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The hyper order and fixed points of solutions of a class of linear differential equations

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Abstract: In this paper, we precise the hyper order of solutions for a class of higher order linear differential equations and investigate the exponents of convergence of the fixed points of solutions and their first derivatives for the second order case. These results generalize those of Nan Li and Lianzhong Yang and of Chen and Shon.

Keywords: Linear differential equations; Hyper order; Fixed points.

MSC: 34M10, 30D35.

1. Introduction

In this paper, we use standard notations from the value distribution theory of meromorphic functions (see [1–3]). We suppose that f is a meromorphic function in whole complex plane $\mathbb{C}$. In addition, we denote the order of growth of f by $\rho(f)$, and use the notation $\rho_2(f)$ to denote the hyper-order of f, defined by

$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f.

To give the precise estimate of fixed points, we denote the exponent of convergence of fixed points by $\tau(f)$, which is defined by

$$\tau(f) = \lambda(f-z) = \limsup_{r\to+\infty} \frac{\log N\left(r, \frac{1}{f-z}\right)}{\log r}$$

and the hyper-exponent of convergence of fixed points and distinct fixed points are denoted by $\tau_2(f)$ and $\overline{\tau}_2(f)$ and are defined by

$$\tau_2(f) = \lambda_2(f-z) = \limsup_{r\to+\infty} \frac{\log\log N\left(r, \frac{1}{f-z}\right)}{\log r},$$

and

$$\overline{\tau}_2(f) = \overline{\lambda}_2(f-z) = \limsup_{r\to+\infty} \frac{\log\log \overline{N}\left(r, \frac{1}{f-z}\right)}{\log r},$$

respectively, where $N\left(r, \frac{1}{f-z}\right)$ and $\overline{N}\left(r, \frac{1}{f-z}\right)$ are respectively the integrated counting function of fixed points and distinct fixed points of f. We denote the exponent of convergence of zeros (distinct zeros) of f by $\lambda(f)$ $(\overline{\lambda}(f))$ and the hyper-exponent of convergence of zeros (distinct zeros) of f by $\lambda_2(f)$ $(\overline{\lambda}_2(f))$.

Consider the second-order homogeneous linear differential equation

$$f'' + P(e^z)f' + Q(e^z)f = 0, \tag{1}$$

where $P(w)$ and $Q(w)$ are not constants polynomials in $w = e^z$ $(z \in \mathbb{C})$. It's well-known that every solution of Equation (1) is entire.

Suppose $f \not\equiv 0$ is a solution of (1). If f satisfies the condition

$$\limsup_{r\to+\infty} \frac{\log T(r,f)}{r} = 0,$$

then we say that f is a nontrivial subnormal solution of (1), and if f satisfies the condition [4],

$$\limsup_{r\to+\infty} \frac{\log T(r,f)}{r^n} = 0,$$

then we say that f is a nontrivial n-subnormal solution of (1). In [5], Wittich investigated the subnormal solution of (1), and obtained the form of all subnormal solutions in the following theorem:

Theorem 1. [5] *If $f \not\equiv 0$ is a subnormal solution of* (1)*, then f must have the form*

$$f(z) = e^{cz}(a_0 + a_1 e^z + \cdots + a_m e^{mz}),$$

where $m \geq 0$ is an integer and c, a_0, a_1,...,a_m are constants with $a_0 a_m \neq 0$.

Gundersen and Steinbert [6] refined Theorem 1 and got the following theorem:

Theorem 2. [6] *Under the assumption of Theorem 1, the following statements hold:*

(i) If $\deg P > \deg Q$ and $Q \not\equiv 0$, then any subnormal solution $f \not\equiv 0$ of (7) *must have the form*

$$f(z) = \sum_{k=0}^{m} h_k e^{-kz},$$

where $m \geq 1$ is an integer and h_0, h_1, ..., h_m are constants with $h_0 \neq 0$ and $h_m \neq 0$.

(ii) If $\deg P \geq 1$ and $Q \equiv 0$, then any subnormal solution of Equation (7) *must be constant.*

(iii) If $\deg P < \deg Q$, then the only subnormal solution of (7) *is $f \equiv 0$.*

Chen and Shon [7] investigated more general equation than (7), and got the following theorem: Set

$$a_j(z) = a_{jd_j} z^{d_j} + a_{j(d_j-1)} z^{d_j-1} + \cdots + a_{j1} z + a_{j0}, \ (j = 0, \cdots, n), \tag{2}$$

$$b_k(z) = b_{km_k} z^{m_k} + b_{k(m_k-1)} z^{m_k-1} + \cdots + b_{k1} z + b_{k0}, \ (k = 0, \cdots, s), \tag{3}$$

where $d_j \geq 0$ $(j = 0, \cdots, n)$, $m_k \geq 0$ $(k = 0, \cdots, s)$ are integers, a_{jd_j}, ..., a_{j0}; b_{km_k}, ..., b_{k0} are complex constants such that $a_{jd_j} \neq 0$, $b_{km_k} \neq 0$.

Theorem 3. [7] *Let $a_n(z), ..., a_1(z)$, $a_0(z)$, $b_s(z), ..., b_1(z)$, $b_0(z)$ be polynomials and satisfy* (2) *and* (3)*, and $a_n(z) b_s(z) \neq 0$. Suppose that $P^*(e^z) = a_n(z)e^{nz} + \cdots + a_1(z)e^z + a_0(z)$, $Q^*(e^z) = b_s(z)e^{sz} + \cdots + b_1(z)e^z + b_0(z)$. If $n < s$, then every solution f $(\not\equiv 0)$ of equation*

$$f'' + P^*(e^z)f' + Q^*(e^z)f = 0$$

satisfies $\rho_2(f) = 1$.

Many authors investigated the growth of solutions and the existence of subnormal solutions for some class of higher order linear differential equations (see [4,7–13]). For the higher-order linear homogeneous differential equation

$$f^{(k)} + P_{k-1}(e^z) f^{(k-1)} + \cdots + P_0(e^z) f = 0, \tag{4}$$

where $P_j(e^z)$ $(j = 0, \cdots, k-1)$ are polynomials in z, Yang and Li [11] generalized the result of Theorem 2 to the higher order and obtained the following results: Set

$$a_{jm_j}(z) = a_{jm_jd_{jm_j}} z^{d_{jm_j}} + a_{jm_j(d_{jm_j}-1)} z^{d_{jm_j}-1} + \cdots + a_{jm_j1} z + a_{jm_j0}, \quad (5)$$

where $d_{jm_j} \geq 0$ $(j = 0, \cdots, k-1)$ are integers, $a_{jm_jd_{jmj}}, ..., a_{jm_j0}$ are complex constants, $a_{jm_jd_{jmj}} \neq 0$.

Theorem 4. [11] *Let $a_{jm_j}(z)$ be polynomials and satisfy (5). Suppose that*

$$P_j(e^z) = a_{jm_j}(z)\, e^{m_jz} + \cdots + a_{j1}(z)\, e^z,$$

where $a_{jm_j}(z) \not\equiv 0$. If there exists an integer s $(s \in \{0, \cdots, k-1\})$ satisfying

$$m_s > \max\{m_j : j = 0, \cdots, s-1, s+1, \cdots, k-1\} = m,$$

then every solution $f \not\equiv 0$ of Equation (4) satisfies $\rho_2(f) = 1$ if one of the following condition holds:

(i) $s = 0$ *or* 1.
(ii) $s \geq 2$ *and* $\deg a_{0j}(z) > \deg a_{ij}(z)$ $(i \neq 0)$.

Theorem 5. [11] *Under the assumption of Theorem 4, if $zP_0(e^z) + P_1(e^z) \not\equiv 0$, then we have every solution $f \not\equiv 0$ of Equation (4) satisfies*

$$\tau_2(f) = \overline{\tau}_2(f) = \rho_2(f) = 1.$$

In particular, they also investigated the exponents of convergence of the fixed points of solutions and their first derivatives for a second order Equation (1) and obtained the following theorem:

Theorem 6. [11] *Let $a_n(z), ..., a_1(z)$, $b_s(z), ..., b_1(z)$ be polynomials and satisfy (2) and (3), and $a_n(z)b_s(z) \neq 0$. Suppose that $P(e^z) = a_n(z)e^{nz} + \cdots + a_1(z)e^z$, $Q(e^z) = b_s(z)e^{sz} + \cdots + b_1(z)e^z$. If $n \neq s$, then every solution f $(\not\equiv 0)$ of Equation (1) satisfy $\lambda(f-z) = \lambda(f'-z) = \rho(f) = \infty$ and $\lambda_2(f-z) = \lambda_2(f'-z) = \rho_2(f) = 1$.*

Thus, it is natural to ask what will happen if we change $\exp\{z\}$ in the coefficients of (4) into $\exp\{A(z)\}$? In this paper, we consider the above problem to Theorems 3, 4, 5 and 6, we obtain the following results: We set

$$A(z) = c_nz^n + c_{n-1}z^{n-1} + \cdots + c_1z + c_0,$$

where $n \geq 1$ is an integer and $c_0, ..., c_n$ are complex constants such that $Rec_n > 0$, throughout the rest of this paper.

Theorem 7. *Let $a_{jm_j}(z)$ be polynomials and satisfy (5). Suppose that*

$$P_j(e^{A(z)}) = a_{jm_j}(z)\, e^{m_jA(z)} + \cdots + a_{j1}(z)\, e^{A(z)}, \quad (6)$$

where $a_{jm_j}(z) \not\equiv 0$. If there exists an integer s $(s \in \{0, \cdots, k-1\})$ satisfying

$$m_s > \max\{m_j : j = 0, \cdots, s-1, s+1, \cdots, k-1\} = m,$$

then every solution $f \not\equiv 0$ of equation

$$f^{(k)} + P_{k-1}(e^{A(z)})f^{(k-1)} + \cdots + P_0(e^{A(z)})f = 0 \quad (7)$$

satisfies $\rho(f) = \infty$ and $\rho_2(f) = n$ if one of the following condition holds:

(i) $s = 0$ *or* 1.
(ii) $s \geq 2$ *and* $\deg a_{0j}(z) > \deg a_{ij}(z)$ $(i \neq 0)$.

Example 1. Let $f = e^{e^{z^2}}$ be a solution of the equation

$$f^{(4)} - 2ze^{z^2}f^{(3)} - 12z^2e^{z^2}f'' - 24z^3e^{z^2}f' - [24z^2e^{3z^2} + (96z^2+12)e^{2z^2} + (16z^4+48z^2+12)e^{z^2}]f = 0.$$

Set

$$\begin{aligned}
&P_3(e^{A(z)}) = a_{3,1}(z)e^{A(z)} = -2ze^{z^2},\\
&P_2(e^{A(z)}) = a_{2,1}(z)e^{A(z)} = -12z^2e^{z^2},\\
&P_1(e^{A(z)}) = a_{1,1}(z)e^{A(z)} = -24z^3e^{z^2},\\
&P_0(e^{A(z)}) = a_{0,3}(z)e^{3A(z)} + a_{0,2}(z)e^{2A(z)} + a_{0,1}(z)e^{A(z)} = -24z^2e^{3z^2} - (96z^2+12)e^{2z^2} - (16z^4+48z^2+12)e^{z^2}.
\end{aligned}$$

We remark that $s = 0$ and $m_0 = 3 > \max\{m_j : j = 1,2,3\} = m = 1$. Obviously, the conditions of Theorem 7 are satisfied, we see that $\rho(f) = \infty$ and $\rho_2(f) = n = 2$.

Remark 1. Very recently, Li *et al.*, [4] have investigated n subnormal solutions of the Equation (7) with

$$P_j(e^{A(z)}) = a_{jm_j}e^{m_jA(z)} + \cdots + a_{j1}e^{A(z)} \ \ (j = 0,...,k-1),$$

where $a_{jm_j}, \cdots, a_{j1}$ $(j = 0,...,k-1)$ are complex constants instead of polynomials and obtained some results concerning their growth.

Corollary 1. *Under the assumption of Theorem 7, if $zP_0(e^{A(z)}) + P_1(e^{A(z)}) \not\equiv 0$, then we have every solution $f \not\equiv 0$ of Equation (4) satisfies*

$$\tau(f) = \overline{\tau}(f) = \rho(f) = \infty \text{ and } \tau_2(f) = \overline{\tau}_2(f) = \rho_2(f) = n.$$

In particular, we also investigate the exponents of convergence of the fixed points of solutions and their first derivatives for a second order equation

$$f'' + P(e^{A(z)})f' + Q(e^{A(z)})f = 0, \tag{8}$$

and we obtain the following theorems:

Theorem 8. *Let $a_p(z),..., a_1(z)$, $b_s(z),..., b_1(z)$ be polynomials and satisfy (2) and (3), and $a_p(z)b_s(z) \neq 0$. Suppose that $P(e^{A(z)}) = a_p(z)e^{pA(z)} + \cdots + a_1(z)e^{A(z)}$, $Q(e^{A(z)}) = b_s(z)e^{sA(z)} + \cdots + b_1(z)e^{A(z)}$. If $p \neq s$, then every solution $f\ (\not\equiv 0)$ of Equation (8) satisfies $\lambda(f-z) = \lambda(f'-z) = \rho(f) = \infty$ and $\lambda_2(f-z) = \lambda_2(f'-z) = \rho_2(f) = n$.*

Example 2. Let $f = e^{e^{z^2}}$ be a solution of the equation

$$f'' - 3ze^{z^2}f' + [2z^2e^{2z^2} - (4z^2+2)e^{z^2}]f = 0.$$

Set

$$\begin{aligned}
&P(e^{A(z)}) = a_1(z)e^{A(z)} = -3ze^{z^2},\\
&Q(e^{A(z)}) = b_2(z)e^{2A(z)} + b_1(z)e^{A(z)} = 2z^2e^{2z^2} - (4z^2+2)e^{z^2}.
\end{aligned}$$

It is clear that the conditions of Theorem 8 are satisfied with $p = 1 \neq s = 2$, we see that $\lambda(e^{e^{z^2}} - z) = \lambda(2ze^{z^2}e^{e^{z^2}} - z) = \rho(f) = \infty$ and $\lambda_2(e^{e^{z^2}} - z) = \lambda_2(2ze^{z^2}e^{e^{z^2}} - z) = \rho_2(f) = n = 2$.

Remark 2. If $p = s$, then the conclusions of Theorem 8 does not hold. For instance, consider the following equation

$$f''+\left(\left(z^4+2iz\right)e^{2(1+5i)z^3+2z}+\left(-z^2+(2-i)\,z\right)e^{(1+5i)z^3+z}\right)f' - \left(\left(z^3+2i\right)e^{2(1+5i)z^3+2z}+(-z+2-i)\,e^{(1+5i)z^3+z}\right)f=0. \tag{9}$$

We can easily see that (9) has solution $f(z) = z$ which satisfies $\rho(f) = 0 \neq \infty$ and $\rho_2(f) = 0 \neq n = 3$. In this example, we have $p = s = 2$, $A(z) = (1+5i)\,z^3 + z$, $a_2(z) = z^4 + 2iz$, $a_1(z) = -z^2 + (2-i)\,z$, $b_2(z) = -\left(z^3 + 2i\right)$ and $b_1(z) = -(-z+2-i)$.

Theorem 9. *Let $a_p(z), ..., a_1(z), a_0(z), b_s(z), ..., b_1(z), b_0(z)$ be polynomials and satisfy (2) and (3), and $a_p(z)b_s(z) \neq 0$. Suppose that*

$$P^*(e^{A(z)}) = a_p(z)e^{pA(z)} + \cdots + a_1(z)e^{A(z)} + a_0(z),$$
$$Q^*(e^{A(z)}) = b_s(z)e^{sA(z)} + \cdots + b_1(z)e^{A(z)} + b_0(z).$$

If $p < s$, then every solution $f\ (\not\equiv 0)$ of equation

$$f'' + P^*(e^{A(z)})f' + Q^*(e^{A(z)})f = 0 \tag{10}$$

satisfies $\rho(f) = \infty$ and $\rho_2(f) = n$.

Example 3. Let $f = e^z e^{e^z}$ be a solution of the equation

$$f'' + (e^{z+1} - 3)f' + [(-e^{-2} - e^{-1})e^{2(z+1)} - e^{z+1} + 2]f = 0.$$

Set

$$P^*(e^{A(z)}) = a_1(z)e^{A(z)} + a_0(z) = e^{z+1} - 3,$$
$$Q^*(e^{A(z)}) = b_2(z)e^{2A(z)} + b_1(z)e^{A(z)} + b_0(z) = (-e^{-2} - e^{-1})e^{2(z+1)} - e^{z+1} + 2.$$

It is clear that the conditions of Theorem 9 are satisfied with $p = 1 < s = 2$, here we have $\rho(f) = \infty$ and $\rho_2(f) = n = 1$.

Remark 3. If $p \geq s$, then the conclusions of Theorem 9 does not hold. For instance, consider the following equation

$$f''-\left(\left(2z^2+3z\right)e^{(1-i)z^2+2z+i}+iz^3-z^2+(1+i)\,z\right)f' + \left((2z+3)\,e^{(1-i)z^2+2z+i}+iz^2-z+1+i\right)f=0. \tag{11}$$

It is easy to see that (11) has solution $f(z) = z$ which satisfies $\rho(f) = 0 \neq \infty$ and $\rho_2(f) = 0 \neq n = 2$. In this example, we have $p = s = 1$, $A(z) = (1-i)\,z^2 + 2z + i$, $a_1(z) = -\left(2z^2 + 3z\right)$, $a_0(z) = -\left(iz^3 - z^2 + (1+i)\,z\right)$, $b_1(z) = 2z + 3$ and $b_0(z) = iz^2 - z + 1 + i$.

Remark 4. Setting $c_n = 1, c_{n-1} = \cdots = c_0 = 0$ and $n = 1$, in Theorem 7, Corollary 1, Theorem 8 and Theorem 9, we obtain Theorem 4, Theorem 5, Theorem 6 and Theorem 3 respectively.

2. Auxiliary Lemmas

Recall that

$$A(z) = c_n z^n + c_{n-1}z^{n-1} + \cdots + c_0,, c_l = \alpha_l e^{i\theta_l},\ z = re^{i\theta},\ Rec_n > 0,$$

we set $\delta_l(A,\theta) = Re(c_l(e^{i\theta})^l) = \alpha_l \cos(\theta_l + l\theta)$, and $H_{l,0} = \{\theta \in [0,2\pi) : \delta_l(A,\theta) = 0\}$, $H_{l,+} = \{\theta \in [0,2\pi) : \delta_l(A,\theta) > 0\}$, $H_{l,-} = \{\theta \in [0,2\pi) : \delta_l(A,\theta) < 0\}$, for $l = 1,\cdots,n$, throughout the rest of this paper. Obviously, if $\delta_n(A,\theta) \neq 0$, as $r \to +\infty$, we get

$$|e^{A(z)}| = e^{\delta_n(A,\theta)r^n + \cdots + \delta_1(A,\theta)r + Rec_0} = e^{\delta_n(A,\theta)r^n(1+o(1))}. \tag{12}$$

Lemma 1. [3] *Let $f_j(z)$ $(j = 1,\cdots,n)$ $(n \geq 2)$ be meromorphic functions, $g_j(z)$ $(j = 1,\cdots,n)$ be entire functions, and satisfy*

(i) $\sum_{j=1}^{n} e^{g_j(z)} \equiv 0$;
(ii) when $1 \leq j \leq k \leq n$, *then* $g_i(z) - g_k(z)$ *is not a constant;*
(iii) when $1 \leq j \leq n$, $1 \leq h \leq k \leq n$,

then

$$T(r,f_j) = o\{T(r,e^{g_h - g_k})\} \quad (r \to +\infty, r \notin E),$$

where $E \subset (1,\infty)$ is of finite linear measure or logarithmic measure. Also, $f_j(z) \equiv 0$ $(j = 1,\cdots,n)$.

Lemma 2. *Let $A(z)$, $P_j(e^{A(z)})$, m_j, m_s, m and $a_{ij}(z)$ satisfy the hypotheses of Theorem 7, then Equation (7) has no constant polynomial solution.*

Proof. Suppose that $f_0(z) = b_l z^l + \cdots + b_1 z + b_0$ $(l \geq 1)$ is a nonconstant polynomial solution of (7), where $b_l \neq 0, \cdots, b_0$ are complex constants.

If $l \geq s$, then $f^{(s)} \not\equiv 0$. Taking $z = r$, we have

$$|e^{A(z)}| = \left|e^{A(r)}\right| = \left|e^{c_n r^n + c_{n-1}r^{n-1} + \cdots + c_0}\right| = e^{Rec_n r^n + Rec_{n-1} r^{n-1} + \cdots + Rec_0} = e^{Rec_n r^n(1+o(1))}. \tag{13}$$

Substituting f_0 into (7) and using (13), we conclude that

$$\begin{aligned}
&|a_{sm_s dsm_s} r^{d_{sm_s}} e^{m_s Rec_n r^n(1+o(1))}| b_l l(l-1)\cdots(l-s+1)| r^{l-s}(1+o(1)) \leq |-P_s(e^{A(r)}) f_0^{(s)}(r)| \\
&\leq |f^{(k)}(r)| + |P_{k-1}(e^{A(r)}) f_0^{(k-1)}(r)| + \cdots + |P_{s+1}(e^{A(r)}) f_0^{(s+1)}(r)| + |P_{s-1}(e^{A(r)}) f_0^{(s-1)}(r)| + \cdots + |P_0(e^{A(r)}) f_0(r)| \\
&\leq M_0 r^d e^{mRec_n r^n(1+o(1))}(1+o(1)),
\end{aligned} \tag{14}$$

where $d = \max\{d_{jm_j} : j = 0,\cdots,s-1,s+1,\cdots,k-1\}$ and $M_0 > 0$ is some constant. Since $m_s > m$, we see that (3) is a contradiction. Obviously, when $s = 0$ or 1, we can get that the Equation (7) has nonconstant polynomial solution from the above process. If $l < s$, then

$$P_l(e^{A(z)}) f_0^{(l)}(z) + \cdots + P_0(e^{A(z)}) f_0(z) = 0. \tag{15}$$

Set $\max\{m_j : j = 0,\cdots,l\} = h$. If $m_j < h$, then we can rewrite

$$P_j(e^{A(z)}) = a_{jh}(z) e^{hA(z)} + \cdots + a_{j(m_j+1)}(z) e^{(m_j+1)A(z)} + a_{jm_j}(z) e^{m_j A(z)} + \cdots + a_{j1}(z) e^{A(z)}$$

for $j = 0,\cdots,l$, where $a_{jh}(z) = \cdots = a_{j(m_j+1)}(z) = 0$. Thus, we conclude by (15) that

$$\begin{aligned}
&[a_{lh}(z) f_0^{(l)} + a_{(l-1)h}(z) f_0^{(l-1)} + \cdots + a_{0h}(z) f_0] e^{hA(z)} + \cdots + [a_{lj}(z) f_0^{(l)} + a_{(l-1)j}(z) f_0^{(l-1)} \\
&+ \cdots + a_{0j}(z) f_0] e^{jA(z)} + \cdots + [a_{l1}(z) f_0^{(l)} + a_{(l-1)1}(z) f_0^{(l-1)} + \cdots + a_{01}(z) f_0] e^{A(z)} = 0.
\end{aligned} \tag{16}$$

Set

$$Q_j(z) = a_{lj}(z) f_0^{(l)} + a_{(l-1)j}(z) f_0^{(l-1)} + \cdots + a_{0j} f_0 \quad (j = 1,\cdots,h). \tag{17}$$

Since f_0 and $a_{ij}(z)$ are polynomials, we see that

$$m(r,Q_j) = o\{m(r,e^{(\alpha-\beta)A(z)})\}, \quad (1 \leq \beta < \alpha \leq h). \tag{18}$$

By Lemma 1 and (16) -(18), we conclude that

$$Q_1(z) \equiv Q_2(z) \equiv \cdots \equiv Q_h(z) \equiv 0. \tag{19}$$

Since $\deg f_0 > \deg f_0' > \cdots > \deg f_0^{(l)}$ and $\deg a_{0j}(z) > \deg a_{ij}(z)$ $(i \neq 0)$, so by (16) and (19), we get a contradiction. □

Lemma 3. [14,15] *Let $f(z)$ be an entire function and suppose that $|f^{(k)}(z)|$ is unbounded on some ray $\arg z = \theta$. Then, there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, \cdots)$, where $r_m \to +\infty$ such that $f^{(k)}(z_m) \to \infty$ and*

$$\left| \frac{f^{(j)}(z_m)}{f^{(k)}(z_m)} \right| \leq |z_m|^{k-j} (1 + o(1)) \qquad (j = 0, \cdots, k-1).$$

Lemma 4. [16] *Let $f(z)$ be a transcendental meromorphic function of finite order ρ. Let $\Gamma = \{(k_1, j_1), (k_2, j_2), \cdots, (k_m, j_m)\}$ denote a set of distinct pairs of integers satisfying $k_i > j_i \geq 0$ $(i = 1, 2, \cdots, m)$ and let $\varepsilon > 0$ be a given constant. Then, there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero such that if $\theta \in [0, 2\pi) \setminus E_1$, then there is a constant $R_1 = R_1(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \geq R_1$ and for all $(k, j) \in \Gamma$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho - 1 + \varepsilon)}.$$

Lemma 5. [17] *Let $f(z)$ be an entire function with $\rho(f) = \rho < \infty$. Suppose that there exists a set $E_2 \subset [0, 2\pi)$ that has linear measure zero, such that for any ray $\arg z = \theta_0 \in [0, 2\pi) \setminus E_2$, $|f(re^{i\theta_0})| \leq Mr^k$ $(M = M(\theta_0) > 0$ is a constant and $k > 0$ is a constant independent of $\theta_0)$. Then $f(z)$ is a polynomial with $\deg f \leq k$.*

Lemma 6. [16] *Let f be a transcendental meromorphic function, and $\alpha > 1$ be a given constant. Then, there exists a set $E_3 \subset (1, \infty)$ with finite logarithmic measure and a constant $C > 0$ that depends only on α and i, j $(i, j \in \mathbb{N})$, such that for all z satisfying $|z| = r \notin E_3 \cup [0, 1]$,*

$$\left| \frac{f^{(j)}(z)}{f^{(i)}(z)} \right| \leq C \left(\frac{T(\alpha r, f)}{r} (\log^{\alpha} r) \log T(\alpha r, f) \right)^{j-i}. \tag{20}$$

Remark 5. From the proof of Lemma 6 ([16, Theorem 3]), we can see that the exceptional set E_4 equals $\{|z| : z \in (\cup_{n=1}^{+\infty} O(a_n))\}$, where $a_n (n = 1, 2, \cdots)$ denote all zeros and poles of $f^{(i)}$, and $O(a_n)$ denote sufficiently small neighborhoods of a_n. Hence, if $f(z)$ is a transcendental entire function and z is a point that satisfies $|f(z)|$ to be sufficiently large, then the point $z \notin E_4$ satisfies (20). For details, see , [9, Remark 2.10].

Lemma 7. [10,18] *Let $A_0, \cdots, A_{k-1}$ be entire functions of finite order. If $f(z)$ is a solution of equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_0 f = 0,$$

then $\rho_2(f) \leq \max\{\rho(A_j) : j = 0, \cdots, k-1\}$.

Lemma 8. [19] *Let $g(z)$ be an entire function of infinite order with the hyper-order $\rho_2(g) = \rho$, and let $\nu(r)$ be the central index of g. Then,*

$$\limsup_{r \to +\infty} \frac{\log \log \nu(r)}{\log r} = \rho_2(g) = \rho.$$

Lemma 9. [7] *Let $f(z)$ be an entire function that satisfies $\rho(f) = \rho (n < \rho < \infty)$; or $\rho(f) = \infty$ and $\rho_2 = 0$; or $\rho_2 = \alpha (0 < \alpha < \infty)$, and a set $E_5 \subset [1, \infty)$ has a finite logarithmic measure. Then, there exists a sequence $\{z_k = r_k e^{i\theta_k}\}$ such that $|f(z_k)| = M(r_k, f)$, $\theta_k \in [0, 2\pi)$, $\lim_{k \to \infty} \theta_k = \theta_0 \in [0, 2\pi)$, $r_k \notin E_5$, and $r_k \to \infty$, such that*

(i) *if $\rho(f) = \rho$ $(n < \rho < \infty)$, then for any given $\varepsilon_1 (0 < \varepsilon_1 < \frac{\rho - n}{2})$,*

$${r_k}^{\rho - \varepsilon_1} < \nu(r_k) < {r_k}^{\rho + \varepsilon_1};$$

(ii) if $\rho(f) = \infty$ and $\rho_2(f) = 0$, then for any given $\varepsilon_2 (0 < \varepsilon_2 < \frac{1}{2})$, and for any large M (> 0), we have, as r_k is sufficiently large,

$$r_k{}^M < \nu(r_k) < \exp\{r_k{}^{\varepsilon_2}\};$$

(iii) if $\rho_2(f) = \alpha (0 < \alpha < \infty)$, then for any given $\varepsilon_3 (0 < \varepsilon_3 < \alpha)$,

$$\exp\{r_k{}^{\alpha-\varepsilon_3}\} < \nu(r_k) < \exp\{r_k{}^{\alpha+\varepsilon_3}\}.$$

Lemma 10. [20] *Let g be a non-constant entire function, and let $0 < \delta < 1$. There exists a set $E_6 \subset [1, \infty)$ of finite logarithmic measure with the following property. For $r \in [1, \infty) \setminus E_6$, the central index $\nu(r)$ of g satisfies*

$$\nu(r) \le (\log M(r, g))^{1+\delta}.$$

Lemma 11. [21,22] *Let A_0, ..., A_{k-1}, $F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_0 f = F,$$

with $\rho(f) = +\infty$ and $\rho_2(f) = \rho$, then f satisfies $\overline{\lambda}(f) = \lambda(f) = \rho(f) = \infty$ and $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho$.

Lemma 12. [14] *Let $\varphi : [0, +\infty) \to \mathbb{R}$ and $\psi : [0, +\infty) \to \mathbb{R}$ be monotone non-decreasing functions such that $\varphi(r) \le \psi(r)$ for all $r \notin E_7 \cup [0, 1]$, where $E_7 \subset (1, +\infty)$ is a set of finite logarithmic measure. Let $\gamma > 1$ be a given constant. Then there exists a $r_1 = r_1(\gamma) > 0$ such that $\varphi(r) \le \psi(\gamma r)$ for all $r > r_1$.*

3. Proofs of the results

Proof of Theorem 7. Suppose that $f \not\equiv 0$ is a solution of (7), then f is an entire function. By Lemma 2, we see that f is transcendental.

First step. We prove that $\rho(f) = \infty$.

Suppose, to the contrary, that $\rho(f) = \rho < \infty$. By Lemma 4, for any given $\varepsilon > 0$, there exists a set $E_1 \subset [0, 2\pi)$ with linear measure zero, such that if $\theta \in [0, 2\pi) \setminus E_1$, then there exists a constant $R_1 = R_1(\theta) > 1$, such that for all z satisfying $\arg z = \theta$ and $|z| = r > R_1$, we have

$$\left| \frac{f^{(j)}(z)}{f^{(s)}(z)} \right| \le r^{(\rho-1+\varepsilon)(j-s)} \qquad j = s+1, \cdots, k. \tag{21}$$

Case 1. Take a ray $\arg z = \theta \in H_{n,+} \setminus E_1$, then $\delta_n(A, \theta) > 0$. We assume that $|f^{(s)}(re^{i\theta})|$ is bounded on the ray $\arg z = \theta$. If $|f^{(s)}(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 3, there exists a sequence $\{z_t = r_t e^{i\theta}\}$ such that as $r_t \to +\infty$, $f^{(s)}(z_t) \to \infty$ and

$$\left| \frac{f^{(i)}(z_t)}{f^{(s)}(z_t)} \right| \le r_t{}^{s-i}(1 + o(1)) \le 2 r_t{}^s, \qquad i = 0, \cdots, s-1. \tag{22}$$

By (7), we get

$$|P_s(e^{A(z_t)})| \le \left| \frac{f^{(k)}(z_t)}{f^{(s)}(z_t)} \right| + \sum_{j=0 j\neq s}^{k-1} |P_j(e^{A(z_t)})| \left| \frac{f^{(j)}(z)}{f^{(s)}(z)} \right|. \tag{23}$$

For $r_t \to +\infty$, we have

$$\begin{aligned}
\left|P_s(e^{A(z_t)})\right| &= |a_{sm_s}(z_t) e^{m_s A(z_t)} + \cdots + a_{s1}(z_t) e^{A(z_t)}| \\
&\ge |a_{sm_s}(z_t) e^{m_s A(z_t)}| - \left|a_{s(m_s-1)}(z_t) e^{m_{s-1} A(z_t)} + \cdots + a_{s1}(z_t) e^{A(z_t)}\right| \\
&\ge |a_{sm_s}(z_t) e^{m_s A(z_t)}| - \left[\left|a_{s(m_s-1)}(z_t) e^{m_{s-1} A(z_t)}\right| + \cdots + \left|a_{s1}(z_t) e^{A(z_t)}\right|\right] \\
&= |a_{sm_s d_{sm_s}}| r_t^{d_{sm_s}} e^{m_s \delta_n(A,\theta) r_t^n (1+o(1))} (1+o(1)) - [|a_{s(m_s-1)d_{s(m_s-1)}}| r_t^{d_{s(m_s-1)}} e^{(m_s-1)\delta_n(A,\theta) r_t^n (1+o(1))} (1+o(1)) \\
&\quad + \cdots + |a_{s1d_{s1}}| r_t^{d_{s1}} e^{\delta_n(A,\theta) r_t^n (1+o(1))} (1+o(1))]
\end{aligned}$$

$$\geq \frac{1}{2}|a_{sm_sd_{sm_s}}|r_t^{d_{sm_s}}e^{m_s\delta_n(A,\theta)r_t^n(1+o(1))}(1+o(1)), \tag{24}$$

and

$$\begin{aligned}\left|P_j(e^{A(z_t)})\right| &= |a_{jm_j}(z_t)e^{m_jA(z_t)}+\cdots+a_{j1}(z_t)e^{A(z_t)}| \\ &\leq |a_{jm_jd_{jm_j}}|r_t^{d_{jm_j}}e^{m_j\delta_n(A,\theta)r_t^n(1+o(1))}(1+o(1))+\cdots+|a_{jm_j1}|r_t^{d_{j1}}e^{\delta_n(A,\theta)r_t^n(1+o(1))}(1+o(1)) \\ &\leq 2|a_{jm_jd_{jm_j}}|r_t^d e^{m\delta_n(A,\theta)r_t^n(1+o(1))}(1+o(1)),\ \ (j\neq s),\end{aligned} \tag{25}$$

where $d=\max\{d_{jm_j}: j=0,\cdots,s-1,s+1,\cdots,k-1\}$. Substituting (21), (22), (24), (25) into (23), we obtain that for sufficiently large r_t

$$\frac{1}{2}|a_{sm_sd_{sm_s}}|{r_t}^{d_{sm_s}}e^{m_s\delta_n(A,\theta)r_t^n(1+o(1))}(1+o(1))\leq C_0r_t^{d+k\rho}e^{m\delta_n(A,\theta)(1+o(1)){r_t}^n}(1+o(1)), \tag{26}$$

where $C_0>0$ is a constant. From (26), we can get a contradiction by $m_s>m$ and $\delta_n(A,\theta)>0$, so

$$|f(re^{i\theta})|\leq Mr^s\leq M_1r^k, \qquad M_1>0, \tag{27}$$

on the ray $\arg z=\theta\in H_{n,+}\setminus E_1$.

Case 2. Now, we take a ray $\arg z=\theta\in H_{n,-}$, then $\delta_n(A,\theta)<0$. If $|f^{(k)}(re^{i\theta})|$ is unbounded on the ray $\arg z=\theta$, then by Lemma 3, there exists a sequence $\{z_t=r_te^{i\theta}\}$ such that as $r_t\to+\infty$, $f^{(s)}(z_t)\to\infty$ and

$$\left|\frac{f^{(i)}(z_t)}{f^{(k)}(z_t)}\right|\leq {r_t}^{k-i}(1+o(1))\leq 2{r_t}^k, \qquad i=0,\cdots,k-1. \tag{28}$$

By (7), we get

$$-1=P_{k-1}(e^{A(z_t)})\frac{f^{(k-1)}(z_t)}{f^{(k)}(z_t)}+\cdots+P_0(e^{A(z_t)})\frac{f(z_t)}{f^{(k)}(z_t)}. \tag{29}$$

For $r_t\to+\infty$, we have

$$\begin{aligned}\left|P_j(e^{A(z_t)})\right| &= |a_{jm_j}(z_t)e^{m_jA(z_t)}+\cdots+a_{j1}(z_t)e^{A(z_t)}| \\ &\leq |a_{jm_jd_{jm_j}}|r^{d_{jm_j}}e^{m_j\delta_n(A,\theta)r_t^n(1+o(1))}(1+o(1))+\cdots+|a_{jm_j1}|r^{d_{j1}}e^{\delta_n(A,\theta)r_t^n(1+o(1))}(1+o(1)) \\ &\leq 2|a_{jm_j1}|r^de^{\delta_n(A,\theta)r_t^n(1+o(1))}(1+o(1))\ \ (j=0,...,k-1).\end{aligned} \tag{30}$$

Substituting (28) and (30) into (29), we obtain that for sufficiently large r_t

$$1\leq C_1{r_t}^{k+d}e^{\delta_n(A,\theta){r_t}^n(1+o(1))}(1+o(1)), \qquad C_1>0. \tag{31}$$

Since $\delta_n(A,\theta)<0$, when $r_t\to+\infty$, by (31), we get $1\leq 0$, this is a contradiction. Hence

$$|f(re^{i\theta})|\leq M_2r^k, \qquad M_2>0, \tag{32}$$

on the ray $\arg z=\theta\in H_{n,-}\setminus E_1$. From Lemma 5, (27) and (32), we know that $f(z)$ is a polynomial, which contradicts the assertion that $f(z)$ is transcendental. Therefore, $\rho(f)=\infty$.

Step 2. We prove that $\rho_2(f)=n$. By Lemma 7 and $\rho(P_j(e^{A(z)}))=n$ $(j=0,\cdots,k-1)$, we see that $\rho_2(f)\leq \max\{\rho(P_j(e^{A(z)}))\}=n$.

Now, we suppose that there exists a solution f_0 satisfies $\rho_2(f_0)=\alpha<n$. Then we have

$$\limsup_{r\to+\infty}\frac{\log T(r,f_0)}{r^n}=0. \tag{33}$$

By Lemma 6, we see that there exists a subset $E_3 \subset (1,\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin E_3 \cup [0,1]$,

$$\left|\frac{f_0{}^{(j)}(z)}{f_0(z)}\right| \le C[T(2r,f_0)]^{k+1}, \qquad j=1,\cdots,k, \tag{34}$$

where $C(>0)$ is some constant. From the Wiman-Valiron theory, there is a set $E_8 \subset (1,\infty)$ having finite logarithmic measure, such that we can choose a z satisfying $|z| = r \notin [0,1] \cup E_8$ and $|f_0(z)| = M(r,f_0)$, then we get

$$\frac{f_0{}^{(j)}(z)}{f_0(z)} = \left(\frac{\nu(r)}{z}\right)^j (1+o(1)), \qquad j=1,\cdots,k-1, \tag{35}$$

where $\nu(r)$ is the central index of $f_0(z)$. By Lemma 9, we see that there exists a sequence $\{z_t = r_t e^{i\theta_t}\}$ such that $|f_0(z_t)| = M(r_t,f_0)$, $\theta_t \in [0,2\pi)$, with $r_t \notin [0,1] \cup E_5 \cup E_8$, $r_t \to +\infty$ and for any sufficiently large $M_3(>2k+3)$

$$\nu(r_t) > r_t{}^{M_3} > r_t. \tag{36}$$

Case 1. Suppose $\theta_0 \in H_{n,+}$. Since $\delta_n(A,\theta) = \alpha_n \cos(\theta_n + n\theta)$ is a continuous function of θ, by $\theta_t \to \theta_0$ we get $\lim_{t\to\infty} \delta_n(A,\theta_t) = \delta_n(A,\theta_0) > 0$. Therefore, there exists a constant $N(>0)$ such that as $t > N$,

$$\delta_n(A,\theta_t) \ge \frac{1}{2}\delta_n(A,\theta_0) > 0.$$

By (33), for any given $\varepsilon_1 (0 < \varepsilon_1 < \frac{1}{2^{n+1}(k+1)}\delta_n(A,\theta_0))$, and $t > N$,

$$[T(2r_t,f_0)]^{k+1} \le e^{\varepsilon_1 (k+1)(2r_t)^n} \le e^{\frac{1}{2}\delta_n(A,\theta_t) r_t{}^n}. \tag{37}$$

By (34), (35) and (37), we have

$$\left(\frac{\nu(r_t)}{r_t}\right)^{k-s}(1+o(1)) = \left|\frac{f_0^{(k-s)}(z_t)}{f_0(z_t)}\right| \le C[T(2r_t,f_0)]^{k+1} \le Ce^{\frac{1}{2}\delta_n(A,\theta_0)r_t{}^n}. \tag{38}$$

By (7), we get

$$-\frac{f_0^{(s)}(z_t)}{f_0(z_t)}P_s(e^{A(z_t)}) = \frac{f_0^{(k)}(z_t)}{f_0(z_t)} + \sum_{j=0, j\neq s}^{k-1} P_j(e^{A(z_t)})\frac{f_0^{(j)}(z_t)}{f_0(z_t)}. \tag{39}$$

Substituting (24), (25) and (35) into (39), we get for sufficiently large r_t,

$$\begin{aligned}
&\left(\frac{\nu(r_t)}{r_t}\right)^s \frac{1}{2}|a_{sm_s d_{sm_s}}| r_t{}^{d_{sm_s}} e^{m_s\delta_n(A,\theta_t) r_t{}^n (1+o(1))}(1+o(1)) \\
&\quad \le \left(\frac{\nu(r_t)}{r_t}\right)^k (1+o(1)) + \sum_{j=0, j\neq s}^{k-1} 2|a_{jm_j d_{jm_j}}| r_t^d e^{m\delta_n(A,\theta) r_t^n (1+o(1))}\left(\frac{\nu(r_t)}{r_t}\right)^j (1+o(1)).
\end{aligned} \tag{40}$$

By (36), (38) and (40), we get

$$\begin{aligned}
&|a_{sm_s d_{sm_s}}| r_t{}^{d_{sm_s}} e^{m_s\delta_n(A,\theta_t) r_t{}^n (1+o(1))}(1+o(1)) \\
&\quad \le 2\left(\frac{\nu(r_t)}{r_t}\right)^{k-s}(1+o(1)) + \sum_{j=0, j\neq s}^{k-1} 4|a_{jm_j d_{jm_j}}| r_t^d e^{m\delta_n(A,\theta) r_t^n (1+o(1))}\left(\frac{\nu(r_t)}{r_t}\right)^{j-s}(1+o(1)) \\
&\quad \le 2\left(\frac{\nu(r_t)}{r_t}\right)^{k-s}(1+o(1)) + \sum_{j=0}^{s-1} 4|a_{jm_j d_{jm_j}}| r_t^d e^{m\delta_n(A,\theta) r_t^n (1+o(1))}\left(\frac{\nu(r_t)}{r_t}\right)^{j-s}(1+o(1)) \\
&\qquad + \sum_{j=s+1}^{k-1} 4|a_{jm_j d_{jm_j}}| r_t^d e^{m\delta_n(A,\theta) r_t^n (1+o(1))}\left(\frac{\nu(r_t)}{r_t}\right)^{j-s}(1+o(1))
\end{aligned}$$

$$\leq C_2 {r_t}^d e^{m\delta_n(A,\theta)r_t^n(1+o(1))} \left(\frac{\nu(r_t)}{r_t}\right)^{k-s}(1+o(1)),$$

where $C_2 > 0$ is a constant. From this inequality and (38), it follows that

$$|a_{sm_sd_{sm_s}}|{r_t}^{d_{sm_s}} e^{(m_s-m)\delta_n(A,\theta_t){r_t}^n(1+o(1))}(1+o(1)) \leq C_2 {r_t}^d \left(\frac{\nu(r_t)}{r_t}\right)^{k-s}(1+o(1)) \leq CC_2|a_{jm_jd_{jm_j}}|{r_t}^d e^{\frac{1}{2}\delta_n(A,\theta_0){r_t}^n}. \tag{41}$$

Since $m_s - m \geq 1 > \frac{1}{2}$ and $\delta(A,\theta_t) \geq \frac{1}{2}\delta_n(A,\theta_0) > 0$, we see that (41) is a contradiction.
Case 2. Suppose $\theta_0 \in H_{n,-}$. Since $\delta_n(A,\theta)$ is a continuous function of θ, by $\theta_t \to \theta_0$ we get $\lim_{t\to\infty}\delta_n(A,\theta_t) = \delta_n(A,\theta_0) < 0$. Therefore, there exists a constant $N(>0)$ such that as $t > N$,

$$\delta_n(A,\theta_t) \leq \frac{1}{2}\delta_n(A,\theta_0) < 0.$$

By (7), we can write

$$e^{-m_sA(z_t)}\frac{{f_0}^{(k)}(z_t)}{f_0(z_t)} = e^{-m_sA(z_t)}P_{k-1}(e^{A(z_t)})\frac{f^{(k-1)}(z_t)}{f_0(z_t)} + \cdots + e^{-m_sA(z_t)}P_0(e^{A(z_t)}). \tag{42}$$

From (6) and $\delta_n(A,\theta_t) < 0$, we get

$$\begin{aligned}|e^{-m_sA(z_t)}P_j(e^{A(z_t)})| &= \left|e^{-m_sA(z_t)}\left(a_{jm_j}(z_t)e^{m_jA(z_t)} + \cdots + a_{j1}(z_t)e^{A(z_t)}\right)\right| \\ &= \left|a_{jm_j}(z_t)e^{-(m_s-m_j)A(z_t)} + \cdots + a_{j1}(z_t)e^{-(m_s-1)A(z_t)}\right| \\ &\leq C_3{r_t}^{d_{j1}}e^{-(m_s-1)\delta(A,\theta_t){r_t}^n(1+o(1))}(1+o(1)),\end{aligned} \tag{43}$$

where $C_3 > 0$ is a constant. Substituting (35) and (43) into (42), we get

$$e^{-m_s\delta(A,\theta_t){r_t}^n(1+o(1))}\nu(r_t) \leq C_4{r_t}^{d+k}e^{-(m_s-1)\delta(A,\theta_t){r_t}^n(1+o(1))}(1+o(1)), \tag{44}$$

where $C_4 > 0$ is a constant. By substituting (36) into (44), we have

$${r_t}^{M_3}e^{-m_s\delta(A,\theta_t){r_t}^n(1+o(1))} \leq C_4{r_t}^{d+k}e^{-(m_s-1)\delta(A,\theta_t){r_t}^n(1+o(1))}(1+o(1)). \tag{45}$$

Since $\delta(A,\theta_t) \leq \frac{1}{2}\delta_n(A,\theta_0) < 0$, we see (45) is also a contradiction.
Case 3. Suppose $\theta_0 \in H_{n,0}$. Since $\theta_t \to \theta_0$, for any given ε_2 $(0 < \varepsilon_2 < \frac{1}{10n})$, there exists as integer N (> 0), such that as $t > N$, $\theta_t \in [\theta_0 - \varepsilon_2, \theta_0 + \varepsilon_2]$, and

$$z_t = r_te^{i\theta_t} \in \overline{\Omega} = \{z : \theta_0 - \varepsilon_2 \leq \arg z \leq \theta_0 + \varepsilon_2\}.$$

By Lemma 6, we se that there exist a subset $E_3 \subset (1,\infty)$ having logarithmic measure $lmE_3 < \infty$, and a constant $C > 0$ such that for all z satisfying $|z| = r \notin E_3 \cup [0,1]$,

$$\left|\frac{{f_0}^{(i)}(z)}{{f_0}^{(s)}(z)}\right| \leq C[T(2r,{f_0}^{(s)})]^{k-s+1}, \qquad i = s+1,\cdots,k, \tag{46}$$

Now, we consider the growth of $f_0(re^{i\theta})$ on a ray $\arg z = \theta \in \overline{\Omega}\setminus\{\theta_0\}$. By the properties of cosine function, we suppose without loss of generality that $\delta_n(A,\theta) > 0$ for $\theta \in [\theta_0 - \varepsilon_2, \theta_0)$ and $\delta_n(A,\theta) < 0$ for $\theta \in (\theta_0, \theta_0 + \varepsilon_2]$.
Subcase 3.1 For a fixed $\theta \in [\theta_0 - \varepsilon_2, \theta_0)$, we have $\delta_n(A,\theta) > 0$. Since $\rho_2(f_0) < n$, we get that f_0 satisfies (33). From $T(r,{f_0}^{(s)}) < (s+1)T(r,f_0) + S(r,f_0)$, where $S(r,f) = o(T(r,f))$, as $r \to +\infty$ outside of a possible

exceptional set of finite logarithmic measure, we get that ${f_0}^{(s)}$ also satisfies (33). So for any given ε_2 satisfying $0 < \varepsilon_2 < \frac{1}{2^{n+1}(k-s+1)}\delta_n(A,\theta)$, we have

$$[T(2r_t, f_0^{(s)})]^{k-s+1} \leq e^{\varepsilon_2(k-s+1)(2r_t)^n} \leq e^{\frac{1}{2}\delta_n(A,\theta_0){r_t}^n}. \tag{47}$$

We assert that $|f_0^{(s)}(re^{i\theta})|$ is bounded on the ray $\arg z = \theta \in [\theta_0 - \varepsilon_2, \theta_0)$. If $|f^{(s)}(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, then, by Lemma 3, there exists a sequence $\{y_j = R_j e^{i\theta}\}$ such that as $R_j \to \infty$, $f_0^{(s)}(y_j) \to \infty$ and

$$\left|\frac{f_0^{(i)}(y_j)}{f_0^{(s)}(y_j)}\right| \leq {R_j}^{s-i}(1+o(1)) \leq 2{R_j}^s, \quad i = 0, \cdots, s-1. \tag{48}$$

By Remark 5 and $f_0^{(s)}(y_j) \to \infty$, we know that $|y_j| = R_j \notin E_4$. By (46) and (47), we have for sufficiently large j,

$$\left|\frac{f_0^{(j)}(y_j)}{f_0^{(s)}(y_j)}\right| \leq C[T(2R_j, f_0^{(s)})]^{k-s+1} \leq Ce^{\frac{1}{2}\delta_n(A,\theta_0)R_j^n}, \quad j = s+1, \cdots, k. \tag{49}$$

Substituting (24), (25), (48) and (49) into (23)

$$\begin{aligned}
&\frac{1}{2}|a_{sm_s d_{sm_s}}|{R_j}^{d_{sm_s}} e^{m_s\delta_n(A,\theta)R_j^n(1+o(1))}(1+o(1)) \\
&= |P_s(e^{A(y_j)})| \\
&\leq \left|\frac{f^{(k)}(y_j)}{f^{(s)}(y_j)}\right| + \sum_{j=0 j\neq s}^{k-1} |P_j(e^{A(y_j)})| \left|\frac{f^{(j)}(y_j)}{f^{(s)}(y_j)}\right| \\
&= \left|\frac{f^{(k)}(y_j)}{f^{(s)}(y_j)}\right| + \sum_{j=0}^{s-1} |P_j(e^{A(y_j)})| \left|\frac{f^{(j)}(y_j)}{f^{(s)}(y_j)}\right| + \sum_{j=s+1}^{k-1} |P_j(e^{A(y_j)})| \left|\frac{f^{(j)}(y_j)}{f^{(s)}(y_j)}\right| \\
&\leq Ce^{\frac{1}{2}\delta_n(A,\theta_0)R_j^n} + \sum_{j=0}^{s-1} 4|a_{jm_j d_{jm_j}}|R_j^d e^{m\delta_n(A,\theta)R_j^n(1+o(1))}{R_j}^s(1+o(1)) \\
&\quad + \sum_{j=s+1}^{k-1} 2|a_{jm_j d_{jm_j}}|R_j^d e^{m\delta_n(A,\theta)R_j^n(1+o(1))} Ce^{\frac{1}{2}\delta_n(A,\theta_0)R_j^n} \\
&\quad \leq C_5 R_j^d e^{(\frac{1}{2}+m)\delta_n(A,\theta){R_j}^n},
\end{aligned} \tag{50}$$

where $C_5 > 0$ is a constant, which yields a contradiction by $m_s - m \geq 1 > \frac{1}{2}$ and $\delta_n(A,\theta) > 0$. Hence $|f_0^{(s)}(re^{i\theta})|$ is bounded on the ray $\arg z = \theta$, so

$$|f_0(re^{i\theta})| \leq M_4 r^s, \quad M_4 > 0, \tag{51}$$

on the ray $\arg z = \theta \in [\theta_0 - \varepsilon_4, \theta_0)$.

Subcase 3.2 For a fixed $\theta \in (\theta_0, \theta_0 + \varepsilon_2]$, we have $\delta_n(A,\theta) < 0$. Using a reasoning similar to that in Subcase 3.1, we obtain

$$|f_0(re^{i\theta})| \leq M_5 r^k, \quad M_5 > 0, \tag{52}$$

on the ray $\arg z = \theta \in (\theta_0, \theta_0 + \varepsilon_4]$. By (51) and (52), we see that on the ray $\arg z = \theta \in \overline{\Omega} \setminus \{\theta_0\}$,

$$|f_0(re^{i\theta})| \leq M_5 r^k, \quad M_5 > 0. \tag{53}$$

But since $\rho(f_0(re^{i\theta})) = \infty$ and $\{z_t = r_t e^{i\theta_t}\}$ satisfies $|f_0(z_t)| = M(r_t, f_0)$, we see that, for any large $M_6(> k)$, as t is sufficiently large,

$$|f_0(z_t)| = |f_0(z_t)| = |f_0(z_t)| = |f_0(r_t e^{i\theta_t})| \geq \exp\{r_t^{M_6}\}. \tag{54}$$

Since $z_t \in \overline{\Omega}$, by (53) and (54), we see that $\theta_t = \theta_0$ as $t \to \infty$. Therefore, $\delta_n(A, \theta_t) = 0$ as $t \to \infty$. Thus, for sufficiently large t,

$$\begin{aligned} |P_j(e^{z_t})| &= |a_{jm_j}(z_t)e^{m_j A(z_t)} + a_{jm_{j-1}}(z_t)e^{m_{j-1}A(z_t)} + \cdots + a_{j1}(z_t)e^{A(z_t)}| \\ &\leq |a_{jm_j}(z_t)| + |a_{jm_{j-1}}(z_t)| + \cdots + |a_{j1}(z_t)| \leq C_6 r^d, \end{aligned} \tag{55}$$

where $j = 0, \cdots, k-1$ and $C_6 > 0$ is a constant. By (7), (35) and (55), we get that

$$|-(\frac{\nu(r_t)}{z_t})^k(1+o(1))| = |-\frac{f_0^{(k)}(z_t)}{f_0(z_t)}| \leq C_7 r^d (\frac{\nu(r_t)}{z_t})^{k-1}(1+o(1)),$$

i.e.,

$$\nu(r_t)(1+o(1)) \leq C_7 r^{d+1}(1+o(1)), \tag{56}$$

where $C_7 > 0$ is a constant. Substituting (36) into (56), we obtain also a contradiction. So we have $\rho_2(f) = n$. □

Proof of Corollary 1. From Theorem 7, we get $\rho(f) = \infty$ and $\rho_2(f) = n$. Let $g = f - z$, then $f = g + z$. Substituting it into (7), we have

$$g^{(k)} + P_{k-1}(e^{A(z)})g^{(k-1)} + \cdots + P_0(e^{A(z)})g = -zP_0(e^{A(z)}) - P_1(e^{A(z)}).$$

Since $-zP_0(e^{A(z)}) - P_1(e^{A(z)}) \not\equiv 0$, from Lemma 11, $\rho(g) = \infty$ and $\rho_2(g) = n$ we conclude $\overline{\lambda}(g) = \lambda(g) = \rho(g) = \infty$ and $\overline{\lambda}_2(g) = \lambda_2(g) = \rho_2(g) = n$. So $\overline{\tau}(f) = \tau(f) = \rho(f) = \infty$ and $\overline{\tau}_2(f) = \tau_2(f) = \rho_2(f) = n$. □

Proof of Theorem 8. From Theorem 7, we get $\rho(f) = \infty$ and $\rho_2(f) = n$.

(i) Let $g = f - z$, then $f = g + z$. Substituting it into (8), we have

$$g'' + P(e^{A(z)})g' + Q(e^{A(z)})g = -P(e^{A(z)}) - zQ(e^{A(z)}).$$

Since $p \neq s$, we get $-P(e^{A(z)}) - Q(e^{A(z)})z \not\equiv 0$. From Lemma 11, we obtain $\lambda(g) = \rho(g) = \rho(f) = \infty$ and $\lambda_2(g) = \rho_2(g) = \rho_2(f) = n$. So $\lambda(f - z) = \infty$ and $\lambda_2(f - z) = n$.

(ii) Differentiating both sides of (8), we get that

$$f''' + P(e^{A(z)})f'' + [(P(e^{A(z)}))' + Q(e^{A(z)})]f' + (Q(e^{A(z)}))'f = 0. \tag{57}$$

By (8), we have

$$f = -\frac{f'' + P(e^{A(z)})f'}{Q(e^{A(z)})}. \tag{58}$$

Substituting (58) into (57), we get

$$f''' + [(P(e^{A(z)}))' - \frac{(Q(e^{A(z)}))'}{Q(e^{A(z)})}]f'' + [(P(e^{A(z)}))' + Q(e^{A(z)}) - \frac{(Q(e^{A(z)}))'}{Q(e^{A(z)})}P(e^{A(z)})]f' = 0. \tag{59}$$

Let $g = f' - z$, then $f' = g + z$, $f'' = g' + 1$, $f''' = g''$. Substituting these into (59), we get that

$$\begin{aligned} & g'' + [(P(e^{A(z)}))' - \frac{(Q(e^{A(z)}))'}{Q(e^{A(z)})}]g' + [(P(e^{A(z)}))' + Q(e^{A(z)}) - \frac{(Q(e^{A(z)}))'}{Q(e^{A(z)})}P(e^{A(z)})]g \\ &= -P(e^{A(z)}) + \frac{(Q(e^{A(z)}))'}{Q(e^{A(z)})} - [(P(e^{A(z)}))' + Q(e^{A(z)}) - \frac{(Q(e^{A(z)}))'}{Q(e^{A(z)})}P(e^{A(z)})]z \\ &= h(z). \end{aligned} \tag{60}$$

Next, we prove that $h(z)\not\equiv 0$. If $h(z)\equiv 0$, then

$$-P(e^{A(z)})+\frac{(Q(e^{A(z)}))'}{Q(e^{A(z)})}\equiv [(P(e^{A(z)}))'+Q(e^{A(z)})-\frac{(Q(e^{A(z)}))'}{Q(e^{A(z)})}P(e^{A(z)})]z.$$

Since $Q(z)\not\equiv 0$, we have

$$(Q(e^{A(z)}))'-(Q(e^{A(z)}))^2z\equiv P(e^{A(z)})Q(e^{A(z)})+[(P(e^{A(z)}))'Q(e^{A(z)})-(Q(e^{A(z)}))'P(e^{A(z)})]z. \tag{61}$$

Suppose $p>s$. By taking $z=r$, we have

$$P(e^{A(r)})=a_p(r)e^{pA(r)}+\cdots+a_1(r)e^{A(r)},\quad \text{and}\quad Q(e^{A(r)})=b_s(r)e^{sA(r)}+\cdots+b_1(r)e^{A(r)}.$$

We get

$$\begin{aligned}(P(e^{A(r)}))'&=\sum_{j=1}^{p}(a_j'(r)+jA'(r)a_j(r))e^{jA(r)}\\&=(a_p'(r)+pA'(r)a_p(r))e^{pA(r)}+\cdots+(a_1'(r)+A'(r)a_1(r))e^{A(r)}\end{aligned}$$

and

$$\begin{aligned}(Q(e^{A(r)}))'&=\sum_{j=1}^{s}(b_j'(r)+jA'(r)b_j(r))e^{jA(r)}\\&=(b_s'(r)+sA'(r)b_s(r))e^{sA(r)}+\cdots+(b_1'(r)+A'(r)b_1(r))e^{A(r)}.\end{aligned}$$

So, we obtain

$$\begin{aligned}&|P(e^{A(r)})Q(e^{A(r)})+(P(e^{A(r)}))'Q(e^{A(r)})r-(Q(e^{A(r)}))'P(e^{A(r)})r|\\&=|a_p(r)b_s(r)+(p-s)rA'(r)a_p(r)b_s(r)+(a_p'(r)b_s(r)-a_p(r)b_s'(r))r|e^{(p+s)Rec_nr^n(1+o(1))}(1+o(1)).\end{aligned}$$

Since $a_p(r)$, $b_s(r)$ and $A(r)$ are polynomials and $p>s$, we get

$$\deg((p-s)rA'(r)a_p(r)b_s(r))>\deg[a_p(r)b_s(r)+(a_p'(r)b_s(r)-a_p(r)b_s'(r))r].$$

So, we have

$$|(p-s)rA'(r)a_p(r)b_s(r)+a_p(r)b_s(r)+(a_p'(r)b_s(r)-a_p(r)b_s'(r))r|=Mr^{d_1}(1+o(1))\not\equiv 0,$$

where $M>0$ and $d_1>0$ are some constants. It follows that

$$|P(e^{A(r)})Q(e^{A(r)})+(P(e^{A(r)}))'Q(e^{A(r)})r-(Q(e^{A(r)}))'P(e^{A(r)})r|=Mr^{d_1}e^{(p+s)Rec_nr^n(1+o(1))}(1+o(1)).$$

From (61), we have

$$\begin{aligned}Mr^{d_1}e^{(p+s)Rec_nr^n(1+o(1))}(1+o(1))&=|P(e^{A(r)})Q(e^{A(r)})+(P(e^{A(r)}))'Q(e^{A(r)})r-(Q(e^{A(r)}))'P(e^{A(r)})r|\\&=|(Q(e^{A(r)}))'-(Q(e^{A(r)}))^2r|\le M_1r^{d_2}e^{2sRec_nr^n(1+o(1))}(1+o(1)),\end{aligned}$$

where $M_1>0$ and $d_2>0$ are some constants, which is a contradiction. So we have $h(z)\not\equiv 0$. If $p<s$, by (61) for $z=r$ we have

$$\begin{aligned}M_2r^{d_3}e^{2sRec_nr^n(1+o(1))}(1+o(1))&=[(Q(e^{A(r)}))^2+(P(e^{A(r)}))'Q(e^{A(r)})-(Q(e^{A(r)}))'P(e^{A(r)})]r\\&=\left|(Q(e^{A(r)}))'-P(e^{A(r)})Q(e^{A(r)})\right|\\&\le M_3r^{d_4}e^{(p+s)Rec_nr^n(1+o(1))}(1+o(1)),\end{aligned}$$

where $M_2 > 0$, $d_3 > 0$, $M_3 > 0$ and $d_4 > 0$ are some constants.This is a contradiction. So, we obtain $h(z) \not\equiv 0$. Hence, if $p \neq s$ we have $h(z) \not\equiv 0$. From Lemma 11, we get $\lambda(g) = \rho(g) = \rho(f' - z) = \rho(f) = \infty$ and $\lambda_2(g) = \rho_2(g) = \rho_2(f' - z) = \rho_2(f) = n$.

□

Proof of Theorem 9. Suppose that $f \not\equiv 0$ is a solution of (10). Since $\rho(P^*) = \rho(Q^*) = n$, then by Lemma 7, we see that

$$\rho_2(f) \leq \max\{\rho(P^*), \rho(Q^*)\} = n. \tag{62}$$

By Lemma 6, we se that there exist a subset $E_3 \subset (1, \infty)$ having logarithmic measure $lmE_3 < \infty$, and a constant $C > 0$ such that for all z satisfying $|z| = r \notin E_3 \cup [0, 1]$,

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq C[T(2r, f)]^{j+1}, \quad j = 1, 2. \tag{63}$$

Taking $z = r$, in (2) and (3), we obtain that for sufficiently large r

$$\begin{aligned}\left|P^*\left(e^{A(r)}\right)\right| &= \left|a_p(r)e^{pA(r)} + \cdots + a_1(r)e^{A(r)} + a_0(r)\right| \\ &\leq 2\left|a_{pd_p}\right| r^{d_p} e^{pRec_n r^n(1+o(1))}(1 + o(1)),\end{aligned} \tag{64}$$

and

$$\begin{aligned}\left|Q^*\left(e^{A(r)}\right)\right| &= \left|b_s(r)e^{sA(r)} + \cdots + b_1(r)e^{A(r)} + b_0(r)\right| \\ &\geq \frac{1}{2}\left|b_{sm_s}\right| r^{m_s} e^{sRec_n r^n(1+o(1))}(1 + o(1)).\end{aligned} \tag{65}$$

Substituting (63)–(65) into (10), we deduce that for all z satisfying $|z| = r \notin E_3 \cup [0, 1]$

$$\begin{aligned}\frac{1}{2}\left|b_{sm_s}\right| r^{m_s} e^{sRec_n r^n(1+o(1))}(1 + o(1)) &\leq \left|\frac{f''(z)}{f(z)} + P^*\left(e^{A(z)}\right)\frac{f'(z)}{f(z)}\right| \\ &\leq \left|\frac{f''(z)}{f(z)}\right| + \left|P^*\left(e^{A(z)}\right)\right|\left|\frac{f'(z)}{f(z)}\right| \\ &\leq C[T(2r, f)]^3 + 2\left|a_{pd_p}\right| r^{d_p} e^{pRec_n r^n(1+o(1))} C[T(2r, f)]^2(1 + o(1)) \\ &\leq 3C\left|a_{pd_p}\right| r^{d_p} e^{pRec_n r^n(1+o(1))}[T(2r, f)]^3(1 + o(1)).\end{aligned} \tag{66}$$

By (66), we deduce that for all z satisfying $|z| = r \notin E_3 \cup [0, 1]$

$$\left|b_{sm_s}\right| r^{m_s - d_p} e^{(s-p)Rec_n r^n(1+o(1))}(1 + o(1)) \leq 6C\left|a_{pd_p}\right| [T(2r, f)]^3(1 + o(1)). \tag{67}$$

Since $s - p > 0$, by (67) and Lemma 12, we get

$$\rho(f) \geq \limsup_{r\to+\infty} \frac{\log T(r, f)}{\log r} = +\infty, \; \rho_2(f) \geq \limsup_{r\to+\infty} \frac{\log\log T(r, f)}{\log r} = n. \tag{68}$$

From (62) and (68) we obtain $\rho(f) = +\infty$ and $\rho_2(f) = n$. □

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