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Some fixed point theorems for F-expansive mapping in generalized metric spaces

Mohamed Rossafi[1,*] and Abdelkarim Kari[2]

[1] LASMA Laboratory Department of Mathematics Faculty of Sciences, Dhar El Mahraz University Sidi Mohamed Ben Abdellah, Fes, Morocco.

[2] Laboratory of Algebra, Analysis and Applications Faculty of Sciences Ben M'Sik, Hassan II University, Casablanca, Morocco.; abdkrimkariprofes@gmail.com.

[*] Correspondence: mohamed.rossafi@usmba.ac.ma

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Abstract: In this paper, we present the notion of generalized F-expansive mapping in complete rectangular metric spaces and study various fixed point theorems for such mappings. The findings of this paper, generalize and improve many existing results in the literature.

Keywords: Fixed point; rectangular metric spaces; F-expansive mappings.

MSC: 47H10, 54H25.

1. Introduction

The fixed point theory is a very interesting research area in due to its wide range of applicability, to resolve diverse problems emanating from the theory of nonlinear differential equations and integral equations.

Wardowski [1], generalized the famous Banach theorem [2] for $F-$contraction on metric spaces, several mathematicians extended this new notion for contraction on metric spaces [3–6].

The concept of a rectangular metric space was introduced by Branciari in [7]. After that, several interesting results about the existence of fixed points in rectangular metric spaces have been obtained [8–11]. Recently, Kari *et al.*, [12], obtained some results for generalized $\theta-\phi-$expansive mapping in rectangular metric spaces.

In 1984, Wang *et al.*, [13], presented some interesting work on expansion mappings in metric spaces. Recently, Kumar *et al.*, [14], introduced a new concept of $(\alpha, \psi)-$expansive mappings and established some fixed point theorems for such mapping in complete rectangular metric spaces.

In this paper, inspired by the idea of $F-$contraction introduced by Wardowski [1] in metric spaces, we presented generalized $F-$expansive mapping and establish various fixed point theorems for such mappings in complete rectangular metric spaces. Our theorems extend, generalize and improve many existing results.

2. Preliminaries

By an expansion mappings [13] on a metric space (X, d), we understand a mapping $T : X \to X$ satisfying for all $x, y \in X$:

$$d(Tx, Ty) \geq kd(x, y),$$

where k is a real in $]1, +\infty[$.

In 2000 Branciari [7] introduced the concept of rectangular metric spaces.

Definition 1. [7] Let X be a non-empty set and $d : X \times X \to \mathbb{R}^+$ be a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, on has

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all distinct points $x, y \in X$;

(iii) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$(the rectangular inequality).

Then (X,d) is called an rectangular metric space.

Definition 2. [15] Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty[$. We say that T is a triangular (α, η)-admissible mapping if

(T_1) $\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1, x, y \in X$;
(T_2) $\eta(x,y) \leq 1 \Rightarrow \eta(Tx, Ty) \leq 1, x, y \in X$;

$$(T_3) \begin{cases} \alpha(x,y) \geq 1 \\ \alpha(y,z) \geq 1 \end{cases} \Rightarrow \alpha(x,z) \geq 1 \text{ for all } x, y, z \in X;$$
$$(T_4) \begin{cases} \eta(x,y) \leq 1 \\ \eta(y,z) \leq 1 \end{cases} \Rightarrow \eta(x,z) \leq 1 \text{ for all } x, y, z \in X.$$

Definition 3. [15] Let (X,d) be a rectangular metric space and let $\alpha, \eta : X \times X \to [0, +\infty[$ be two mappings. Then

(a) T is $\alpha-$continuous mapping on (X,d), if for given point $x \in X$ and sequence $\{x_n\}$ in X, $x_n \to x$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, imply that $Tx_n \to Tx$.
(b) T is η sub$-$continuous mapping on (X,d), if for given point $x \in X$ and sequence $\{x_n\}$ in X, $x_n \to x$ and $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$, imply that T$x_n \to Tx$.
(c) T is (α, η) $-$continuous mapping on (X,d), if for given point $x \in X$ and sequence $\{x_n\}$ in X, $x_n \to x$ and $\alpha(x_n, x_{n+1}) \geq 1$ or $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$, imply that $Tx_n \to Tx$.

Recently Hussain *et al.*, gives the following definition [16]:

Definition 4. [16] Let $d(X,d)$ be a rectangular metric space and let $\alpha, \eta : X \times X \to [0, +\infty[$ be two mappings. The space X is said to be

(a) $\alpha-$complete, if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, converges in X.
(b) $\eta-$ sup $-$complete, if every Cauchy sequence $\{x_n\}$ in X with $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$, converges in X.
(c) (α, η) $-$complete, if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ or $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$, converges in X.

Definition 5. [16] Let (X,d) be a rectangular metric space and let $\alpha, \eta : X \times X \to [0, +\infty[$ be two mappings. The space (X,d) is said to be

(a) (X,d) is α -regular, if $x_n \to x$, where $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, implies $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.
(b) (X,d) is $\eta-$sub -regular, if $x_n \to x$, where $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$, implies $\eta(x_n, x) \leq 1$ for all $n \in \mathbb{N}$.
(c) (X,d) is (α, η)-regular, if $x_n \to x$, where $\alpha(x_n, x_{n+1}) \geq 1$ or $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$, imply that $\alpha(x_n, x) \geq 1$ or $\eta(x_n, x) \leq 1$ for all $n \in \mathbb{N}$.

The following definition introduced by Wardowski [1]:

Definition 6. [1] Let $\mathbb{F}$ be the family of all functions $F\colon \mathbb{R}^+ \to \mathbb{R}$ such that

(i) F is strictly increasing;
(ii) for each sequence $(x_n)_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty} x_n = 0$, if and only if $\lim_{n\to\infty} F(x_n) = -\infty$;
(iii) there exists $k \in]0,1[$ such that $\lim_{x\to 0} x^k F(x) = 0$.

Recently, Piri and Kuman [4] extended the result of Wardowski [1] by changing the condition (iii) in the Definition 6 as follow:

Definition 7. [4] Let Γ be the family of all functions $F\colon \mathbb{R}^+ \to \mathbb{R}$ such that

(i) F is strictly increasing;
(ii) for each sequence $(x_n)_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty} x_n = 0,$ if and only if $\lim_{n\to\infty} F(x_n) = -\infty;$
(iii) F is continuous.

3. Fixed point theorem on rectangular metric spaces

We introduce a new notion of generalized $F-$expansive mapping in the context of rectangular metric spaces as follows:

Definition 8. Let (X, d) be a rectangular metric space and $T : X \to X$ be a given mapping. T is said to be generalized $F-$expansive mapping if there exists $F \in \mathbb{F}$ and $\tau > 0$ such that

$$M(x,y) > 0 \Rightarrow F(d(Tx,Ty)) - \tau \geq F(M(x,y)), \; for\ all\ x,y \in X, \tag{1}$$

where $M(x,y) = \min\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty)\}.$

Theorem 1. *Let (X,d) be a $(\alpha-\eta)$-complete generalized metric space, and $T : X \to X$ be a bijective, generalized F-expansive mapping satisfying following conditions*

(i) T^{-1} is a triangular $(\alpha,\eta)-$admissible mapping;
(ii) there exists $x_0 \in X$ such that $\alpha\left(x_0, T^{-1}x_0\right) \geq 1$ or $\eta\left(x_0, T^{-1}x_0\right) \leq 1;$
(iii) T is a $(\alpha,\eta)-$continuous.

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(z,u) \geq 1$ or $\eta(z,u) \leq 1$ for all $z,u \in Fix(T)$.

Proof. Let $x_0 \in X$ such that $\alpha\left(x_0, T^{-1}x_0\right) \geq 1$ or $\eta\left(x_0, T^{-1}x_0\right) \leq 1$. We define the sequence $\{x_n\}$ in X by $x_n = Tx_{n+1}$, for all $n \in \mathbb{N}$.

Since T^{-1} is an triangular $(\alpha,\eta)-$admissible mapping, then

$$\alpha(x_0,x_1) = \alpha\left(x_0, T^{-1}x_0\right) \geq 1 \Rightarrow \alpha\left(T^{-1}x_0, T^{-1}x_1\right) = \alpha(x_1,x_2) \geq 1,$$

or

$$\eta(x_0,x_1) = \eta\left(x_0, T^{-1}x_0\right) \leq 1 \Rightarrow \eta\left(T^{-1}x_0, T^{-1}x_1\right) = \eta(x_1,x_2) \leq 1.$$

Continuing this process we have

$$\alpha(x_{n-1},x_n) \geq 1,$$

or

$$\eta(x_{n-1},x_n) \leq 1,$$

for all $n \in \mathbb{N}$. By (T_3) and (T_4), one has.

$$\alpha(x_m,x_n) \geq 1 \text{ or } \eta(x_m,x_n) \leq 1, \quad \forall m,n \in \mathbb{N},\ m \neq n. \tag{2}$$

Suppose that there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = Tx_{n_0}$. Then x_{n_0} is a fixed point of T and the prove is finished. Hence, we assume that $x_n \neq Tx_n$, i.e., $d(x_{n-1},x_n) > 0$ for all $n \in \mathbb{N}$.
Step 1: We shall prove

$$\lim_{n\to+\infty} d(x_n,x_{n+1}) = 0.$$

Applying inequality (1) with $x = x_n$ and $y = x_{n+1}$, we obtain

$$F(d(x_{n-1},x_n)) = F(d(Tx_n,Tx_{n+1})) > F(d(Tx_n,Tx_{n+1})) - \tau \geq F(M(x_n,x_{n+1})), \tag{3}$$

where

$$\begin{aligned} M(x_n,x_{n+1}) &= \min\{d(x_n,x_{n+1}), d(x_n,Tx_n), d(x_{n+1},Tx_{n+1}), d(x_n,Tx_{n+1})\} \\ &= \min\{d(x_n,x_{n+1}), d(x_n,x_{n-1}), d(x_{n+1},x_n), d(x_n,x_n)\} \\ &= \min\{d(x_n,x_{n+1}), d(x_n,x_{n-1})\}. \end{aligned}$$

If for some n, $M(x_n, x_{n+1}) = d(x_n, x_{n-1})$, then the inequality (3), we get

$$F(d(x_{n-1}, x_n)) > F(d(Tx_n, Tx_{n+1})) - \tau \geq F(d(x_{n-1}, x_n)). \tag{4}$$

It is a contradiction. Hence $M(x_n, x_{n+1}) = d(x_n, x_{n+1})$. Therefore

$$F(d(Tx_n, Tx_{n+1})) - \tau \geq F(d(x_n, x_{n+1})). \tag{5}$$

Thus,

$$F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) - \tau \text{ for all } n \in \mathbb{N}. \tag{6}$$

Continuing this process, we get

$$F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) - \tau \leq ... \leq F(d(x_0, x_1)) - n\tau. \tag{7}$$

Now, by (6) and the condition F_3 of Definition 2, we deduce that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \tag{8}$$

Taking the limit as $n \to \infty$ in (7) and using the condition F_2, we get

$$\lim_{n \to +\infty} d(x_n, x_{n+1}) = 0. \tag{9}$$

Step 2: Now, we shall prove

$$x_n \neq x_m, \ for \ all \ m, n \in \mathbb{N}, m \neq n. \tag{10}$$

On the contrary, assume that $x_n = x_m$ for some $n = m + k > m$. Indeed, suppose that $x_n = x_m$, so we have

$$x_n = Tx_{n+1} = Tx_{m+1} = x_m.$$

Denote $d_n = d(x_n, x_{n+1})$. By the inequality (8), we have

$$d_n < d_{n-1}.$$

Continuing this process, we get

$$d_m = d_n < d_{n-1} < ... < d_m. \tag{11}$$

Which is a contradiction. Thus (10) hold.
Step 3: We prove shall

$$\lim_{n \to +\infty} d(x_n, x_{n+2}) = 0. \tag{12}$$

Applying inequality (1) with $x = x_n$, $y = x_{n+2}$, we obtain

$$F(d(x_{n-1}, x_{n+1})) = F(d(Tx_n, Tx_{n+2})) > F(d(Tx_n, Tx_{n+2})) - \tau \geq F(M(x_n, x_{n+2})), \tag{13}$$

where

$$\begin{aligned} M(x_n, x_{n+2}) &= \min\{d(x_n, x_{n+2}), d(x_n, Tx_n), d(x_{n+2}, Tx_{n+2}), d(x_n, Tx_{n+2})\} \\ &= \min\{d(x_n, x_{n+2}), d(x_n, x_{n-1}), d(x_{n+2}, x_{n+1}), d(x_n, x_{n+1})\} \\ &= \min\{d(x_n, x_{n+2}), d(x_{n+1}, x_{n+2})\}. \end{aligned}$$

Take $a_n = d(x_n, x_{n+2})$ and $b_n = d(x_{n+1}, x_{n+2})$. Thus, by (13), one can write

$$F(a_{n-1}) = F(d(x_{n-1}, x_{n+1})) = F(d(Tx_n, Tx_{n+2})) > F(d(Tx_n, Tx_{n+2})) - \tau \geq F[M(x_n, x_{n+2})] = F(\min\{a_n, b_n\}).$$

Therefore,

$$a_{n-1} \geq \min\{a_n, b_n\}.$$

Again, by (8)

$$b_{n-1} \geq b_n \geq \min\{a_n, b_n\}.$$

Which implies that

$$\min\{a_n, b_n\} \leq \min\{a_{n-1}, b_{n-1}\}, \ \forall n \in \mathbb{N}.$$

Then the sequence $\min\{a_n, b_n\}_{n\in\mathbb{N}}$ is monotone non increasing. Thus, there exists $\lambda \geq 0$ such that

$$\lim_{n\to\infty} \min\{a_n, b_n\} = \lambda.$$

Assume that $\lambda > 0$. By (9), we have

$$\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \sup \min\{a_n, b_n\} = \lim_{n\to\infty} \min\{a_n, b_n\} = \lambda.$$

Taking the $\lim\sup_n \to \infty$ in (13), and using (F_3), we obtain

$$F(\lambda) = F\left(\lim_{n\to\infty} \sup\{a_{n-1}\}\right) \geq F\left(\lim_{n\to\infty} \sup\{a_n\}\right) > F\left(\lim_{n\to\infty} \sup\{a_n\}\right) - \tau \geq F\left(\lim_{n\to\infty} \min\{a_n, b_n\}\right),$$

which implies that

$$F(\lambda) > F(\lambda) - \tau \geq F(\lambda).$$

Therefore,

$$F(\lambda) < F(\lambda).$$

By (F_1), we get

$$\lambda < \lambda.$$

It is a contradiction, then

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{14}$$

Step 4: We shall prove that $\{x_n\}$ is a Cauchy sequence in (X, d), that is

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0 \text{ for all } n \neq m. \tag{15}$$

If otherwise there exists an $\varepsilon > 0$ for which we can find sequence of positive integers $\left\{x_{n_{(k)}}\right\}_k$ and $\left\{x_{m_{(k)}}\right\}_k$ of $\{x_n\}$ such that, for all positive integers k, $n_{(k)} > m_{(k)} > k$,

$$d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \geq \varepsilon \textit{ and } d\left(x_{m_{(k)}}, x_{n_{(k)-1}}\right) < \varepsilon. \tag{16}$$

Now, using (9), (14), (16) and the rectangular inequality, we find

$$\begin{aligned}
\varepsilon &\leq d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \\
&\leq d\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) + d\left(x_{m_{(k)+1}}, x_{m_{(k)-1}}\right) + d\left(x_{m_{(k)-1}}, x_{n_{(k)}}\right) \\
&< d\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) + d\left(x_{m_{(k)+1}}, x_{m_{(k)}-1}\right) + \varepsilon.
\end{aligned}$$

Then

$$\lim_{k\to\infty} d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \varepsilon. \tag{17}$$

Now, by rectangular inequality, we have

$$d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leq d\left(x_{m_{(k)+1}}, x_{m_{(k)}}\right) + d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) + d\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right). \tag{18}$$

$$d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq d\left(x_{m_{(k)}}, x_{m(k)+1}\right) + d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) + d\left(x_{n_{(k)+1}}, x_{n_{(k)}}\right). \tag{19}$$

$$\varepsilon \leq d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq d\left(x_{m_{(k)}}, x_{n_{(k)-1}}\right) + d\left(x_{n_{(k)-1}}, x_{n_{(k)+1}}\right) + d\left(x_{n_{(k)+1}}, x_{n_{(k)}}\right). \tag{20}$$

Letting $k \to \infty$ in the above inequalities, using (9), (16) and (17), we obtain

$$\lim_{k\to\infty} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) = \varepsilon, \tag{21}$$

and

$$\lim_{k\to\infty} d\left(x_{m_{(k)}}, x_{n_{(k)-1}}\right) = \varepsilon. \tag{22}$$

On the other hand

$$\begin{aligned} M\left(x_{m_{(k)}}, x_{n_{(k)}}\right) &= \min\left\{d\left(x_{m_{(k)}}, x_{n_{(k)}}\right), d\left(x_{m_{(k)}}, Tx_{m_{(k)}}\right), d\left(x_{n_{(k)}}, Tx_{n_{(k)}}\right), d\left(x_{m_{(k)}}, Tx_{n_{(k)}}\right)\right\} \\ &= \min\left\{d\left(x_{m_{(k)}}, x_{n_{(k)}}\right), d\left(x_{m_{(k)}}, x_{m_{(k)-1}}\right), d\left(x_{n_{(k)}}, x_{n_{(k)-1}}\right), d\left(x_{m_{(k)}}, x_{n_{(k)-1}}\right)\right\}. \end{aligned}$$

Letting $k \to \infty$ in the above inequalities and using (9), (17) and (22), we get that

$$\lim_{k\to\infty} M\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \varepsilon. \tag{23}$$

By (21), let $A = \frac{\varepsilon}{2} > 0$, from the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$|d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) - \varepsilon| \leq A \ \forall n \geq n_0.$$

This implies that

$$d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \geq A > 0 \ \forall n \geq n_0,$$

and by (23), let $B = \frac{\varepsilon}{2} > 0$, from the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that

$$M\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \geq B > 0 \ \forall n \geq n_1.$$

Applying (1) with $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$, we obtain

$$F\left(d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right) > F\left(d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right) - \tau \geq F\left(M\left(d\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right)$$

Letting $k \to \infty$ the above inequality and using (F_3), we obtain

$$F\left(\lim_{k\to\infty} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right) > F\left(\lim_{k\to\infty} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right) - \tau \geq F\left(\lim_{k\to\infty} M\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right),$$

Therefore,

$$F(\varepsilon) < F(\varepsilon).$$

It is a contradiction. Then

$$\lim_{n,m\to\infty} d\left(x_m, x_n\right) = 0.$$

It follows that $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is (α, η)-complete and

$$\alpha(x_{n-1}, x_n) \geq 1 \ or \ \eta(x_{n-1}, x_n) \leq 1,$$

for all $n \in \mathbb{N}$, the there exists $z \in X$ such that

$$\lim_{n\to\infty} d\left(x_n, z\right) = 0.$$

Step 5: We show that $d\left(Tz, z\right) = 0$ arguing by contradiction, we assume that

$$d\left(Tz, z\right) > 0.$$

By rectangular inequality we get,

$$d\left(Tx_n, Tz\right) \leq d\left(Tx_n, x_n\right) + d\left(x_n, z\right) + d\left(z, Tz\right), \tag{24}$$

$$d\left(z, Tz\right) \leq d\left(z, x_n\right) + d\left(x_n, Tx_n\right) + d\left(Tx_n, Tz\right). \tag{25}$$

By letting $n \to \infty$ in inequality (24) and (25) we obtain

$$d\left(z, Tz\right) \leq \lim_{n\to\infty} d\left(Tx_n, Tz\right) \leq d\left(z, Tz\right).$$

Therefore,

$$\lim_{n\to\infty} d\left(Tx_n, Tz\right) = d\left(z, Tz\right). \tag{26}$$

Since T is (α, η)-continuous, then $Tx_n \to Tz$ i.e $\lim_{n\to\infty} d\left(Tx_n, Tz\right) = 0$. Hence $d(Tz, z) = 0$, so $Tz = z$.
Step 6: (Uniqueness) Now, suppose that $z, u \in X$ are two fixed points of T such that $u \neq z$ and $\alpha(z, u) \geq 1$ or $\eta(z, u) \leq 1$. Therefore, we have

$$d\left(Tz, Tu\right) = d\left(z, u\right) > 0.$$

Applying (1) with $x = z$ and $y = u$, we have

$$F\left(d\left(Tu, Tz\right)\right) - \tau \geq F\left(M\left(z, u\right)\right),$$

where

$$M\left(z, u\right) = min\left\{d\left(z, u\right), d\left(z, Tz\right), d\left(u, Tu\right), d\left(z, Tu\right)\right\} = d\left(z, u\right).$$

Therefore, we have

$$F\left(d\left(z, u\right)\right) > F\left(d\left(u, z\right)\right) - \tau \geq F\left(d\left(z, u\right)\right).$$

It is a contradiction. Therefore $u = z$. □

Theorem 2. *Let $\alpha, \eta : X \times X \to \mathbb{R}^+$ be two function and let (X, d) be a (α, η) −complete rectangular metric space. Let $T : X \to X$ be a bijective mapping satisfying the following conditions:*

(i) T^{-1} is a triangular (α, η)-admissible mapping;
(ii) T is a generalized $(\alpha, \eta) - F$-expansive mapping ;
(iii) $\alpha\left(z, T^{-1}z\right) \geq 1$ or $\eta\left(z, T^{-1}z\right) \leq 1$, for all $z \in Fix\left(T\right)$.

Then T has a fixed point.

Proof. Let $z \in$ Fix $\left(T^n\right)$ for some fixed $n > 1$. As $\alpha\left(z, T^{-1}z\right) \geq 1$ or $\eta\left(z, T^{-1}z\right) \leq 1$ and T^{-1} is a triangular (α, η)-admissible mapping, then

$$\alpha\left(T^{-1}z, T^{-2}z\right) \geq 1 \text{ or } \eta\left(T^{-2}z, T^{-1}z\right) \leq 1.$$

Continuing this process, we have

$$\alpha\left(T^{-n}z, T^{-n-1}z\right) \geq 1 \text{ or } \eta\left(T^{-n}z, T^{-n-1}z\right) \leq 1,$$

for all $n \in \mathbb{N}$. By $(T3)$ and $(T4)$, we get

$$\alpha\left(T^{-m}z, T^{-n}z\right) \geq 1 \text{ or } \eta\left(T^{-m}z, T^{-n}z\right) \leq 1, \ \ \forall\, m, n \in \mathbb{N},\ n \neq m.$$

Since T is a bijective mapping, then $T^{-n}z = z = T^n z$ for all $n \in \mathbb{N}$ and $z \in$ Fix (T). Therefore,

$$\alpha\left(T^m z, T^n z\right) \geq 1 \text{ or } \eta\left(T^m z, T^n z\right) \leq 1, \ \ \forall\, m, n \in \mathbb{N},\ n \neq m.$$

Assume that $z \notin$ Fix (T), i.e. $d\left(z, Tz\right) > 0$. Then, we have

$$d\left(z, Tz\right) = d\left(T^n z, Tz\right) = d\left(TT^{n-1}z, Tz\right).$$

Applying (1) with $x = z$ and $y = T^{n-1}z$, we obtain

$$F\left(d\left(z,Tz\right)\right) - \tau = F\left(d\left(TT^{n-1}z,Tz\right)\right) - \tau \geq F\left(M\left(T^{n-1}z,z\right)\right),$$

where

$$M\left(T^{n-1}z,z\right) = \min\left\{d\left(z,T^{n-1}z\right),d\left(z,Tz\right),d\left(T^{n-1}z,TT^{n-1}z\right),d\left(T^{n-1}z,Tz\right)\right\}. \tag{27}$$

Letting $n \to \infty$ in (27), we obtain

$$\lim_{n\to+\infty} M\left(T^{n-1}z,z\right) = d\left(z,Tz\right)$$

Now, using (F_3), we get

$$F\left(d\left(z,Tz\right)\right) - \tau \geq F\left(d\left(z,Tz\right)\right).$$

It is a contradiction. Then $z \in Fix(T)$. $\square$

Example 1. Let $X = [1,+\infty[$ and $d : X \times X \to [0,+\infty[$ define by

$$d\left(x,y\right) = |x-y|.$$

Then (X,d) is a metric space and rectangular metric space. Define mapping $T : X \to X$ and $\alpha,\eta : X \times X \to [0,+\infty[$ by

$$T(x) = x^2$$

and

$$\alpha\left(x,y\right) = \frac{x+y}{\max\left\{x,y\right\}+1},$$

$$\eta\left(x,y\right) = \frac{|x-y|}{\max\left\{x,y\right\}+1}.$$

Then, T is an (α,η) $-$continuous triangular (α,η) $-$admissible mapping and T is a bijective mapping.

Let $F\left(t\right) = ln(t)$, $\tau = ln(2)$. Evidently, $(\alpha\left(x,y\right) \geq 1$ or $(x,y) \leq 1)$ and $\min\{d\left(x,y\right),d\left(x,Tx\right),d\left(y,Ty\right),d\left(y,Tx\right)\} > 0$ are when $x \neq y \neq 1$.

Now, consider the following two cases:

Case 1: $(x > y > 1)$

As

$$d(Tx,Ty) = x^2-y^2,\ F(d(Tx,Ty)) = ln(x^2-y^2) = ln(x-y)+ln(x+y).$$

Thus,

$$F(d(Tx,Ty)) - \tau = ln(x^2-y^2) - ln(2) = ln(x-y)+ln(x+y)-ln(2).$$

We have

$$F(d(x,y)) = ln(x-y).$$

On the other hand

$$F(d(x,y)) - F(d(Tx,Ty)) + \tau = ln(x^2-y^2) = ln(x-y) - ln(x-y) - ln(x+y) + ln(2) = ln(x-y)+ln(2).$$

Since $x,y \in]1,+\infty[$, then

$$-ln(x+y)+ln(2) \leq 0.$$

Which implies that

$$F(d(Tx,Ty)) - \tau \geq F(d(x,y)) \geq F\left[\min\{d\left(x,y\right),d\left(x,Tx\right),d\left(y,Ty\right),d\left(y,Ty\right)\right].$$

Case 2: $(y > x > 1)$

As

$$d(Ty,Tx) = y^2-x^2,\ F(d(Ty,Tx)) = ln(y^2-x^2) = ln(y-x)+ln(y+x),$$

Thus,

$$F(d(Ty,Tx)) - \tau = ln(y^2 - x^2) - ln(2) = ln(y-x) + ln(y+x) - ln(2).$$

We have

$$F(d(x,y)) = ln(y-x).$$

On the other hand

$$F(d(y,x)) - F(d(Ty,Tx)) + \tau = ln(y-x) - ln(y-x) - ln(y+x) + ln(2) = -ln(x-y) + ln(2).$$

Since $y, x \in \left]1, +\infty\right[$, then

$$ln(y-x) + ln(2) \leq 0.$$

Which implies that

$$F(d(Ty,Tx)) - \tau \geq F(d(y,x)) \geq F\left[\min\{d\left(y,x\right), d\left(y,Ty\right), d\left(x,Tx\right), d\left(x,Ty\right)\right].$$

Hence, the condition (1) is satisfied. Therefore, T has a unique fixed point $z = 1$.

Theorem 3. *Let $\alpha, \eta : X \times X \to \mathbb{R}^+$ be two functions and let $d\left(X,d\right)$ be a $\left(\alpha,\eta\right)-$complete rectangular metric space. Let $T : X \to X$ be a bijective mapping satisfying the following assertions:*

(i) T^{-1} is triangular $\left(\alpha,\eta\right)-$admissible;
(ii) T is a generalized $\left(\alpha,\eta\right) - F-$expansive mapping;
(iii) there exists $x_0 \in X$ such that $\alpha\left(x_0, T^{-1}x_0\right) \geq 1$ or $\eta\left(x_0, T^{-1}x_0\right) \leq 1$;
(iv) $\left(X,d\right)$ is a $\left(\alpha,\eta\right)$-regular rectangular metric space.

Then T has a fixed point. Moreover, T has a unique fixed point whenever $\alpha\left(z,u\right) \geq 1$ or $\eta\left(z,u\right) \leq 1$ for all $z, u \in Fix\left(T\right)$.

Proof. Let $x_0 \in X$ such that $\alpha\left(x_0, T^{-1}x_0\right) \geq 1$ or $\eta\left(x_0, T^{-1}x_0\right) \leq 1$. Similar to the proof of Theorem 1, we can conclude that

$$\left(\alpha\left(x_n, x_{n+1}\right) \geq 1 \text{ or } \eta\left(x_n, x_{n+1}\right) \leq 1\right), \text{ and } x_n \to z \text{ as } n \to \infty,$$

and from inequality (26), we have

$$\lim_{n\to\infty} d(Tx_n, Tz) = d(z, Tz).$$

From (iv) $\alpha\left(x_n, z\right) \geq 1$ or $\eta\left(x_n, z\right) \leq 1$, hold for $n \in \mathbb{N}$.

Suppose that $Tz = x_{n_{0-1}} = Tx_{n_0}$ for some $n_0 \in \mathbb{N}^*$. From Theorem 1 we know that the members of the sequence $\{x_n\}$ are distinct. Hence, we have $Tz \neq Tx_n$, i.e. $d\left(Tz, Tx_n\right) > 0$ for all $n > n_0$. Thus, we can apply (1) to x_n and z for all $n > n_0$ to get

$$F\left(d\left(Tz, Tx_n\right)\right) - \tau \geq F\left(M\left(z, x_n\right)\right), \ \forall n \geq n_0,$$

where

$$\begin{aligned} M\left(z, x_n\right) &= \min\left\{d\left(z, x_n\right), d\left(z, Tz\right), d\left(x_n, Tx_n\right), d\left(z, Tx_n\right)\right\} \\ &= \min\left\{d\left(z, x_n\right), d\left(z, Tz\right), d\left(x_n, x_{n-1}\right), d\left(z, x_{n-1}\right)\right\}. \end{aligned}$$

Therefore,

$$F\left(d\left(Tz, Tx_n\right)\right) - \tau \geq F\left(\min\left\{d\left(z, x_n\right), d\left(z, Tz\right), d\left(x_n, x_{n-1}\right), d\left(z, x_{n-1}\right)\right\}\right). \tag{28}$$

By letting $n \to \infty$ in inequality (28), we obtain

$$\lim_{n\to\infty} F\left(d\left(Tz, Tx_n\right)\right) > \lim_{n\to\infty} F\left(d\left(Tz, Tx_n\right)\right) - \tau \geq \lim_{n\to\infty} F\left(\min\left\{d\left(z, x_n\right), d\left(z, Tz\right), d\left(x_n, x_{n-1}\right), d\left(z, x_{n-1}\right)\right\}\right).$$

Since F is continuous function and $\lim_{n\to+\infty} M(z,x_n) = d(z,Tz)$, we conclude that

$$F\left(d(z,Tz)\right) > F\left(d(z,Tz)\right),$$

which implies that

$$d(z,Tz) < d(z,Tz).$$

It is a contradiction. Hence $Tz = z$. The proof of the uniqueness is similarly to that of Theorem 1. □

Corollary 1. *Let $\alpha, \eta : X \times X \to [0,+\infty[$ be two functions, (X,d) be a (α,η)-complete rectangular metric space and $T : X \to X$ be a bijective mapping. Suppose that for all $x,y \in X$ with $\alpha(x,y) \geq 1$ or $\eta(x,y) \leq 1$ and $M(x,y) > 0$ we have*

$$F\left(d\left(Tx,Ty\right)\right) - \tau \geq F\left(d\left(x,y\right)\right).$$

Then T has a fixed point, if

(i) T^{-1} *is a triangular* (α,η) *−admissible mapping;*
(ii) *there exists* $x_0 \in X$ *such that* $\alpha\left(x_0, T^{-1}x_0\right) \geq 1$ *or* $\eta\left(x_0, T^{-1}x_0\right) \leq 1$;
(iii) *Tis a* (α,η) *−continuous; or*
(iv) (X,d) *is an* (α,η)*-regular rectangular metric space.*

Moreover, T has a unique fixed point when $\alpha\left(z,u\right) \geq 1$ or $\eta\left(z,u\right) \leq 1$ for all $z,u \in Fix(T)$.

4. Fixed point theorem on rectangular metric spaces endowed with a partial order

Definition 9. [16] Let $(X,d,\preceq)$ be an ordered rectangular metric space and $T : X \to X$ be a mapping. Then

1) (X,d) is said to be O-complete, if every Cauchy $\{n_n\}$ in X with $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ or $x_n \succeq x_{n+1}$ for all $n \in \mathbb{N}$, converges in X.
2) (X,d) is said to be O-regular, if for each sequence $\{n_n\}$ in X $\{x_n\} \to x$ and $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ or $x_n \succeq x_{n+1}$ for all $n \in \mathbb{N}$ imply that $\{n_n\} \preceq x$ or $\{n_n\} \succeq x$ respectively.
3) T is said to be O-continuous, if for given $x \in X$ and sequence $\{n_n\}$ with $x_n \succeq x_{n+1}$ or $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$, $\{n_n\} \to x \Rightarrow Tx_n \to Tx$.

Definition 10. Let $(X,d,\preceq)$ be an ordered rectangular metric spaces and $T : X \to X$ be a mapping. We say that T be an ordered F- expansive mapping, if for all $x,y \in X$ with $x \preceq y$ or $x \succeq y$ such that

$$M(x,y) > 0 \Rightarrow F\left(d\left(Tx,Ty\right)\right) - \tau \geq F\left(M(x,y)\right),$$

where $M(x,y) = \min\left\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty)\right\}$.

Theorem 4. *Let $(X,d,\preceq)$ be an O-complete partially ordered rectangular metric space. Let $T : X \to X$ be a bijective self mapping on X satisfying the following assertions:*

(i) T^{-1} *is monotone ;*
(ii) T *is an ordered F- expansive mapping;*
(iii) *there exists* $x_0 \in X$ *such that* $x_0 \preceq T^{-1}x_0$ *or* $x_0 \succeq T^{-1}x_0$
(iv) *either T is O-continuous; or*
(v) (X,d) *is O-regular.*

Then T has a fixed point. Moreover, T has a unique fixed point whenever $z \preceq u$ or $z \succeq u$ for all $z,u \in Fix\left(T\right)$.

Proof. Define the mapping $\alpha : X \times X \to [0,+\infty[$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x \preceq y \\ 0 & \text{otherwise} \end{cases}$$

and the mapping $\eta : X \times X \to [0, +\infty[$ by

$$\eta(x,y) = \begin{cases} 1 & \text{if } x \succeq y \\ 0 & \text{otherwise} \end{cases}$$

Using condition (iii) we have

$$x_0 \preceq T^{-1}x_0 \Rightarrow \alpha(x_0, T^{-1}x_0) \geq 1,$$

or

$$x_0 \succeq T^{-1}x_0 \Rightarrow \eta(x_0, T^{-1}x_0) \leq 1.$$

Owing to the monotonicity of T^{-1}, we get

$$\alpha(x,y) \geq 1 \Rightarrow x \preceq y \Rightarrow T^{-1}x \preceq T^{-1}y \Rightarrow \alpha(T^{-1}x, T^{-1}y) \geq 1,$$

or

$$\eta(x,y) \leq 1 \Rightarrow x \succeq y \Rightarrow T^{-1}x \succeq T^{-1}y \Rightarrow \eta(T^{-1}x, T^{-1}y) \leq 1.$$

Therefore, (T_1) and (T_2) hold.

On the other hand, if

$$\begin{cases} \alpha(x,y) \geq 1 \\ \alpha(x,y) \geq 1 \end{cases} \Rightarrow \begin{cases} x \preceq y \\ y \preceq z \end{cases}$$

or

$$\begin{cases} \eta(x,y) \leq 1 \\ \eta(x,y) \leq 1 \end{cases} \Rightarrow \begin{cases} x \succeq y \\ y \succeq z \end{cases}$$

Since (X,d) be an O-complete partially ordered rectangular metric space, we conclude that

$$x \preceq z \text{ } or \text{ } x \succeq z \Rightarrow \alpha(x,z) \geq 1 \text{ } or \text{ } \eta(x,z) \leq 1.$$

Thus, (T_3) and (T_4) hold. This shows that T^{-1} is a triangular (α, η) $-$admissible mapping then

$$(\alpha\left(x_n, x_{n+1}\right) \geq 1 \text{ or } \eta\left(x_n, x_{n+1}\right) \leq 1).$$

Now, if T is O-continuous, then $x_n \preceq x_{n+1}$ or $x_n \succeq x_{n+1} \Rightarrow \alpha(x_n, x_{n+1}) \geq 1$ or $\eta(x_n, x_{n+1}) \leq 1$ and $x_n \to z$ as $n \to \infty$ with $z \in X \Longrightarrow Tx_n \to Tx$. The existence and uniqueness of a fixed point follows from Theorem 1.

Now, suppose that follow $(X, d, \preceq)$ is O-regular. Let $\{x_n\}$ be a sequence such that

$$\{n_n\} \preceq x \text{ } or \text{ } \{n_n\} \succeq x,$$

which implies that

$$(\alpha\left(x_n, x\right) \geq 1 \text{ or } \eta\left(x_n, x\right) \leq 1),$$

for all n and $x_n \to x$ as $n \to \infty$. This shows that (X,d) is $(\alpha, \eta)-$ regular. Thus, the existence and uniqueness of fixed point from Theorem 3. $\square$

Corollary 2. *Let $(X, d, \preceq)$ be an O-complete partially ordered rectangular metric spaces. Further, let $T : X \to X$ be a bijective self mapping on X be such that T^{-1} is a monotone mapping and there exist $k \in]0,1[$ such that $kd(Tx, Ty) \geq d(x,y)$, for all $x, y \in X$ with $x \preceq y$ or $x \succeq y$. Also, suppose that the following conditions hold:*

(i) there exists $x_0 \in X$ such that $x_0 \preceq T^{-1}x_0$ or $x_0 \succeq T^{-1}x_0$;
(ii) either T is O-continuous; or
(iii) X is O-regular.

Then T has a fixed point. Moreover, T has a unique fixed point whenever $z \preceq u$ or $z \succeq u$ for all $z, u \in Fix\,(T)$.

Proof. Let $F(t) = ln(t)$ for all $t \in]0, +\infty[$ and $\tau = \frac{1}{ln(k)}$. Clearly $F \in \mathbb{F}$ and $\tau > 0$. We prove that T is a generalized F-expansive mapping. Indeed, Since

$$kd(Tx, Ty) \geq d(x, y).$$

We have

$$\begin{aligned} ln\left[k.d(Tx, Ty)\right] &= ln\left[d(Tx, Ty)\right] + ln(k) \\ &= ln\left(d(Tx, Ty)\right) - \frac{1}{ln(k)} \\ &\geq ln\left[d(x, y)\right] \\ &\geq ln\left[\min\{d(x, y), d(x, Tx), d(y, Ty), d(Tx, y)\}\right]. \end{aligned}$$

As in the proof of Theorems 1 and 4, T has a unique fixed point $x \in X$. $\square$

Corollary 3. *Let $(X, d, \preceq)$ be an O-complete partially ordered rectangular metric spaces. Further, let $T : X \to X$ be a bijective self mapping on X such that T^{-1} is a monotone mapping and there exist $\alpha \in \left]0, \frac{1}{2}\right[$ such that*

$$\alpha d(Tx, Ty) \geq \frac{d(x, Tx) + d(y, Ty)}{2}$$

for all $x, y \in X$ with $x \preceq y$ or $x \succeq y$. Also suppose that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq T^{-1}x_0$ or $x_0 \succeq T^{-1}x_0$;
(ii) either T is O-continuous; or
(iii) X is O-regular.

Then T has a fixed point. Moreover, T has a unique fixed point whenever $z \preceq u$ or $z \succeq u$ for all $z, u \in Fix(T)$.

Proof. Let $F(t) = ln(t)$ for all $t \in]0, +\infty[$ and $\tau = \frac{1}{ln(2\alpha)}$. Clearly $F \in \mathbb{F}$ and $\tau > 0$. We prove that T is a generalized F-expansive mapping. Indeed, since

$$\alpha d(Tx, Ty) \geq \frac{d(x, Tx) + d(y, Ty)}{2}.$$

We have

$$\begin{aligned} ln\left[2\alpha.d(Tx, Ty)\right] &= ln\left[d(Tx, Ty)\right] + ln(2\alpha) \\ &= ln\left(d(Tx, Ty)\right) - \frac{1}{ln(2\alpha)} \\ &\geq ln\left[d(x, Tx) + d(y, Ty)\right] \\ &\geq ln\left[\min\{d(x, y), d(x, Tx), d(y, Ty), d(Tx, y)\}\right]. \end{aligned}$$

As in the proof of Theorems 1 and 4, T has a unique fixed point $x \in X$. $\square$

Corollary 4. *Let $(X, d, \preceq)$ be an O-complete partially ordered rectangular metric spaces. Further, let $T : X \to X$ be a bijective self mapping on X, such that T^{-1} is a monotone mapping and there exist $\lambda \in \left]0, \frac{1}{3}\right[$ such that*

$$\alpha d(Tx, Ty) \geq \frac{d(x, y) + d(x, Tx) + d(y, Ty)}{3}$$

for all $x, y \in X$ with $x \preceq y$ or $x \succeq y$. Also suppose that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq T^{-1}x_0$ or $x_0 \succeq T^{-1}x_0$;
(ii) either T is O-continuous; or
(iii) X is O-regular.

Then T has a fixed point. Moreover, T has a unique fixed point whenever $z \preceq u$ or $z \succeq u$ for all $z, u \in Fix(T)$.

Proof. Let $F(t) = ln(t)$ for all $t \in]0, +\infty[$, and $\tau = \frac{1}{ln(3\alpha)}$. Clearly $F \in \mathbb{F}$ and $\tau > 0$. We prove that T is a F-expansive mapping. Indeed, since

$$\lambda d(Tx, Ty) \geq \frac{d(x,y) + d(x,Tx) + d(y,Ty)}{3}.$$

We have

$$\begin{aligned} ln\left[3\lambda.d(Tx,Ty)\right] &= ln\left[d(Tx,Ty)\right] + ln(3\lambda) \\ &= ln\left(d(Tx,Ty)\right) - \frac{1}{ln(3\lambda)} \\ &\geq ln\left[d(x,y) + d(x,Tx) + d(y,Ty)\right] \\ &\geq ln\left[\min\{d(x,y), d(x,Tx), d(y,Ty), d(Tx,y)\}\right]. \end{aligned}$$

As in the proof of Theorems 1 and 4, T has a unique fixed point $x \in X$. □

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