2\in I$ with $a_1<a_2$. If $g(a_1)=g\Big(\frac{a_1+a_2}{2}\Big)=g(a_2)$ and $|g''|^2$ is exponentially convex on $[a_1,a_2]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2}g(x)\,dx-g\Big(\frac{a_1+a_2}{2}\Big)\right|\\
&\leq (a_2-a_1)^2\left(\frac{1}{162}\right)^{\frac{1}{2}}\left[\left(\frac{59}{31104}\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^2+\frac{133}{31104}\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^2\right)^{\frac{1}{2}}\right.\\
&\quad\left.+\left(\frac{133}{31104}\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^2+\frac{59}{31104}\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^2\right)^{\frac{1}{2}}\right].
\end{aligned}$$

2.2. Simpson's inequality for exponentially quasi-convex

Theorem 3. *Let $g : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I° such that $g'' \in L_1[a_1, a_2]$, where $a_1, a_2 \in I$ with $a_1 < a_2$. If $|g''|$ is exponentially quasi-convex on $[a_1, a_2]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\Big[g(a_1) + 4g\Big(\frac{a_1+a_2}{2}\Big) + g(a_2)\Big] - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} g(x)dx\right| \\
&\leq \frac{(a_2-a_1)^2}{81}\max\left\{\frac{|g''(a_2)|}{e^{\alpha a_2}}, \frac{|g''(a_1)|}{e^{\alpha a_1}}\right\}.
\end{aligned}$$

Proof. From Lemma 1 and by using the exponentially quasi-convexity of $|g''|$, we get

$$\begin{aligned}
&\left|\frac{1}{6}\Big[g(a_1) + 4g\Big(\frac{a_1+a_2}{2}\Big) + g(a_2)\Big] - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} g(x)dx\right| \\
&\leq (a_2-a_1)^2 \int_0^1 |k(t)| \left|g''(ta_2 + (1-t)a_1)\right| dt \\
&\leq (a_2-a_1)^2 \int_0^{\frac{1}{2}} \left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right| \max\left\{\frac{|g''(a_2)|}{e^{\alpha a_2}}, \frac{|g''(a_1)|}{e^{\alpha a_1}}\right\} dt \\
&\quad + (a_2-a_1)^2 \int_{\frac{1}{2}}^1 \left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right| \max\left\{\frac{|g''(a_2)|}{e^{\alpha a_2}}, \frac{|g''(a_1)|}{e^{\alpha a_1}}\right\} dt \\
&= (a_2-a_1)^2 (I_1 + I_2),
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{2}} \left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right| \max\left\{\frac{|g''(a_2)|}{e^{\alpha a_2}}, \frac{|g''(a_1)|}{e^{\alpha a_1}}\right\} dt \\
&= \max\left\{\frac{|g''(a_2)|}{e^{\alpha a_2}}, \frac{|g''(a_1)|}{e^{\alpha a_1}}\right\} \int_0^{\frac{1}{2}} \left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right| dt \\
&= \frac{1}{162}\max\left\{\frac{|g''(a_2)|}{e^{\alpha a_2}}, \frac{|g''(a_1)|}{e^{\alpha a_1}}\right\},
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \int_{\frac{1}{2}}^1 \left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right| \max\left\{\frac{|g''(a_2)|}{e^{\alpha a_2}}, \frac{|g''(a_1)|}{e^{\alpha a_1}}\right\} dt \\
&= \max\left\{\frac{|g''(a_2)|}{e^{\alpha a_2}}, \frac{|g''(a_1)|}{e^{\alpha a_1}}\right\} \int_{\frac{1}{2}}^1 \left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right| dt \\
&= \frac{1}{162}\max\left\{\frac{|g''(a_2)|}{e^{\alpha a_2}}, \frac{|g''(a_1)|}{e^{\alpha a_1}}\right\},
\end{aligned}$$

which completes the proof. □

Corollary 4. *Let $g : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I° such that $g'' \in L_1[a_1, a_2]$, where $a_1, a_2 \in I$ with $a_1 < a_2$. If $g(a_1) = g\Big(\frac{a_1+a_2}{2}\Big) = g(a_2)$ and $|g''|$ is exponentially quasi-convex on $[a_1, a_2]$, then the following inequality holds:*

$$\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2} g(x)dx - g\Big(\frac{a_1+a_2}{2}\Big)\right| \leq \frac{(a_2-a_1)^2}{81}\max\left\{\frac{|g''(a_2)|}{e^{\alpha a_2}}, \frac{|g''(a_1)|}{e^{\alpha a_1}}\right\}.$$

Theorem 4. *Let $g : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I° such that $g'' \in L_1[a_1, a_2]$, where $a_1, a_2 \in I$ with $a_1 < a_2$. If $|g''|^q$ is exponentially quasi-convex on $[a_1, a_2]$ and $q \geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\Big[g(a_1) + 4g\Big(\frac{a_1+a_2}{2}\Big) + g(a_2)\Big] - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} g(x)dx\right| \\
&\leq \frac{(a_2-a_1)^2}{81}\left(\max\left\{\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q, \left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

Proof. Suppose that $q \geq 1$. From Lemma 1, we have

$$\begin{aligned}
&\left|\frac{1}{6}\Big[g(a_1)+4g\Big(\frac{a_1+a_2}{2}\Big)+g(a_2)\Big]-\frac{1}{a_2-a_1}\int_{a_1}^{a_2} g(x)dx\right| \\
&\leq (a_2-a_1)^2\int_0^1 |k(t)|\left|g''(ta_2+(1-t)a_1)\right| dt \\
&= (a_2-a_1)^2\int_0^{\frac{1}{2}} \left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right| \left|g''(ta_2+(1-t)a_1)\right| dt \\
&\quad + (a_2-a_1)^2\int_{\frac{1}{2}}^1 \left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right| \left|g''(ta_2+(1-t)a_1)\right| dt.
\end{aligned}$$

Using the Hölder's inequality for functions

$$\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|^{1-\frac{1}{q}}$$

and

$$\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|^{\frac{1}{q}} \left|g''(ta_2+(1-t)a_1)\right|$$

for the first integral and the functions

$$\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|^{1-\frac{1}{q}}$$

and

$$\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|^{\frac{1}{q}} \left|g''(ta_2+(1-t)a_1)\right|,$$

for the second integral, from the above relation we get the inequalities:

$$\begin{aligned}
&\left|\frac{1}{6}\Big[g(a_1)+4g\Big(\frac{a_1+a_2}{2}\Big)+g(a_2)\Big]-\frac{1}{a_2-a_1}\int_{a_1}^{a_2} g(x)dx\right| \\
&\leq (a_2-a_1)^2\Big(\int_0^{\frac{1}{2}} \left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right| dt\Big)^{1-\frac{1}{q}} \\
&\quad \times \Big(\int_0^{\frac{1}{2}} \left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right| \left|g''(ta_2+(1-t)a_1)\right|^q dt\Big)^{\frac{1}{q}} \\
&\quad + (a_2-a_1)^2\Big(\int_{\frac{1}{2}}^1 \left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right| dt\Big)^{1-\frac{1}{q}} \\
&\quad \times \Big(\int_{\frac{1}{2}}^1 \left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right| \left|g''(ta_2+(1-t)a_1)\right|^q dt\Big)^{\frac{1}{q}}.
\end{aligned}$$

Since $|g''|^q$ is exponentially quasi-convex, therefore we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right| \left|g''(ta_2+(1-t)a_1)\right|^q dt \\
&\leq \int_0^{\frac{1}{2}} \left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right| \max\left\{\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q, \left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\right\} dt \\
&= \max\left\{\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q, \left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\right\} \int_0^{\frac{1}{2}} \left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right| dt \\
&= \frac{1}{162}\max\left\{\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q, \left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\right\}
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left|g''(ta_2+(1-t)a_1)\right|^q dt\\
&\leq \int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\max\left\{\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q,\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\right\}dt\\
&=\max\left\{\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q,\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\right\}\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|dt\\
&=\frac{1}{162}\max\left\{\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q,\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\right\}.
\end{aligned} \tag{5}$$

From (4) and (5), we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[g(a_1)+4g\Big(\frac{a_1+a_2}{2}\Big)+g(a_2)\right]-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}g(x)dx\right|\\
&\leq (a_2-a_1)^2\Big(\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|dt\Big)^{1-\frac{1}{q}}\left(\frac{1}{162}\max\left\{\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q,\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\right\}\right)^{\frac{1}{q}}\\
&\quad+(a_2-a_1)^2\Big(\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|dt\Big)^{1-\frac{1}{q}}\left(\frac{1}{162}\max\left\{\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q,\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\right\}\right)^{\frac{1}{q}}\\
&=2(a_2-a_1)^2\left(\frac{1}{162}\right)^{1-\frac{1}{q}}\left(\frac{1}{162}\right)^{\frac{1}{q}}\left(\max\left\{\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q,\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\right\}\right)^{\frac{1}{q}}\\
&=\frac{(a_2-a_1)^2}{81}\left(\max\left\{\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q,\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\right\}\right)^{\frac{1}{q}},
\end{aligned}$$

where we used the fact that

$$\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|dt=\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|dt=\frac{1}{162}.$$

The proof is complete. □

Corollary 5. *Let $g: I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I° such that $g''\in L_1[a_1,a_2]$, where $a_1,a_2\in I$ with $a_1<a_2$. If $g(a_1)=g\Big(\frac{a_1+a_2}{2}\Big)=g(a_2)$ and $|g''|^q$ is exponentially quasi-convex on $[a_1,a_2]$ and $q\geq 1$, then the following inequality holds:*

$$\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2}g(x)dx-g\Big(\frac{a_1+a_2}{2}\Big)\right|\leq\frac{(a_2-a_1)^2}{81}\left(\max\left\{\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^q,\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^q\right\}\right)^{\frac{1}{q}}.$$

Corollary 6. *Let $g: I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I° such that $g''\in L_1[a_1,a_2]$, where $a_1,a_2\in I$ with $a_1<a_2$. If $g(a_1)=g\Big(\frac{a_1+a_2}{2}\Big)=g(a_2)$ and $|g''|^2$ is exponentially quasi-convex on $[a_1,a_2]$, then the following inequality holds:*

$$\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2}g(x)dx-g\Big(\frac{a_1+a_2}{2}\Big)\right|\leq\frac{(a_2-a_1)^2}{81}\left(\max\left\{\left|\frac{g''(a_2)}{e^{\alpha a_2}}\right|^2,\left|\frac{g''(a_1)}{e^{\alpha a_1}}\right|^2\right\}\right)^{\frac{1}{2}}.$$

3. Applications to special means

We now consider the following special means for positive real numbers a_1 and a_2.

1. The arithmetic mean: $\mathcal{A}=\mathcal{A}(a_1,a_2)=\frac{a_1+a_2}{2}$.
2. The harmonic mean:

$$\mathcal{H}=\mathcal{H}(a_1,a_2)=\frac{2a_1a_2}{a_1+a_2}.$$

3. The logarithmic mean:

$$\mathcal{L} = \mathcal{L}(a_1, a_2) = \begin{cases} a_1 & \text{if} \quad a_1 = a_2 \\ \frac{a_2 - a_1}{\ln a_2 - \ln a_1} & \text{if} \quad a_1 \neq a_2. \end{cases}$$

4. The p-logarithmic mean:

$$\mathcal{L}_p = \mathcal{L}_p(a_1, a_2) = \begin{cases} \left[\frac{a_2^{p+1} - a_1^{p+1}}{(p+1)(a_2 - a_1)}\right]^{\frac{1}{p}} & \text{if} \quad a_1 \neq a_2 \\ a_1 & \text{if} \quad a_1 = a_2 \end{cases}, \quad p \in \mathbb{R} \setminus \{-1, 0\}.$$

It is well know that $\mathcal{L}_p$ is monotonic nondecreasing over $p \in \mathbb{R}$ with $\mathcal{L}_{-1} = \mathcal{L}$. In particular, we have the following inequalities

$$\mathcal{H} \leq \mathcal{L} \leq \mathcal{A}$$

Some new inequalities are derived for the above means.

Proposition 1. *Let $a_1, a_2 \in \mathbb{R}$, $0 < a_1 < a_2$. Then, we have*

$$\left| \frac{1}{3}\mathcal{A}(a_1^4, a_2^4) + \frac{2}{3}\mathcal{A}^4(a_1, a_2) - \mathcal{L}_4^4(a_1, a_2) \right| \leq \frac{2(a_2 - a_1)^2}{27} \left[\frac{a_2^2}{e^{\alpha a_2}} + \frac{a_1^2}{e^{\alpha a_1}} \right].$$

Proof. The assertion follows from Theorem 1 and a simple computation applied to $g(x) = \frac{x^4}{12}, \; x \in [a_1, a_2]$, where $|g''|$ is exponentially convex mapping. □

Proposition 2. *Let $a_1, a_2 \in \mathbb{R}$, $0 < a_1 < a_2$. Then, we have*

$$\left| \frac{1}{3}\mathcal{A}(a_1^{r+1}, a_2^{r+1}) + \frac{2}{3}\mathcal{A}^{r+1}(a_1, a_2) - \mathcal{L}_{r+1}^{r+1}(a_1, a_2) \right| \leq \frac{r(r+1)(a_2 - a_1)^2}{81} \max \left\{ \frac{a_2^{r-1}}{e^{\alpha a_2}}, \frac{a_1^{r-1}}{e^{\alpha a_1}} \right\}.$$

Proof. This time we use Theorem 3 and a simple computation applied to $g(x) = \frac{x^{r+1}}{r+1}, \; r \geq 1, \; x \in [a_1, a_2]$. Here, the function $|g''(x)| = rx^{r-1}$ is increasing and exponentially quasi-convex. □

4. Applications to Simpson's formula

Let $g : [a_1, a_2] \to \mathbb{R}$ and $\mathbf{P}$ be a partition of the interval $[a_1, a_2]$; i.e.

$$\mathbf{P} : a_1 = s_0 < s_1 < \cdots < s_{n-1} < s_n = a_2; \qquad h_i = \frac{(s_{i+1} - s_i)}{2}.$$

Now, for the given Simpson's quadrature:

$$S(g, \mathbf{P}) = \sum_{i=0}^{n-1} \frac{g(s_i) + 4g(s_i + h_i) + g(s_{i+1})}{3} h_i,$$

it is well known that if g is differentiable such that $g^{(4)}(x)$ exist on (a_1, a_2) and $K = \max_{x \in [a_1, a_2]} |g^{(4)}(x)| < \infty$, then

$$I = \int_{a_1}^{a_2} g(s)ds = S(g, \mathbf{P}) + E_s(g, \mathbf{P}), \tag{6}$$

where the approximation error $E_s(g, \mathbf{P})$ of the integral I by Simpson's formula $S(g, \mathbf{P})$ satisfies

$$|E_s(g, \mathbf{P})| \leq \frac{K}{90} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^5. \tag{7}$$

It is clear that if the function g is not four times differentiable or the fourth derivative is not bounded on (a_1, a_2), then (7) cannot be applied.

Theorem 5. *Let $g : I \subset [0, \infty) \to \mathbb{R}$ be a twice differentiable mapping on I° such that $g'' \in L_1[a_1, a_2]$, where $a_1, a_2 \in I$ with $a_1 < a_2$. If $|g''|$ is exponentially convex on $[a_1, a_2]$, then in (6) for every division $\mathbf{P}$ of $[a_1, a_2]$, the following holds:*

$$|E_s(g, \mathbf{P})| \leq \frac{1}{162} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3 \left[\frac{|g''(s_{i+1})|}{e^{\alpha s_{i+1}}} + \frac{|g''(s_i)|}{e^{\alpha s_i}} \right].$$

Proof. Applying Theorem 1 on the subintervals $[s_i, s_{i+1}]$, $(i = 0, 1, 2, \cdots, n-1)$ of the division $\mathbf{P}$, we get

$$\left| \frac{(s_{i+1} - s_i)}{6} \left[g(s_i) + 4g\left(\frac{s_{i+1} - s_i}{2} \right) + g(s_{i+1}) \right] - \int_{s_i}^{s_{i+1}} g(s) ds \right| \leq \frac{(s_{i+1} - s_i)^3}{162} \left[\frac{|g''(s_{i+1})|}{e^{\alpha s_{i+1}}} + \frac{|g''(s_i)|}{e^{\alpha s_i}} \right].$$

Adding over i for 0 to $n-1$ and taking into account that $|g''|$ is exponentially convex, we have:

$$\left| S(g, \mathbf{P}) - \int_{a_1}^{a_2} g(s) ds \right| \leq \sum_{i=0}^{n-1} \frac{(s_{i+1} - s_i)^3}{162} \left[\frac{|g''(s_{i+1})|}{e^{\alpha s_{i+1}}} + \frac{|g''(s_i)|}{e^{\alpha s_i}} \right],$$

which completes the proof. □

Corollary 7. *If $\alpha = 0$, we get*

$$|E_s(g, \mathbf{P})| \leq \frac{1}{162} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3 \left[g''(s_i) + g''(s_{i+1}) \right].$$

Theorem 6. *Let $g : I \subset [0, \infty) \to \mathbb{R}$ be a twice differentiable mapping on I° such that $g'' \in L_1[a_1, a_2]$, where $a_1, a_2 \in I$ with $a_1 < a_2$. If $|g''|$ is exponentially quasi-convex on $[a_1, a_2]$, then in (6) for every division $\mathbf{P}$ of $[a_1, a_2]$, the following holds:*

$$|E_s(g, \mathbf{P})| \leq \frac{1}{81} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3 \max \left\{ \frac{|g''(s_{i+1})|}{e^{\alpha s_{i+1}}}, \frac{|g''(s_i)|}{e^{\alpha s_i}} \right\}.$$

Proof. Applying Theorem 3 and proceeding as in the proof of Theorem 5, we obtain the desired result. □

Proposition 3. *Let $g : I \subset [0, \infty) \to \mathbb{R}$ be a twice differentiable mapping on I° such that $g'' \in L_1[a_1, a_2]$, where $a_1, a_2 \in I$ with $a_1 < a_2$. If $|g''|^q$ is exponentially convex on $[a_1, a_2]$ and $q \geq 1$, the following holds:*

$$|E_s(g, \mathbf{P})| \leq \left(\frac{1}{162} \right)^{1 - \frac{1}{q}} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3 \left[M_\alpha^q(g''(s_i), g''(s_{i+1})) \right],$$

where

$$M_\alpha^q\left(g''(s_i), g''(s_{i+1})\right) = \left(\frac{59}{31104} \left| \frac{g''(s_{i+1})}{e^{\alpha s_{i+1}}} \right|^q + \frac{133}{31104} \left| \frac{g''(s_i)}{e^{\alpha s_i}} \right|^q \right)^{\frac{1}{q}} + \left(\frac{133}{31104} \left| \frac{g''(s_{i+1})}{e^{\alpha s_{i+1}}} \right|^q + \frac{59}{31104} \left| \frac{g''(s_i)}{e^{\alpha s_i}} \right|^q \right)^{\frac{1}{q}}.$$

Proof. The proof is immediate by using Theorem 2. □

Proposition 4. *Let $g : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on I° such that $g'' \in L_1[a_1,a_2]$, where $a_1,a_2 \in I$ with $a_1 < a_2$. If $|g''|^q$ is exponentially quasi-convex on $[a_1,a_2]$ and $q \geq 1$, the following holds:*

$$|E_s(g,\mathbf{P})| \leq \frac{1}{81}\sum_{i=0}^{n-1}(s_{i+1}-s_i)^3\left(\max\left\{\left|\frac{g''(s_{i+1})}{e^{\alpha s_{i+1}}}\right|^q,\left|\frac{g''(s_i)}{e^{\alpha s_i}}\right|^q\right\}\right)^{\frac{1}{q}}.$$

Proof. The proof follows by applying Theorem 4. □

5. Conclusion

Several inequalities of the Simpson's type for exponentially convex and exponentially quasi-convex functions are hereby established. For $\alpha = 0$, we recapture results in [7]. Furthermore, we presented some applications to special means and to Simpson's formula. We look forward to further investigation in this direction.

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