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Global existence and decay of solutions for p-biharmonic parabolic equation with logarithmic nonlinearity

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Abstract: In this paper, we study the initial boundary value problem for a p-biharmonic parabolic equation with logarithmic nonlinearity. By using the potential wells method and logarithmic Sobolev inequality, we obtain the existence of the unique global weak solution. In addition, we also obtain decay polynomially of solutions.

Keywords: Parabolic equation; p-biharmonic; Global existence; Logarithmic nonlinearity.

MSC: 35B40; 35G31; 35K25.

1. Introduction

In this paper, we investigate the existence of global and decay of solutions for the p-biharmonic parabolic equation with logarithmic nonlinearity

$$\begin{cases} u_t + \Delta\left(|\Delta u|^{p-2}\Delta u\right) - \Delta u_t = u\,|u|^{q-2}\ln|u|, & x\in\Omega,\ t>0,\\ u(x,t) = \Delta(x,t) = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1}$$

where Ω is bounded domain $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, p, q are positive constants, $2<p<q<p\left(1+\frac{4}{n}\right)$, and $u_0 \in \left(W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega)\right)\setminus\{0\}$. The term $\Delta\left(|\Delta u|^{p-2}\Delta u\right)$ is called a p-biharmonic operator.

Studies of logarithmic nonlinearity have a long history in physics as it occurs naturally in different areas of physics such as supersymmetric field theories, inflationary cosmology, nuclear physics, optics and quantum mechanics [1,2]. Peng and Zhou [3] studied the following heat equation with logarithmic nonlinearity

$$u_t - \Delta u_t = |u|^{p-2}u\ln|u|.$$

They obtained the global existence and blow-up of solutions. Also, they discussed the upper bound of blow-up time under suitable conditions. Nhan and Truong [4] studied the following nonlinear pseudo-parabolic equation

$$u_t - \Delta u_t - div\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{p-2}u\ln|u|.$$

They obtained results as regard the existence or non-existence of global solutions. Also, He *et al.*, [5] proved the decay and the finite time blow-up for weak solutions of the equation. Cao and Liu [6] studied the following nonlinear evolution equation with logarithmic source

$$u_t - \Delta u_t - div\left(|\nabla u|^{p-2}\nabla u\right) - k\Delta u_t = |u|^{p-2}u\ln|u|.$$

They established the existence of global weak solutions. Moreover, they considered global boundedness and blowing-up at ∞.

Wang and Liu [7] considered the following p-biharmonic parabolic equation with the logarithmic nonlinearity

$$u_t + \Delta\left(|\Delta u|^{p-2}\Delta u\right) = |u|^{q-2}u\ln|u|$$

They studied existence of weak solutions by potential well method, blow up at finite time by concative method.

Recently some authors studied the hyperbolic and parabolic equation with logarithmic source term (see [8–20]). This paper is organized as follows: In the §**??**, we introduce some lemma which will be needed later. In §**??**, under some conditions, we obtain the unique global weak solution of the problem (1). Meanwhile, we find that the solution is decay polynomially.

It is necessary to note that prence of the logarithmic nonlinearity causes some difficulties in deploying the potential well method. In order to handle this situation we need the following logarithmic Sobolev inequality which was introduced by ([4,21,22]).

Proposition 1. *Let u be any function in $H^1(\mathbb{R}^n)$ and $\mu > 0$ be any number. Then*

$$p\int_{\mathbb{R}^n}|u(x)|^p\ln\left(\frac{|u(x)|}{\|u(x)\|_{L^p(\mathbb{R}^n)}}\right)dx + \frac{n}{p}\ln\left(\frac{p\mu e}{n\mathcal{L}_p}\right)\int_{\mathbb{R}^n}|u(x)|^p\,dx \le \mu\int_{\mathbb{R}^n}|\nabla u(x)|^p\,dx. \tag{2}$$

where

$$\mathcal{L}_p = \frac{p}{n}\left(\frac{p-1}{e}\right)^{p-1}\pi^{-\frac{p}{2}}\left[\frac{\Gamma\left(\frac{\pi}{2}+1\right)}{\Gamma\left(n\frac{p-1}{p}+1\right)}\right]^{\frac{p}{n}}.$$

2. Preliminaries

For simplicity, we denote

$$\|u\|_s = \|u\|_{L^s(\Omega)},\ \|u\|_{W_0^{2,p}(\Omega)} = \|u\|_{2,s} = \left(\|\Delta u\|_s^s + \|\nabla u\|_s^s + \|u\|_s^s\right)^{\frac{1}{s}},$$

for $1 < s < \infty$ (see [23,24], for details). We also use notation X_0 to denote $\left(W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega)\right)\setminus\{0\}$ and $W^{-2,p'}(\Omega)$ to denote the dual space of $W^{2,s}(\Omega)$, where s' is Hölder conjugate functional of $s > 1$.

Let us introduce the energy functional J and Nehari functional I defined on X_0 as follow

$$J(u) = \frac{1}{p}\|\Delta u\|_p^p - \frac{1}{q}\int_\Omega |u|^q\ln|u|\,dx + \frac{1}{q^2}\|u\|_q^q, \tag{3}$$

and

$$I(u) = \|\Delta u\|_p^p - \int_\Omega |u|^q\ln|u|\,dx. \tag{4}$$

By (3) and (4), we get

$$J(u) = \frac{1}{q}I(u) + \left(\frac{1}{p}-\frac{1}{q}\right)\|\Delta u\|_p^p + \frac{1}{q^2}\|u\|_q^q. \tag{5}$$

Let

$$\mathcal{N} = \{u \in X_0 : I(u) = 0\},$$

be the Nehari manifold. Thus, we may define

$$d = \inf_{u\in\mathcal{N}} J(u). \tag{6}$$

d is positive and is obtained by some $u \in \mathcal{N}$. Then it is obvious that

$$M = \frac{1}{p^2}\left(\frac{p^2e}{n\mathcal{L}_p}\right)^{\frac{n}{p}}.$$

From [4], we know $d \ge M$.

The local existence of the weak solutions can be obtained via the standard parabolic theory. It is easy to obtain the following equality

$$\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} ds + J(u(t)) \le J(u_0), \quad 0 \le t \le T. \tag{7}$$

Lemma 1. *Let $u \in X_0$. Then we possess*

(i) $\lim_{\lambda\to 0^+} j(\lambda) = 0$ *and* $\lim_{\lambda\to+\infty} j(\lambda) = -\infty$;
(ii) there is a unique $\lambda^* > 0$ *such that* $j'(\lambda^*) = 0$;
(iii) $j(\lambda)$ *is increasing on* $(0, \lambda^*)$, *decreasing on* $(\lambda^*, +\infty)$ *and attains the maximum at* λ^*;
(iv) $I(\lambda u) > 0$ *for* $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ *for* $\lambda^* < \lambda < +\infty$ *and* $I(\lambda^* u) = 0$.

Proof. For $u \in X_0$, by the definition of j, we get

$$\begin{aligned} j(\lambda) &= \frac{1}{p}\|\Delta(\lambda u)\|_p^p - \frac{1}{q}\int_\Omega |\lambda u|^q \ln|\lambda u|\, dx + \frac{1}{q^2}\|\lambda u\|_q^q \\ &= \frac{\lambda^p}{p}\|\Delta u\|_p^p - \frac{\lambda^q}{q}\int_\Omega |u|^q \ln|u|\, dx + \frac{\lambda^q}{q}\ln\lambda\, \|u\|_q^q + \frac{\lambda^q}{q^2}\|u\|_q^q. \end{aligned} \tag{8}$$

It is clear that *(i)* holds due to $\int_\Omega |u|^q\, dx \neq 0$. We have

$$\begin{aligned} \frac{d}{d\lambda} j(\lambda) &= \lambda^{p-1}\|\Delta u\|_p^p - \lambda^{q-1}\int_\Omega |u|^q \ln|u|\, dx - \lambda^{q-1}\ln\lambda\, \|u\|_q^q, \\ &= \lambda^{p-1}\left(\|\Delta u\|_p^p - \lambda^{q-p}\int_\Omega |u|^q \ln|u|\, dx - \lambda^{q-p}\ln\lambda\, \|u\|_q^q\right). \end{aligned}$$

Since $\lambda > 0$, let $\varphi(\lambda) = \lambda^{1-p} j'(\lambda)$, through direct calculation, we get

$$\varphi'(\lambda) = -\lambda^{q-p-1}\left((q-p)\int_\Omega |u|^q \ln|u|\, dx + (q-p)\ln\lambda\, \|u\|_q^q + \|u\|_q^q\right).$$

Hence, there exists a

$$\lambda^* = \exp\left(\frac{(p-q)\int_\Omega |u|^q \ln|u|\, dx - \|u\|_q^q}{(q-p)\|u\|_q^q}\right) > 0,$$

such that $\varphi'(\lambda) > 0$ on $(0, \lambda^*)$, $\varphi'(\lambda) < 0$ on $(\lambda^*, +\infty)$ and on $\varphi'(\lambda) = 0$. So, $\varphi(\lambda)$ is increasing on $(0, \lambda^*)$, decreasing on $(\lambda^*, +\infty)$. Since $\lim_{\lambda\to 0^+} \varphi(\lambda) = \|\nabla u\|^2 > 0$, $\lim_{\lambda\to+\infty} \varphi(\lambda) = -\infty$, there exists a unique $\lambda^* > 0$ such that $\varphi(\lambda^*) = 0$, i.e., $j'(\lambda^*) = 0$. So *(ii)* holds. Then, $j'(\lambda) = \lambda\varphi(\lambda)$ is positive on $(0, \lambda^*)$, negative on $(\lambda^*, +\infty)$. Thus, $j(\lambda)$ is increasing on $(0, \lambda^*)$, decreasing on $(\lambda^*, +\infty)$ and attains the maximum at λ^*. So *(iii)* holds. The last property, *(iv)*, is only a simple corallary of the fact that

$$\begin{aligned} I(\lambda u) &= \|\Delta(\lambda u)\|_p^p - \int_\Omega |\lambda u|^q \ln|\lambda u|\, dx \\ &= \lambda^p\|\Delta u\|_p^p - \lambda^q\int_\Omega |u|^q \ln|u|\, dx - \lambda^q \ln\lambda\, \|u\|_q^q \\ &= \lambda j'(\lambda). \end{aligned}$$

Thus, $I(\lambda u) > 0$ for $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ for $\lambda^* < \lambda < +\infty$ and $I(\lambda^* u) = 0$. So *(iv)* holds. The proof is complete.

Next we denote

$$R := \left(\frac{p^2 e}{n\mathcal{L}_p}\right)^{n/p^2}.$$

□

Lemma 2. *(i) if* $I(u) > 0$ *then* $0 < \|u\|_p < R$,
(ii) if $I(u) < 0$ *then* $\|u\|_p > R$,
(iii) if $I(u) = 0$ *then* $\|u\|_p \ge R$.

Proof. By the definition of $I(u)$, we get

$$\begin{aligned} I(u) &= \|\Delta u\|_p^p - \int_\Omega |u|^q \ln|u|\,dx \\ &\geq \|\Delta u\|_p^p - \int_\Omega |u|^p \left(\ln\frac{|u|}{\|u\|_p} + \ln\|u\|_p\right) dx \\ &\geq \left(1-\frac{\mu}{p}\right)\|\Delta u\|_p^p + \left(\frac{n}{p^2}\ln\left(\frac{p\mu e}{n\mathcal{L}_p}\right) - \ln\|u\|_p\right)\|u\|_p^p. \end{aligned}$$

Choosing $\mu = p$, we have

$$I(u) \geq \left(\frac{n}{p^2}\ln\left(\frac{p^2 e}{n\mathcal{L}_p}\right) - \ln\|u\|_p\right)\|u\|_p^p.$$

(i) if $I(u) > 0$, then

$$\ln\|u\|_p < \ln\left(\frac{p^2 e}{n\mathcal{L}_p}\right)^{\frac{n}{p^2}},$$

that's mean

$$\|u\|_p < \left(\frac{p^2 e}{n\mathcal{L}_p}\right)^{\frac{n}{p^2}} = R,$$

and (ii) if $I(u) < 0$, we obtain

$$\|u\|_p > \left(\frac{p^2 e}{n\mathcal{L}_p}\right)^{\frac{n}{p^2}} = R,$$

property (iii) we can argue similarly the proof of (ii).

The proof of lemma is complete. □

Lemma 3. [25] *For any $u \in W_0^{1,p}(\Omega)$, $p \geq 1$, $r \geq 1$ and $p_* = \frac{np}{n-p}$, the inequality*

$$\|u\|_q \leq C\|\nabla u\|_p^\theta \|u\|_r^{1-\theta},$$

is valid, where

$$\theta = \left(\frac{1}{r}-\frac{1}{q}\right)\left(\frac{1}{n}-\frac{1}{p}+\frac{1}{r}\right)^{-1},$$

and for $p \geq n = 1$, $r \leq q \leq \infty$; for $n > 1$ and $p < n$, $q \in [r, p_*]$ if $r < p_*$ and $q \in [p_*, r]$ if $r \geq p_*$ for $p = n > 1$, $r \leq q \leq \infty$; for $p > n > 1$, $r \leq q \leq \infty$.

Here, the constant C depends on n, p, q and r.

Lemma 4. [26] *Let $f : R^+ \to R^+$ be a nonincreasing function and σ is a nonnegative constant such that*

$$\int_t^{+\infty} f^{1+\sigma}(s)ds \leq \frac{1}{\omega} f^\sigma(0) f(t), \quad \forall t \geq 0.$$

Hence

(a) $f(t) \leq f(0)e^{1-\omega t}$, for all $t \geq 0$, whenever $\sigma = 0$,

(b) $f(t) \leq f(0)\left(\frac{1+\sigma}{1+\omega\sigma t}\right)^{\frac{1}{\sigma}}$, for all $t \geq 0$, whenever $\sigma > 0$.

3. Main results

Now as in ([4]), we introduce the follows sets:

$$\begin{aligned} \mathcal{W}_1 &= \{u \in X_0 : J(u) < d\},\ \mathcal{W}_2 = \{u \in X_0 : J(u) = d\},\ \mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2, \\ \mathcal{W}_1^+ &= \{u \in \mathcal{W}_1 : I(u) > 0\},\ \mathcal{W}_2^+ = \{u \in \mathcal{W}_2 : I(u) > 0\},\ \mathcal{W}^+ = \mathcal{W}_1^+ \cup \mathcal{W}_2^+, \\ \mathcal{W}_1^- &= \{u \in \mathcal{W}_1 : I(u) < 0\},\ \mathcal{W}_2^- = \{u \in \mathcal{W}_2 : I(u) < 0\},\ \mathcal{W}^- = \mathcal{W}_1^- \cup \mathcal{W}_2^-. \end{aligned}$$

Definition 1. (Maximal Existence Time). Assume that u be weak solutions of problem (1). We define the maximal existence time $T_{\max}$ as follows

$$T_{\max} = \sup\{T > 0 : u(t) \text{ exists on } [0, T]\}.$$

Then

(i) If $T_{\max} < \infty$, we say that u blows up in finite time and $T_{\max}$ is the blow-up time;
(ii) If $T_{\max} = \infty$, we say that u is global.

Definition 2. (Weak solution). We define a function $u \in L^{\infty}(0, T; X_0)$ with $u_t \in L^2(0, T; H_0^1(\Omega))$ to be a weak solution of problem (1) over $[0, T]$, if it satisfies the initial condition $u(0) = u_0 \in X_0$, and

$$\langle u_t, w\rangle + \left\langle |\Delta u|^{p-1}, \Delta w\right\rangle + \langle \nabla u_t, \nabla w\rangle = \int_{\Omega} u\,|u|^{q-2} \ln(|u|)\, w dx,$$

for all $w \in X_0$, and for a.e. $t \in [0, T]$.

Theorem 1. *(Global Existence). Let $u_0 \in \mathcal{W}^+$, $0 < J(u_0) < M$ and $I(u) > 0$. Then there is a unique global weak solution u of (1) satisfying $u(0) = u_0$. We have $u(t) \in \mathcal{W}^+$holds for all $0 \le t < +\infty$, and the energy estimate*

$$\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} ds + J(u(t)) \le J(u_0), \quad 0 \le t \le +\infty.$$

Also, the solution decay polynomially provided $u_0 \in \mathcal{W}_1^+$.

Proof. The Faedo-Galerkin's methods is used. In the space $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$, we take a bases $\{w_j\}_{j=1}^{\infty}$ and define the finite orthogonal space

$$V_m = span\{w_1, w_2, ..., w_m\}.$$

Let u_{0m} be an element of V_m such that

$$u_{0m} = \sum_{j=1}^{m} a_{mj} w_j \to u_0 \quad \text{strongly in } W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega), \tag{9}$$

as $m \to \infty$. We construct the following approximate solution $u_m(x,t)$ of the problem (1)

$$u_m(x,t) = \sum_{j=1}^{m} a_{mj}(t) w_j(x), \tag{10}$$

where the coefficients a_{mj} $(1 \le j \le m)$ satisfy the ordinary differential equations

$$\int_{\Omega} u_{mt} w_i dx + \int_{\Omega} |\Delta u_m|^{p-1} \Delta w_i dx + \int_{\Omega} \nabla u_{mt} \nabla w_i dx = \int_{\Omega} u\,|u_m|^{q-2} \ln(|u_m|)\, w_i dx, \tag{11}$$

for $i \in \{1, 2, ..., m\}$, with the initial condition

$$a_{mj}(0) = a_{mj}, \quad j \in \{1, 2, ..., m\}. \tag{12}$$

We multiply both sides of (11) by a'_{mi}, sum for $i = 1, ..., m$ and integrating with respect to time variable on $[0, t]$, we get

$$\int_0^t \|u_{ms}(s)\|^2_{H_0^1(\Omega)} ds + J(u_m(t)) \le J(u_{0m}), \quad 0 \le t \le T_{\max}, \tag{13}$$

where $T_{\max}$ is the maximal existence time of solution $u_m(t)$. We shall prove that $T_{\max} = +\infty$. From (9), (13) and the continuity of J, we obtain

$$J(u_m(0)) \to J(u_{0m}), \text{ as } m \to \infty, \tag{14}$$

Thanks to $J(u_0) < d$ and the continuity of functional J, it follows from (14) that

$$J(u_{0m}) < d, \quad \text{for sufficiently large } m.$$

And therefore, from (13), we obtain

$$\int_0^t \|u_{ms}(s)\|_{H_0^1(\Omega)}^2 \, ds + J(u_m(t)) < d, \quad 0 \le t \le T_{\max}, \tag{15}$$

for sufficiently large m. Next, we will study

$$u_m(t) \in \mathcal{W}_1^+, \quad t \in [0, T_{\max}), \tag{16}$$

for sufficiently large m. We assume that (16) does not process and think that there exists a sufficiently small time t_0 such that $u_m(t_0) \notin \mathcal{W}_1^+$. Then, by continuity of $u_m(t_0) \in \partial\mathcal{W}_1^+$. So, we get

$$J(u_m(t_0)) = d, \tag{17}$$

and

$$I(u_m(t_0)) = 0. \tag{18}$$

Nevertheless, by definition of d, we see that (17) could not consist by (15) while if (18) holds then, we get

$$J(u_m(t_0)) \ge \inf_{u \in \mathcal{N}} J(u) = d,$$

which also contradicts with (15). Moreover, we have (16), i.e., $J(u_m(t)) < d$, and $I(u_m(t)) > 0$, for any $t \in [0, T_{\max})$, for sufficiently large m. Then, from (5), we obtain

$$\begin{aligned} d &> J(u_m(t)) \\ &= \frac{1}{q} I(u_m) + \left(\frac{1}{p} - \frac{1}{q}\right) \|\Delta u_m\|_p^p + \frac{1}{q^2} \|u_m\|_q^q \\ &\ge \left(\frac{1}{p} - \frac{1}{q}\right) \|\Delta u_m\|_p^p + \frac{1}{q^2} \|u_m\|_q^q, \end{aligned}$$

which gives

$$\|u_m(t)\|_q^q < q^2 d, \tag{19}$$

and

$$\|\Delta u_m\|_p^p < \frac{pq}{q-p} d. \tag{20}$$

Since $u_m(x,t) \in \mathcal{W}_1^+$ for m large enough, it follows from (5) that $J(u_m) \ge 0$ for s large enough. So, by (15) it follows for m large enough

$$\int_0^t \|u_{ms}(s)\|_{H_0^1(\Omega)}^2 \, ds < d. \tag{21}$$

By (20), we know that

$$T_{\max} = +\infty.$$

It follows from (19) and (21) that there exist a function X_0 and a subsequence of $\{u_m\}_{j=1}^{\infty}$ is indicated by $\{u_m\}_{j=1}^{\infty}$ such that

$$u_m \to u \ \text{ weakly* in } L^\infty(0, \infty; W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)), \tag{22}$$

$$u_{mt} \to u_t \ \text{ weakly in } L^2(0, \infty; H_0^1(\Omega)), \tag{23}$$

$$|\Delta u|^{p-2} \Delta u \to \chi \text{ weakly in } L^\infty\left(0, \infty; W^{-2,p'}(\Omega)\right).$$

By (22), (23) and Aubin-Lions compactness theorem, we obtain

$$u_m \to u \ \text{ strongly in } C([0, +\infty]; L^2(\Omega)).$$

This yields that

$$u_m |u_m|^{q-2} \ln |u_m| \to u |u|^{q-2} \ln |u| \ \text{ a.e. } (x,t) \in \Omega \times (0, +\infty). \tag{24}$$

Moreover, since

$$\alpha^{r-1} \ln \alpha = -(e(r-1))^{-1} \ \text{ for } \ \alpha > 1,$$

and
$$\ln\alpha = 2\ln\left(\alpha^{\frac{1}{2}}\right) \leq 2\alpha^{\frac{1}{2}} \quad \text{for} \ \ \alpha > 0.$$

By (19), we have

$$\begin{aligned}
\int_\Omega \left(|u_m(t)|^{q-1}\ln|u_m(t)|\right)^{\frac{2q}{2q-1}} dx &= \int_{\Omega_1} \left(|u_m(t)|^{q-1}\ln|u_m(t)|\right)^{\frac{2q}{2q-1}} dx + \int_{\Omega_2} \left(|u_m(t)|^{q-1}\ln|u_m(t)|\right)^{\frac{2q}{2q-1}} dx \\
&\leq [e(r-1)]^{-\frac{2q}{2q-1}}|\Omega| + 2^{\frac{2q}{2q-1}}\int_{\Omega_2}|u_m(t)|^{\frac{2q\left(q-1+\frac{1}{2}\right)}{2q-1}} dx \\
&= [e(r-1)]^{-\frac{2q}{2q-1}}|\Omega| + 2^{\frac{2q}{2q-1}}\int_{\Omega_2}|u_m(t)|^q dx \\
&\leq C_d := [e(r-1)]^{-\frac{2q}{2q-1}}|\Omega| + 2^{\frac{2q}{2q-1}}q^2 d, \qquad (25)
\end{aligned}$$

where
$$\Omega_1 = \{x\in\Omega : |u_m(t)| \leq 1\}, \ \text{ and } \ \Omega_2 = \{x\in\Omega : |u_m(t)| \geq 1\}.$$

Hence, it follows from (24) and (25) that

$$u_m|u_m|^{q-2}\ln|u_m| \to u|u|^{q-2}\ln|u| \quad \text{weakly* in } L^\infty(0,+\infty; L^{\frac{2q}{2q-1}}(\Omega)).$$

Then integrating (11) respect to t for $0 \leq t < \infty$, we obtain

$$\langle u_t, w\rangle + \langle\chi(t), \Delta w\rangle + \langle\nabla u_t, \nabla w\rangle = \int_\Omega u|u|^{p-2}\ln(|u|)\, w dx,$$

for all $w \in W_0^{2,p}(\Omega)$ and for almost every $t \in [0,\infty]$. Finally, well known arguments of the theory of monotone operators implied
$$\chi = |\Delta u|^{p-2}\Delta u,$$
which yields
$$\langle u_t, w\rangle + \left\langle |\Delta u|^{p-1}, \Delta w\right\rangle + \langle\nabla u_t, \nabla w\rangle = \int_\Omega u|u|^{p-2}\ln|u|\, w dx.$$
for all $w \in W_0^{2,p}(\Omega)$ and for a.e. $t \in [0,\infty]$.

Finally, we discuss the decay results.

Thanks to $u(t) \in \mathcal{W}_1^+$, we deduce from (13) that

$$\left(\frac{1}{p} - \frac{1}{q}\right)\|\Delta u\|_p^p + \frac{1}{q^2}\|u\|_q^q \leq J(u(t)) \leq J(u_0), \quad t \in [0,T].$$

By using (5) and Proposition 1, we put $p\left(\frac{J(u_0)}{M}\right)^{\frac{p}{n}} < \mu < p$, we know

$$\begin{aligned}
I(u(t)) &\geq \left(1 - \frac{\mu}{p}\right)\|\Delta u\|_p^p + \left(\frac{n}{p^2}\ln\left(\frac{p\mu e}{n\mathcal{L}_p}\right) - \ln\|u(t)\|_p\right)\|u(t)\|_p^p \\
&\geq \left(1 - \frac{\mu}{p}\right)\|\Delta u\|_p^p + \frac{1}{p}\ln\left(\frac{M}{J(u_0)}\left(\frac{\mu}{p}\right)^{\frac{n}{p}}\right)\|u(t)\|_p^p \\
&= C_1\|u(t)\|_{W^{2,p}(\Omega)}^p.
\end{aligned}$$

Integrating the $I(u(s))$ with respect to s over (t,T), we obtain

$$\begin{aligned}
\int_t^T I(u(s))ds &= -\int_t^T\int_\Omega u_s(s)u(s)dxds - \int_t^T\int_\Omega \nabla u_s(s)\nabla u(s)dxds \\
&= \frac{1}{2}\|u(t)\|_{H_0^1(\Omega)}^2 - \frac{1}{2}\|u(T)\|_{H_0^1(\Omega)}^2 \\
&\leq C_2\|u(t)\|_{W^{2,p}(\Omega)}^2. \qquad (26)
\end{aligned}$$

where C_2 stand by the best constant in the embedding $W^{2,p}(\Omega) \hookrightarrow\to H_0^1(\Omega)$ From (26), we have

$$\int_t^T \|u(t)\|_{W^{2,p}(\Omega)}^p \, ds \leq \frac{1}{\omega}\|u(t)\|_{W^{2,p}(\Omega)}^2 \text{ for all } t \in [0,T]. \tag{27}$$

Let $T \to +\infty$ in (27), we can get

$$\int_t^\infty \|u(t)\|_{W^{2,p}(\Omega)}^p \, ds \leq \frac{1}{\omega}\|u(t)\|_{W^{2,p}(\Omega)}^2 .$$

From Lemma 5, we have $f(t) = \|u(t)\|_{W^{2,p}(\Omega)}^2$, $\sigma = \frac{p}{2} - 1$, $f(0) = 1$

$$\|u(t)\|_{W^{2,p}(\Omega)} \leq \|u_0\|_{W^{2,p}(\Omega)} \left(\frac{p}{2 + \omega \|u_0\|_{W_0^{2,p}(\Omega)}^{p-2} (p-2)\, t} \right)^{\frac{1}{p-2}}, \quad t \geq 0.$$

The above inequality implies that the solution u decays polynomially. □

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