Open Journal of
Mathematical Analysis

PSR Press

Article

Multiplier properties for the AP-Henstock integral

Kwancheol Shin[1,*] and JU Han Yoon[1]

[1] Department of Mathematics Education Chungbuk National University Cheongju 28644, Republic of Korea.
* Correspondence: kcshin3623@gmail.com

Academic Editor: Mikail Et
Received: 2 January 2021; Accepted: 22 April 2022; Published: 22 June 2022.

Abstract: In this paper, we investigate some properties of the AP-Henstock integral on a compact set and prove that the product of an AP-Henstock integrable function and a function of bounded variation is AP-Henstock integrable. Furthermore, we prove that the product of an AP-Henstock integrable function and a regulated function is also AP-Henstock integrable. We also define the AP-Henstock integral on an unbounded interval, investigate some properties, and show similar multiplier properties.

Keywords: AP-Henstock integral; Theory of integration; Henstock integral.

MSC: 26A39; 26A42; 26A24.

---

1. Introduction and Main Results

The Henstock integral of real-valued functions was first defined by Henstock[1,2]. It is a direct generalization of the Riemann integral because it uses the concept of the tagged partition and the Riemann sum. In Henstock integral, the concept of the norm of a tagged partition in the Riemann integral is replaced by the positive gauge function. Therefore, the definition of the Henstock integral is as simple as the definition of the Riemann integral. On the other hand, to introduce the Lebesgue integral, a good amount of measure theory is required. It is one of the reasons why the Henstock integral is simpler than the Lebesgue integral. However, the Henstock integral is also a generalization of the Lebesgue integral. Every Lebesgue integrable function is Henstock integrable, and both integrals are the same. One of the Lebesgue integral deficits is that not every continuous function that is differentiable everywhere, possible except for a countable number of points, is recovered from its derivative by the Lebesgue integrable. In this sense, we say that the Lebesgue integral does not recover a function from its derivative. On the other hand, the Henstock integral overcomes this drawback: every continuous function $F : [a, b] \to R$ that is differentiable everywhere except for countable number of points on $[a, b]$ can be recovered from its derivative by the Henstock integral, and $\int_a^x F' = F(x) - F(a)$.

The approximate Henstock integral (AP-Henstock integral) [3] further generalizes the Henstock integral by using the concept of the approximate derivative [4], and the gauge function in the Henstock integral is generalized to *the choice* in the AP-Henstock integral. Every Henstock integrable function is AP-Henstock integrable, and the integrals are the same. Furthermore, the AP-Henstock integral recovers an approximate continuous function from its approximate derivative:

Theorem 1. [4] *If $F : [a, b] \to R$ is approximately continuous on $[a, b]$ and approximately differentiable everywhere except for countable number of points on $[a, b]$, then the approximate derivative of F, denoted by F'_{ap}, is AP-Henstock integrable, and $\int_a^x F'_{ap} = F(x) - F(a)$ for any $x \in [a, b]$.*

For the detailed introduction of the AP-Henstock integral, the reader is referred to [5–9].

Although the space of integrable functions is closed under the addition and the scalar multiplication, the product of two integrable functions is not necessarily integrable. Therefore, it is an important question of what kind of properties of an integrable function guarantees the integrability of the product of two integrable functions. We call those properties *the multiplier properties* and the related theorems *the multiplier theorems*. For example, in the case of the Lebesgue, Denjoy, and Henstock integral, the product of an integrable function

$f : [a, b] \to \mathbb{R}$ and a function of bounded variation(which is integrable in any sense mentioned above) $G : [a, b] \to \mathbb{R}$ is integrable, and

$$\int_a^b fG = F(b)G(b) - \int_a^b F dG,$$

where $F(x) = \int_a^x f$, and the last integral is the Riemann-Stieltjes integral. For the Perron integral, a restricted condition on the function of bounded variation is required, see [6]. In this paper, we develop the same kind of multiplier properties that ensure the AP-Henstock integrability of the product of two AP-Henstock integrable functions.

On the other hand, the Henstock integral can be defined on unbounded intervals [10]. It is now known as Hake's theorem that there is no such thing as an "improper integral" for the Henstock integral. By modifying the definition of the Henstock integral on unbounded intervals, we define the AP-Henstock integrals on unbounded intervals. In this setting, we investigate some properties for the AP-Henstock integral and prove that, under some additional conditions, the product of an AP-integrable function and a function of bounded variation. Furthermore, the product of an AP-integrable function and a regulated function is AP-Henstock integrable.

We state the mean value theorem for the Riemann-Stieltjes integral which will be used in the main body of our work.

Theorem 2. [1] *Let f be a continuous functions on $[a, b]$ and let φ be a bounded increasing function on $[a, b]$. Then there exists ξ in $[a, b]$ such that*

$$\int_a^b f d\varphi = f(\xi)(\varphi(b) - \varphi(a)).$$

2. Definition and basic properties

An approximate neighborhood(or ap-nbd) of $x \in [a, b]$ is a measurable set $S_x \subset [a, b]$ containing x as a point of density. For every $x \in E \subset [a, b]$, choose an ap-nbd $S_x \subset [a, b]$ of x. Then we say that $\mathcal{S} = \{S_x : x \in E\}$ is a choice on E. A tagged interval $([u, v], x)$ is said to fine to the choice $\mathcal{S} = \{S_x\}$ if $u, v \in S_x$. Let $\mathcal{P} = \{([x_{i-1}, x_i], t_i) : 1 \leq i \leq n\}$ be a finite collection of non-overlapping tagged intervals. If $([x_{i-1}, x_i], t_i)$ is fine to the choice $\mathcal{S}$ for each $i = 1, \cdots, n$, then we say that $\mathcal{P}$ is $\mathcal{S}$-fine. Let $E \subset [a, b]$. If $\mathcal{P}$ is $\mathcal{S}$-fine and $t_i \in E$ for each $i = 1, \cdots, n$, then $\mathcal{P}$ is said to be $\mathcal{S}$-fine on E. If $\mathcal{P}$ is $\mathcal{S}$-fine and $[a, b] = \cup_{i=1}^n [x_{i-1}, x_i]$, then we say that $\mathcal{P}$ is $\mathcal{S}$-fine partition of $[a, b]$.

Definition 1. [4] A function $f : [a, b] \to \mathbb{R}$ is said to be approximate Henstock integrable (AP-Henstock integrable) on $[a, b]$ if there exists a real number A such that for each $\epsilon > 0$ there is a choice $\mathcal{S}$ on $[a, b]$ such that

$$\left| \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) - A \right| < \epsilon,$$

for each $\mathcal{S}$-fine partition $\mathcal{P} = \{([x_{i-1}, x_i], t_i) : 1 \leq i \leq n\}$ of $[a, b]$. In this case, A is called the AP-Henstock integral of f on $[a, b]$, and we write $A = \int_a^b f$. We denote $\sum_{i=1}^n f(t_i)(x_i - x_{i-1}) = S(\mathcal{P}; f)$ and the collection of all functions that are AP-Henstock integrable on an interval I by $AH(I)$.

Theorem 3. [4] *Let f and g be AP-Henstock integrable functions on $[a, b]$. then for any real numbers α and β, $\alpha f + \beta g$ is AP-Henstock integrable on $[a, b]$ and $\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$.*

Theorem 4. *Let $f : [a, b] \to \mathbb{R}$. If $f = 0$ almost everywhere on $[a, b]$, then $f \in AH([a, b])$ and $\int_a^b f = 0$.*

Proof. The facts that f is Henstock integrable on $[a, b]$ and the integral is 0 are proved in [6]. Since every Henstock integrable function is AP-Henstock integrable and the integrals are the same, $f \in AH([a, b])$ and $\int_a^b f = 0$. □

Corollary 1. *Let $f \in AH([a, b])$ and $g : [a, b] \to \mathbb{R}$. If $g = f$ almost everywhere on $[a, b]$, then $g \in AH([a, b])$ and $\int_a^b g = \int_a^b f$.*

Proof. Since $g - f = 0$ almost everywhere on I, by the theorem, $g - f \in AH([a,b])$ and $\int_a^b (g-f) = 0$. Therefore, $g = (g-f) + f \in AH(I)$ and $\int_a^b g = \int_a^b (g-f) + \int_a^b f = \int_a^b f$. □

Theorem 5. [4] *Let $f : [a,b] \to \mathbb{R}$ be AP-Henstock integrable on $[a,b]$ and let $F(x) = \int_a^x f$ for each $x \in [a,b]$. Then*

1. *the function F is measurable;*
2. *the function F is approximately continuous on $[a,b]$;*
3. *the function F is approximately differentiable almost everywhere on $[a,b]$ and $F'_{ap} = f$ almost everywhere on $[a,b]$; and*
4. *the function f is measurable.*

3. Integral of the translate of a function

In this section, we prove that the translate of an *AP*-Henstock integrable function is *AP*-Henstock integrable.

Let $I := [a,b]$ and let $r \in \mathbb{R}$. We define the r-additive translate of I to be the interval $I_r := [a+r, b+r]$, and the r-additive translate of f to be the function $f_r(y) := f(y-r)$ for all $y \in I_r$. Similarly, if $r > 0$, we define the r-multiplicative translate of I to be the interval $I_{(r)} := [ar, br]$, and the r-multiplicative translate of f to be the function $f_{(r)}(z) := f(z/r)$ for all $z \in I_{(r)}$.

Theorem 6. *(a) If f is AP-Henstock integrable on I, then f_r is AP-Henstock integrable on I_r and $\int_{I_r} f_r = \int_I f$.*
(b) If f is AP-Henstock integrable on I, then $f_{(r)}$ is AP-Henstock integrable on $I_{(r)}$ and $\int_{I_{(r)}} f_{(r)} = r\int_I f$.

Proof. (a) Let $\epsilon > 0$. Since $f \in AH(I)$, there exist a choice $\mathcal{S} = \{S_x : x \in I\}$ on I such that if $\mathcal{P}_1$ is a $\mathcal{S}$-fine partition of I, then $|S(f;\mathcal{P}_1) - \int_I f| \le \epsilon$. Now, we define $\mathcal{S}_\epsilon := \{S_{y-r} + r : S_{y-r} \in \mathcal{S},\ y \in I_r\}$. Suppose that $\mathcal{Q} := \{([y_{i-1}, y_i], s_i)\}_{i=1}^n$ is a $\mathcal{S}_\epsilon$-fine partition of I_r. If we let $x_i := y_i - r$ and $t_i := s_i - r$, then $x_{i-1} \le t_i \le x_i$, $x_{i-1}, x_i \in S_{t_i}$, and $t_i \in I$, whence $\mathcal{P} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a $\mathcal{S}$-fine partition of I. Since $S(f_r;\mathcal{Q}) = S(f;\mathcal{P})$, we infer that

$$\left|S(f_r;\mathcal{Q}) - \int_I f\right| = \left|S(f;\mathcal{P}) - \int_I f\right| \le \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we conclude that $f_r \in AH(I_r)$ and $\int_{I_r} f_r = \int_I f$.

(b) Let $\epsilon > 0$. Since $f \in AH(I)$, there exists a choice $\mathcal{S} = \{S_x : x \in I\}$ of I such that if $\mathcal{P}_1$ is a $\mathcal{S}$-fine partition on I, then $|S(f;\mathcal{P}_1) - \int_I f| \le \epsilon/r$. Now, we define $\mathcal{S}_\epsilon := \{rS_{y/r} : S_{y/r} \in \mathcal{S},\ y \in I_{(r)}\}$, then $\mathcal{S}_\epsilon$ is a choice on $I_{(r)}$. Suppose that $\mathcal{Q} := \{([y_{i-1}, y_i], s_i)\}_{i=1}^n$ is a S_ϵ-fine partition of $I_{(r)}$. If we let $x_i := y_i/r$ and $t_i := s_i/r$, then $x_{i-1} \le t_i \le x_i$, $x_{i-1}, x_i \in S_{t_i}$, and $t_i \in I$, whence $\mathcal{P} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a $\mathcal{S}$-fine partition of I. Since $S(f_{(r)};\mathcal{Q}) = \sum_{i=1}^n f_{(r)}(s_i)(y_i - y_{i-1}) = r\sum_{i=1}^n f(t_i)(x_i - x_{i-1}) = rS(f;\mathcal{P})$, we have

$$\left|S(f_{(r)};\mathcal{Q}) - r\int_I f\right| = r\left|S(f;\mathcal{P}) - \int_I f\right| \le \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we conclude that $f_{(r)} \in AH(I_{(r)})$ and $\int_{I_{(r)}} f_{(r)} = r\int_I f$.
□

4. Multiplier Properties on bounded intervals

It is well known that the product of two *AP*-Henstock integrable functions is not necessarily *AP*-Henstock integrable, even when one of them is bounded or continuous. In this section, we provide some conditions under which the product of two *AP*-Henstock integrable functions is *AP*-Henstock integrable. We also establish the Mean Value Theorems. We start by showing the squeeze theorem for the *AP*-Henstock integral.

Theorem 7. *A function f belongs to $AH(I := [a,b])$ if and only if for every $\epsilon > 0$ there exist functions φ_ϵ and ψ_ϵ in $AH(I)$ with $\varphi_\epsilon(x) \le f(x) \le \psi_\epsilon(x)$ for all $x \in I$, and such that*

$$\int_I (\psi_\epsilon - \varphi_\epsilon) \le \epsilon.$$

Proof. Let $f \in AH(I)$ and let $\epsilon > 0$. We can take $\varphi_\epsilon := \psi_\epsilon := f$. Conversely, assume that for every $\epsilon > 0$ there exist functions φ_ϵ and ψ_ϵ in $AH(I)$ with $\varphi_\epsilon(x) \leq f(x) \leq \psi_\epsilon(x)$ for all $x \in I$, and such that $\int_I(\psi_\epsilon - \varphi_\epsilon) \leq \epsilon$. Then for each $\epsilon > 0$, it follows that $S(\varphi_\epsilon;\mathcal{P}) \leq S(f;\mathcal{P}) \leq S(\psi_\epsilon;\mathcal{P})$ for any tagged partition $\mathcal{P}$ of I. Since $\varphi_\epsilon \in AH(I)$, there exists a choice $\mathcal{S}_1$ on I such that if $\mathcal{P}$ is a $\mathcal{S}_1$-fine partition of I, then $|S(\varphi_\epsilon;\mathcal{P}) - \int_I \varphi_\epsilon| \leq \epsilon$, whence it follows that $\int_I \varphi_\epsilon - \epsilon \leq S(\varphi_\epsilon;\mathcal{P})$. Similarly there exists a choice $\mathcal{S}_2$ on I such that if $\mathcal{P}$ is a $\mathcal{S}_2$-fine partition of I, then $S(\psi_\epsilon;\mathcal{P}) \leq \int_I \psi_\epsilon + \epsilon$. Now let $\mathcal{S} := \{S_1 \cap S_2 : S_1 \in \mathcal{S}_1, S_2 \in \mathcal{S}_2\}$ and let $\mathcal{P}$ be a $\mathcal{S}$-fine tagged partition of I. Then we have

$$\int_I \varphi_\epsilon - \epsilon \leq S(f;\mathcal{P}) \leq \int_I \psi_\epsilon + \epsilon,$$

and if $\mathcal{Q}$ is a $\mathcal{S}$-fine partition of I, then

$$-\int_I \psi_\epsilon - \epsilon \leq -S(f;\mathcal{Q}) \leq -\int_I \varphi_\epsilon + \epsilon.$$

Adding these inequalities, we obtain

$$-\int_I(\psi_\epsilon - \varphi_\epsilon) - 2\epsilon \leq S(f;\mathcal{P}) - S(f;\mathcal{Q}) \leq \int_I(\psi_\epsilon - \varphi_\epsilon) + 2\epsilon.$$

Hence we conclude that

$$|\ S(f;\mathcal{P}) - S(f;\mathcal{Q})\ | \leq \int_I(\psi_\epsilon - \varphi_\epsilon) + 2\epsilon \leq 3\epsilon.$$

Since $\epsilon > 0$ is arbitrary, f satisfies the Cauchy Criterion([4], Theorem 16.6) for the *AP*-Henstock integral. Therefore $f \in AH(I)$. $\square$

Definition 2. [1] Let $I := [a, b]$. A function $f : I \to \mathbb{R}$ is said to be regulated on I if for every $\epsilon > 0$ there exists a step function $s_\epsilon : I \to \mathbb{R}$ such that

$$|\ f(x) - s_\epsilon(x)\ | \leq \epsilon$$

for all $x \in I$.

It easy to see that a function f is regulated on I if and only if there is a sequence $\{s_n\}_{n=1}^{\infty}$ of step functions on I that converges uniformly to f on I.

Theorem 8. *Let $I := [a, b]$. If $f : I \to \mathbb{R}$ is regulated on I, then f is AP-Henstock integrable on I.*

Proof. Let $f : I \to \mathbb{R}$ be regulated on I and let $\epsilon > 0$. Then there exists a step function $s_\epsilon : I \to \mathbb{R}$ such that

$$|\ f(x) - s_\epsilon(x)\ | \leq \epsilon$$

for all $x \in I$. Therefore, we have $s_\epsilon(x) - \epsilon \leq f(x) \leq s_\epsilon(x) + \epsilon$ for all $x \in [a, b]$. If we let $\varphi_\epsilon(x) := s_\epsilon(x) - \epsilon$ and $\psi_\epsilon(x) := s_\epsilon(x) + \epsilon$ for all $x \in I$, then the functions φ_ϵ and ψ_ϵ are *AP*-Henstock integrable on I and $\varphi_\epsilon(x) \leq f(x) \leq \psi_\epsilon(x)$ for $x \in I$. Moreover, since

$$\int_I(\psi_\epsilon - \varphi_\epsilon) \leq 2(b-a)\epsilon,$$

it follows from Theorem 7 that f is *AP*-Henstock integrable on I. $\square$

Theorem 9. *Let $f \in AH(I := [a, b])$ be bounded below and g be regulated on I. Then the product fg belongs to $AH(I)$.*

Proof. Assume that $f(x) \geq 0$ for $x \in I$. It is clear that if s is a step function, then sf belongs to $AH(I)$. Let $A > \int_I f \geq 0$ and let $\epsilon > 0$. Since g is regulated on I, there exists a step function s_ϵ on I such that $|g(x) - s_\epsilon(x)| \leq \frac{\epsilon}{2A}$ for all $x \in I$. Now, we define $\varphi_\epsilon(x) := f(x)\left(s_\epsilon(x) - \frac{\epsilon}{2A}\right)$ and $\psi_\epsilon := f(x)\left(s_\epsilon(x) + \frac{\epsilon}{2A}\right)$ for all $x \in I$, then $\varphi_\epsilon, \psi_\epsilon \in AH(I)$ and it follows that $\varphi_\epsilon(x) \leq f(x)g(x) \leq \psi_\epsilon(x)$ for all $x \in I$, and that

$$\int_I(\psi_\epsilon - \varphi_\epsilon) = \frac{\epsilon}{A}\int_I f \leq \epsilon.$$

Therefore, it follows from Theorem 7 that $fg \in AH(I)$. Now, let $f(x) \geq M$ on I. Then by writing $fg = (f-M)g + Mg$, we see that $fg \in AH(I)$. $\square$

Theorem 10. *Let f and $|f|$ be AP-Henstock integrable on $I := [a,b]$ and let g be a bounded, measurable function on I. Then the product fg is AP-Henstock integrable on I.*

Proof. Let $h := fg$. Since h is measurable, there exists a sequence $\{s_n\}_{n=1}^{\infty}$ of step functions such that $\{s_n\}_{n=1}^{\infty}$ converges to h almost everywhere on I. Let $\overline{s_n}$ be the middle function of $-M|f|$, s_n, and $M|f|$. Then $-M|f| \leq \overline{s_n} \leq M|f|$ and $\{\overline{s_n}\}$ converges to h almost everywhere on I. Therefore, it follows from the Dominated Convergence Theorem for the AP-Henstock integral [9] that $h \in AH(I)$.
$\square$

Theorem 11. *Let $I := [a,b]$, $f \in AH(I)$, φ is of bounded variation on I, and $F(x) := \int_a^x f$ on I. If F is Riemann-Stieltjes integrable with respect to φ on I, then the product $f\varphi$ belongs to $AH(I)$ and*

$$\int_I f\varphi = \int_I \varphi dF = F(b)\varphi(b) - \int_I F d\varphi,$$

where the second and third integrals are the Riemann-Stieltjes integrals.

Proof. Since F is Riemann-Stieltjes integrable with respect to φ, the third integral exists, and the existence of the second integral and the validity of the second equality follows from the well-known integration by part formula for the Riemann-Stieljes integral ([4], Theorem 12.14). Therefore, we only need to show the first equality. To this end, let $\epsilon > 0$. Since φ is Riemann-Stieltjes integrable with respect to F, there exist $\delta > 0$ such that if $\mathcal{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is any tagged partition of I with norm less than 2δ, then

$$\left| \sum_{i=1}^n \varphi(t_i)(F(x_i) - F(x_{i-1})) - \int_I \varphi dF \right| \leq \epsilon.$$

Let $|\varphi(x)| \leq M$ for all $x \in I$. Since $f \in AH(I)$, there exist a choice $\mathcal{S}$ on I such that if $\mathcal{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is $\mathcal{S}$-fine partition of I, then

$$\left| \sum_{i=1}^n \{f(t_i)(x_i - x_{i-1}) - (F(x_i) - F(x_{i-1}))\} \right| \leq \epsilon/2M,$$

and it follows from the Saks-Henstock Lemma for the AP-Henstock integral that

$$\sum_{i=1}^n |f(t_i)(x_i - x_{i-1}) - (F(x_i) - F(x_{i-1}))| \leq \epsilon/M.$$

Define $\mathcal{S}' := \{S_x \cap (x-\delta, x+\delta) : S_x \in \mathcal{S}\}$ and let $\mathcal{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ be $\mathcal{S}'$-fine partition of I. Then,

$$\begin{aligned}
&\left| \sum_{i=1}^n f(t_i)\varphi(t_i)(x_i - x_{i-1}) - \int_I \varphi dF \right| \\
&\leq \left| \sum_{i=1}^n f(t_i)\varphi(t_i)(x_i - x_{i-1}) - \sum_{i=1}^n \varphi(t_i)\left(F(x_i) - F(x_{i-1})\right) \right| + \left| \sum_{i=1}^n \varphi(t_i)\left(F(x_i) - F(x_{i-1})\right) - \int_I \varphi dF \right| \\
&\leq M \sum_{i=1}^n |f(t_i)(x_i - x_{i-1}) - (F(x_i) - F(x_{i-1}))| + \epsilon \leq 2\epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $f\varphi \in AH(I)$ and $\int_I f\varphi = \int_I \varphi dF$. $\square$

Note that the indefinite AP-Henstock integral is only approximately continuous, not necessarily continuous on I. It is shown in [6] (Exercise 12.10) that if two bounded functions F and φ on I share a common point of discontinuity in I, then F is not Rieman-Stieltjes integrable with respect to φ on I. Therefore, the condition in the above theorem that F is Reimann-Stiltjes integrable with respect to φ cannot be removed. On the other hand, it is well-known fact that if F is continuous and φ is of bounded variation on I, then F is Riemann-Stieltjes integrable with respect to φ [6]. Therefore, the following corollary follows.

Corollary 2. *Let f be AP-Henstock integrable on $I := [a,b]$, φ be bounded variation on I, and $F(x) := \int_a^x f$ on I. If F is continuous on I, then $f\varphi \in AH(I)$ and*

$$\int_I f\varphi = \int_I \varphi dF = F(b)\varphi(b) - \int_I F d\varphi,$$

where the second and third integrals are the Riemann-Stieltjes integrals.

In addition to the multiplier theorem above, we provide a version of integration by parts theorem.

Theorem 12. *Let $I := [a,b]$ and let $F, G : I \to \mathbb{R}$ be approximately continuous on I. If $f, g \in AH(I)$, and $F'_{ap} = f$, $G'_{ap} = g$ except for countably many points in I, then $Fg + fG \in AH(I)$ and*

$$\int_I (Fg + fG) = F(b)G(b) - F(a)G(a).$$

Moreover, $Fg \in AH(I)$ if and only if $fG \in AH(I)$, in which case

$$\int_I Fg = F(b)G(b) - F(a)G(a) - \int_I fG.$$

Proof. By the hypothesis, there exist countable sets C_f and C_g such that $F'_{ap}(x) = f(x)$ for $x \in I - C_f$ and $G'_{ap}(x) = g(x)$ for $x \in I - C_g$. Let $C := C_f \cup C_g$ be a countable set. For $x \in I - C$, $(FG)'_{ap} = F'_{ap}G + FG'_{ap} = fG + Fg$. Also, by the hypothesis, FG is approximately continous on I. Therefore, by Theorem 1, $(FG)'_{ap} \in AH(I)$ and $\int_I (FG)'_{ap} = F(g)G(g) - F(a)G(a)$. Since C is a countable set, $fG + Fg \in AH(I)$ and $\int_I fG + Fg = \int_I (FG)'_{ap}$. Moreover, if $Fg \in AH(I)$, since $fG = (Fg + fG) - Fg$, $fG \in AH(I)$.
□

We now establish the Mean Value Theorems for the *AP*-Henstock integral.

Theorem 13. *(First Mean Value Theorem). If f is continuous on $I := [a,b]$, and if $p \in AH(I)$ does not change sign on I, then there exists $\xi \in I$ such that*

$$\int_I fp = f(\xi) \int_I p.$$

Proof. Since f is continuous, f is bounded on I. We invoke the fact that a nonnegative *AP*-Henstock integrable function is Lebesgue integrable and that the product of a Lebesgue integrable function and a bounded, Lebesgue integrable function is Lebesgue integrable. Therefore, p is Lebesgue integrable and fp is Lebesgue integrable on I. If $p \geq 0$, then $mp \leq fp \leq Mp$, where $m := \min\{f(x) : x \in I\}$ and $M := \max\{f(x) : x \in I\}$, and

$$m \int_I p \leq \int_I fp \leq M \int_I p.$$

If $\int_I p = 0$, then the result is trivial; if not, it follows from the Bolzano Intermediate Value Theorem in $\mathbb{R}$. If $p \leq 0$, then the argument is similar. □

Theorem 14. *(Second Mean Value Theorem). If $f \in AH(I := [a,b])$, $F(x) = \int_a^x f$ is continuous on I, and if g is monotone on I, then there exists $\xi \in I$ such that*

$$\int_I fg = g(a) \int_a^{\xi} f + g(b) \int_{\xi}^{b} f.$$

Proof. Since g is of bounded variation, it follows from Theorem 11 that $fg \in AH(I)$ and

$$\int_I fg = \int_I g dF = g(b)F(b) - g(a)F(a) - \int_I F dg,$$

where the third integral is the Riemann-Stieltjes integral. Then, by Theorem 1.2, there exists $\xi \in I$ such that the last two terms equal

$$\begin{aligned} g(b)F(b) - g(a)F(a) - F(\xi)(g(b) - g(a)) &= g(a)(F(\xi) - F(a)) + g(b)(F(b) - F(\xi)) \\ &= g(a)\int_a^{\xi} f + g(b)\int_{\xi}^{b} f. \end{aligned}$$

□

5. Multiplier Properties on unbounded intervals.

In this section, we define the AP-Henstock integral on unbounded intervals and investigate some properties of the integral including some multiplier properties.

We extend any function $f : [a, \infty) \to \mathbb{R}$ to a function defined on $[a, \infty]$ in the extended real numbers $\mathbb{R}^* := \mathbb{R} \cup \{\infty, -\infty\}$ by defining $f(\infty) = 0$. We then take a tagged partition of the interval $[a, \infty]$:

$$\mathcal{P} := \{([x_0, x_1], t_i), \cdots, ([x_{n-1}, x_n], t_n), ([x_n, x_{n+1}], t_{n+1})\},$$

so that $x_0 = a$ and $x_{n+1} = \infty$. A choice $\mathcal{S} = \{S_x : x \in [a, \infty]\}$ on $[a, \infty]$ is a set of ap-nbd $S_x \subset [a, \infty]$ that contains x as a point of density. We require that S_x is bounded for each $x \in \mathbb{R}$ and $S_\infty = [d, \infty]$ for some $d > a$. We say that the tagged partition $\mathcal{P}$ is $\mathcal{S}$-fine if $x_{i-1}, x_i \in S_{t_i}$ for $i = 1, \cdots, n+1$, and $d \le x_n$. Because S_x is bounded for $x \in \mathbb{R}$, $t_{n+1} = \infty$. Define $0 \cdot \infty = 0$ so that the contribution of the final term in $\mathcal{P}$ to the Riemann sum is $f(\infty) \cdot \infty = 0$. Now, we give the definition of the AP-integral of a function $f : [a, \infty] \to \mathbb{R}$.

Definition 3. A function $f : [a, \infty] \to \mathbb{R}$ is AP-Henstock integrable on $[a, \infty)$, or on $[a, \infty]$ if there exists a real number A such that for each $\epsilon > 0$ there is a choice $\mathcal{S}$ on $[a, \infty]$ such that

$$\left| \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - A \right| < \epsilon$$

for each $\mathcal{S}$-fine partition $\mathcal{P} = \{([x_{i-1}, x_i], t_i) : 1 \le i \le n+1\}$ of $[a, \infty]$. In this case, A is called the AP-Henstock integral of f on $[a, \infty)$ and we write $A = \int_a^\infty f$.

The collection of all functions that are AP-Henstock integrable on an interval $[a, \infty)$ will be denoted by $AH([a, \infty))$. The next theorem is the Cauchy Criterion for the AP-Henstock integral on unbounded intervals.

Theorem 15. *Let $I := [a, \infty]$ and let $f : I \to \mathbb{R}$. Then, $f \in AH(I)$ if and only if for any $\epsilon > 0$ there exists a choice $\mathcal{S}_\epsilon$ of I such that if $\mathcal{P}$ and $\mathcal{Q}$ are any partitions of I that are $\mathcal{S}$-fine, then $|S(f; \mathcal{P}) - S(f; \mathcal{Q})| \le \epsilon$.*

Proof. Let $f \in AH([a, \infty])$ with $\tilde{A} := \int_a^\infty f$. Let $\tilde{\mathcal{S}}_\epsilon$ be a choice on I such that if $\mathcal{P}, \mathcal{Q}$ are $\tilde{\mathcal{S}}_\epsilon$-fine partitions of I, then $|S(f; \mathcal{P}) - A| \le \epsilon/2$ and $|S(f; \mathcal{Q}) - A| \le \epsilon/2$, which follows that

$$|S(f; \mathcal{P}) - S(f; \mathcal{Q})| \le |S(f; \mathcal{P}) - \tilde{A}| + |S(f; \mathcal{Q}) - \tilde{A}| \le \epsilon.$$

Now, suppose that for any $\epsilon > 0$ there exists a choice $\mathcal{S}_\epsilon$ on I such that if $\mathcal{P}$ and $\mathcal{Q}$ are any partitions of I that are $\mathcal{S}_\epsilon$-fine of I, then $|S(f; \mathcal{P}) - S(f; \mathcal{Q})| \le \epsilon$. For each $n \in \mathbb{N}$, let $\mathcal{S}_n = \{S_{n,x} : x \in I\}$ be a choice on I such that if $\mathcal{P}$ and $\mathcal{Q}$ are $\mathcal{S}_n$-fine, then $|S(f; \mathcal{P}) - S(f; \mathcal{Q})| \le 1/n$. We may assume that $S_{n+1,x} \subset S_{n,x}$ for all $x \in I$, $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $\mathcal{P}_n$ be a $\mathcal{S}_n$-fine partition of I. If $m > n$, then $\mathcal{P}_m$ and $\mathcal{P}_n$ are $\mathcal{S}_n$-fine. Therefore, for $m > n$, $|S(f; \mathcal{P}_n) - S(f; \mathcal{P}_m)| \le 1/n$, and it follows that $\{S(f; \mathcal{P}_n)\}_{n=1}^\infty$ is a Cauchy sequence. Let $A := \lim_{n\to\infty} S(f; \mathcal{P}_n)$. By taking $m \to \infty$, we have $|S(f; \mathcal{P}_n) - A| \le 1/n$. Now, for any given $\epsilon > 0$, let $K \in \mathbb{N}$ be such that $1/K \le \epsilon/2$. If $\mathcal{Q}$ be a $\mathcal{S}_K$-fine partition of I, then

$$|S(f; \mathcal{Q}) - A| \le |S(f; \mathcal{Q}) - S(f; \mathcal{P}_K)| + |S(f; \mathcal{P}_K) - A| \le 2/K \le \epsilon.$$

□

The following theorem is the additive property of the AP-Henstock integral of a function on $[a, \infty]$.

Theorem 16. *Let $I := [a,\infty]$, $f : I \to \mathbb{R}$ and let $c > a$. Then $f \in AH(I)$ if and only if the restriction of f to $[a,c]$ and $[c,\infty]$ are both integrable. In this case we have*

$$\int_a^\infty f = \int_a^c f + \int_c^\infty f.$$

Proof. Let $I_1 := [a,c]$ and $I_2 := [c,\infty]$. Suppose that $f \in AH(I_1)$ and $f \in AH(I_2)$. Let f_1 be the restriction of f to I_1 and let f_2 be the restriction of f to I_2. Let $A_1 := \int_{I_1} f_1$ and let $A_2 := \int_{I_2} f$. Given $\epsilon > 0$, let $\mathcal{S}'_\epsilon := \{S'_{\epsilon,x} : x \in I_1\}$ be a choice on I_1 and let $\mathcal{S}''_\epsilon := \{S''_{\epsilon,x} : x \in I_2\}$ be a choice on I_2 such that if $\mathcal{P}_1$ is a $\mathcal{S}'_\epsilon$-fine partition of I_1 and $\mathcal{P}_2$ is a $\mathcal{S}''_\epsilon$-fine partition of I_2, then

$$|S(f_1;\mathcal{P}_1) - A_1| \le \tfrac{1}{2}\epsilon \quad \text{and} \quad |S(f_2;\mathcal{P}_2) - A_2| \le \tfrac{1}{2}\epsilon.$$

We define a choice $\mathcal{S}_\epsilon := \{S_{\epsilon,x} : x \in I\}$ on I by

$$S_{\epsilon,x} = \begin{cases} S'_{\epsilon,x} \cap [a,c) & \text{if } x \in [a,c) \\ (S'_{\epsilon,c} \cap [a,c]) \cup (S''_{\epsilon,c} \cap [c,\infty)) & \text{if } x = c \\ S''_{\epsilon,x} \cap (c,\infty) & \text{if } x \in (c,\infty) \\ S''_{\epsilon,\infty} & \text{if } x = \infty. \end{cases}$$

Let $\mathcal{P}$ be a S_ϵ-fine partition of I and suppose that each tag occurs only once. Then the point c must be a tag of an subinterval in $\mathcal{P}$. Let $([u,v],c)$ be the tagged interval in $\mathcal{P}$ of which the tag is c. Then $\mathcal{P}$ is of the form $\mathcal{P}_a \cup ([u,v],c) \cup \mathcal{P}_b$ where the tags of $\mathcal{P}_a$ are less than c and the tags of $\mathcal{P}_b$ are greater than c. Let $\mathcal{P}_1 := \mathcal{P}_a \cup ([u,c],c)$ and let $\mathcal{P}_2 := \mathcal{P}_b \cup ([c,v],c)$. Then $\mathcal{P}_1$ is a $\mathcal{S}'_\epsilon$-fine partition of I_1 and $\mathcal{P}_2$ is a $\mathcal{S}''_\epsilon$-fine partition of I_2. Therefore,

$$|S(f;\mathcal{P}) - A_1 - A_2| \le |S(f_1;\mathcal{P}_1) - A_1| + |S(f_2;\mathcal{P}_2) - A_2|.$$

Since $\epsilon > 0$ is arbitrary, f is integrable on I to $\int_{I_1} f + \int_{I_2} f$.

Now, suppose that $f \in AH([a,\infty])$. For each $\epsilon > 0$, let $\tilde{\mathcal{S}}_\epsilon := \{\tilde{S}_{\epsilon,x} : x \in I\}$ be a choice on I that satisfies the Cauchy Criterion (Theorem 15). Let f_1 denote the restriction of f to I_1 and let $\tilde{\mathcal{S}}'_\epsilon := \{\tilde{S}_{\epsilon,x} \cap I_1 : x \in I_1\}$ be the restriction of $\tilde{\mathcal{S}}_\epsilon$ to I_1. Let $\mathcal{P}_1, \mathcal{Q}_1$ be $\tilde{\mathcal{S}}'_\epsilon$-fine partitions of I_1. By adjoining the same tagged partition of I_2, extend $\mathcal{P}_1, \mathcal{Q}_1$ to partitions $\mathcal{P}, \mathcal{Q}$ of I that are $\tilde{\mathcal{S}}_\epsilon$-fine . Then,

$$|S(f_1;\mathcal{P}_1) - S(f_1;\mathcal{Q}_1)| = |S(f;\mathcal{P}) - S(f;\mathcal{Q})| \le \epsilon.$$

Therefore, by Theorem 15, f_1 is integrable on I_1. In the same way, the restriction of f to I_2 is integrable on I_2 □

Corollary 3. *If $f \in AH([a,\infty])$ and if $[c,d] \subset [a,\infty]$, then the restriction of f to $[c,d]$ is integrable.*

Proof. Let $f \in AH([a,\infty])$ and $[c,d] \subset [a,\infty]$. Then it follows from the theorem that $f \in AH([c,\infty])$, which follows that $f \in AH([c,d])$. □

Theorem 17. *Let $I := [a,\infty]$ and let $f : I \to \mathbb{R}$. Then $f \in AH(I)$ if and only if $f \in AH([a,c])$ for every $c \ge a$ and there exists $A \in \mathbb{R}$ such that*

$$\lim_{c\to\infty} \int_a^c f = A.$$

In this case, $\int_a^\infty f = A$.

Proof. Let $f \in AH(I)$, $\int_a^\infty f = A$, and let $\epsilon > 0$. Then there exists a choice $\mathcal{S} := \{S_x : x \in I\}$ on I such that if $\mathcal{P} = \{([x_{i-1},x_i],t_i)\}_{i=1}^{n+1}$ is a $\mathcal{S}$-fine partition of I, then $|S(f;\mathcal{P}) - A| \le \frac{1}{2}\epsilon$. Let $c \ge x_n$. Since $f \in AH([a,c])$ by Theorem 16, there exists a choice $\mathcal{S}_c := \{S_{c,x} : x \in [a,c]\}$ on $[a,c]$ such that if $\mathcal{P}_c$ is a S_c-fine partition of $[a,c]$, then $\left|S(f;\mathcal{P}_c) - \int_a^c f\right| \le \frac{1}{2}\epsilon$. We may assume that $S_{c,x} \subset S_x$ for all $x \in [a,c]$. Let $\mathcal{P}_c^* := \mathcal{P}_c \cup ([c,\infty],\infty)$, then $\mathcal{P}_c^*$ is a $\mathcal{S}$-fine partition of $[a,\infty]$ such that $S(f;\mathcal{P}_c) = S(f;\mathcal{P}_c^*)$. Therefore,

$$\left|\int_a^c f - A\right| \le \left|\int_a^c f - S(f;\mathcal{P}_c)\right| + |S(f;\mathcal{P}_c^*) - A| \le \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $\lim_{c\to\infty}\int_a^c f = A$.

Now, suppose that $f \in AH([a,c])$ for every $c \geq a$ and that there exists $A \in \mathbb{R}$ such that $\lim_{c\to\infty}\int_a^c f = A$. Take a strictly increasing unbounded sequence $\{c_k\}_{k=0}^{\infty}$ with $c_0 = a$. Given $\epsilon > 0$, let $N \in \mathbb{N}$ be such that if $b \geq c_N$, then $\left|\int_a^b f - A\right| \leq \epsilon$. Since $f \in AH(I_k := [c_{k-1}, c_k])$ for each $k \in \mathbb{N}$, let $\mathcal{S}_k := \{S_{k,x} : x \in I_k\}$ be a choice on I_k such that if $\mathcal{P}_k$ is a $\mathcal{S}_k$-fine partition of I_k, then $\left|S(f;\mathcal{P}_k) - \int_{I_k} f\right| \leq \epsilon/2^k$. We may assume that

1. $S_{1,c_0} \subset \left[c_0, \frac{c_0+c_1}{2}\right]$,

and if $k \geq 1$, that

2. $S_{k+1,c_k} \subset S_{k,c_k} \cap \left(\frac{c_{k-1}+c_k}{2}, \frac{c_k+c_{k+1}}{2}\right)$, and
3. $S_{k,x} \subset \left(\frac{c_{k-1}+x}{2}, \frac{x+c_k}{2}\right)$ for $x \in (c_{k-1}, c_k)$.

Now, in order to define a choice on I, we assign a measurable set S_x to each $x \in I$ by

$$S_x = \begin{cases} S_{k,x} & \text{if } x \in [c_{k-1}, c_k),\ k \in \mathbb{N} \\ [c_N, \infty] & \text{if } x = \infty, \end{cases}$$

so that $\mathcal{S}^* = \{S_x^* : x \in I\}$ be a choice on I. Let $\mathcal{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n+1}$ be a $\mathcal{S}^*$-fine partition of I. By the definition of $\mathcal{S}^*$, the tag for the unbounded subinterval $[x_n, \infty]$ in $\mathcal{P}$ must be ∞ and $c_N \leq x_n$. Now let $s \in \mathbb{N}$ be the smallest positive integer such that $x_n \leq c_s$ so that $N \leq s$. Again, by the the condition (3), for $k = 1, \cdots, s-1$, the point c_k must be the tag for any subinterval in $\mathcal{P}$ that contains c_k, and we may assume that c_k appears as an end point to the intervals. We let

$$\mathcal{Q}_1 := \mathcal{P} \cap [c_0, c_1], \cdots, \mathcal{Q}_{s-1} := \mathcal{P} \cap [c_{s-2}, c_{s-1}], \mathcal{Q}_s := \mathcal{P} \cap [c_{s-1}, x_n].$$

Then, $\mathcal{Q}_k (k = 1, \cdots, s-1)$ is $\mathcal{S}_k$-fine partition of I_k. Therefore, we have

$$\left|S(f;\mathcal{Q}_k) - \int_{I_k} f\right| \leq \frac{\epsilon}{2^k}.$$

Also, since $\mathcal{Q}_s$ is a $\mathcal{S}_s$-fine subpartion of I_s, by the Saks-Henstock Lemma,

$$\left|S(f;\mathcal{Q}_s) - \int_{c_{s-1}}^{x_n} f\right| \leq \frac{\epsilon}{2^s}.$$

Let $\mathcal{Q}_\infty := \{([x_n, \infty], \infty)\}$ so that $S(f;\mathcal{Q}_\infty) = 0$. Now, since $\mathcal{P} = \mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_s \cup \mathcal{Q}_\infty$, we have

$$|S(f;\mathcal{P}) - A| = \left|\sum_{i=1}^{s} S(f;\mathcal{Q}_i) + S(f;\mathcal{Q}_\infty) - A\right| \leq \left|\sum_{i=1}^{s} S(f;\mathcal{Q}_i) - \int_a^{x_n} f\right| + |S(f;\mathcal{Q}_\infty)| + \left|\int_a^{x_n} f - A\right| \leq 2\epsilon$$

Since $\epsilon > 0$ is arbitrary, $f \in AH(I)$ and $\int_I f = A$. □

We give a different version of Cauchy Criterion for $f \in AH([a,\infty])$.

Theorem 18. *Let $f : [a,\infty] \to \mathbb{R}$ be such that $f \in AH([a,c])$ for all $c \geq a$. Then $f \in AH([a,\infty])$ if and only if for every $\epsilon > 0$ there exists $K(\epsilon) \geq a$ such that if $q > p \geq K(\epsilon)$, then $|\int_p^q f| \leq \epsilon$.*

Proof. Suppose that $f \in AH([a,\infty])$. Let $\epsilon > 0$. By the previous theorem, there exists $K(\epsilon) > 0$ such that $\left|\int_a^c f - \int_a^\infty f\right| < \epsilon/2$ for all $c \geq K(\epsilon)$. Let $q > p > K(\epsilon)$, then

$$\left|\int_p^q f\right| = \left|\int_a^q f - \int_a^p f\right| = \left|\int_a^q f - \int_a^\infty f\right| + \left|\int_a^p f - \int_a^\infty f\right| < \epsilon.$$

Conversely, suppose that for any given $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $q > p \geq K(\epsilon)$, then $\left|\int_p^q f\right| \leq \epsilon$. Let $\{x_n\}$ be an unbounded increasing sequence with $x_0 \geq a$. Since for any $x_m \geq x_n \geq K(\epsilon)$, $\left|\int_a^{x_m} f - \int_a^{x_n} f\right| =$

$\left|\int_{x_n}^{x_m} f\right| \le \epsilon$, the sequence $\left\{\int_a^{x_n} f\right\}_{n=1}^{\infty}$ is a Cauchy sequence. Let $\lim_{n\to\infty}\int_a^{x_n} f := A$ and N be an integer such that $x_N \ge K(\epsilon)$ and $\left|\int_a^{x_n} f - A\right| < \epsilon$ whenever $n \ge N$. If $c > x_N$, then

$$\left|\int_a^c f - A\right| = \left|\int_a^{x_N} f - A\right| + \left|\int_{x_N}^c f\right| < 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, $\lim_{c\to\infty}\int_a^c f = A$, and by Theorem 17, $f \in AH(I)$. □

We now consider the multiplier properties for the AP-Henstock integral on unbounded intervals.

Theorem 19. *Let $f \in AH([a,\infty))$ be bounded below and let g be a regulated function on $[a,\infty)$. Then the product $fg \in AH([a,\infty))$.*

Proof. Assume that $f(x) \ge 0$ on $[a,\infty]$. By Corollary 3 and Theorem 9, $fg \in AH([p,q])$ for any $q > p \ge a$. Let s be a step function such that $|g(x) - s(x)| < 1$ for all $x \in [a,\infty]$. Let $\epsilon > 0$. By Theorem 18, there exists $K(\epsilon) > d$ such that $|\int_p^q f| < \epsilon$ whenever $q > p \ge K(\epsilon)$. If $q \ge x \ge p \ge K(\epsilon)$, then $|g(x)| < M$ for some $M > 0$ and $|f(x)g(x)| < Mf(x)$. Since $|fg|$ and Mf are measurable on $[p,q]$, $|fg|$ is Lebesgue integrable and hence $|fg| \in AH([p,q])$. It follows that $\left|\int_p^q fg\right| \le \int_p^q |fg| < \int_p^q f < \epsilon$. Therefore, by Theorem 18, $fg \in AH([a,\infty])$. Now, if $f(x) > \alpha$ on $[a,\infty]$ for some $\alpha < 0$, then since $(f-\alpha)g$, $\alpha g \in AH([a,\infty])$, the result follows from $fg = (f-\alpha)g + \alpha g$. □

Theorem 20. *Let $I := [a,\infty)$ and let $f, \varphi : I \to \mathbb{R}$. Suppose that $f \in AH(I)$, $F(x) := \int_a^x f$ is continuous on I, and that φ is bounded and monotone on I. Then the product $f\varphi \in AH(I)$.*

Proof. Let $\epsilon > 0$. Since φ is bounded on I, there exists $M > 0$ such that $|\varphi(x)| \le M$ for all $x \in I$. By Theorem 18, there exists $K(\epsilon) \ge a$ such that if $q > p \ge K(\epsilon)$, then $|\int_p^q f| \le \epsilon/2M$. Since φ is monotone, it follows from Corollary 2 that $f\varphi \in AH([p,q])$ and from Theorem 14 that there exists $\xi \in [p,q]$ such that

$$\int_p^q f\varphi = \varphi(p)\int_p^{\xi} f + \varphi(q)\int_{\xi}^q f.$$

Thus, if $q > p \ge K(\epsilon)$, then $|\int_p^q f\varphi| \le M(\epsilon/2M) + M(\epsilon/2M) = \epsilon$. Since $\epsilon > 0$ is arbitrary, by Theorem 18, $f\varphi$ is AP-Henstock integrable on I.

□

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: "The authors declare no conflict of interest."

Data Availability: All data required for this research is included within this paper.

Funding Information: No funding is available for this research.

References

[1] Henstock, R. (1963). *Theory of Integration*. Butterworths, London.

[2] Henstock, R. (1968). A Riemann-type integral of Lebesgue power. *Canadian Journal of Mathematics, 20*, 79-87.

[3] Bullen, P. S. (1983). The Burkill approximately continuous integral. *Journal of the Australian Mathematical Society, 35*(2), 236-253.

[4] Bruckner, A. M. (2020). Differentiation of real functions. *SERBIULA (Sistema Librum 2.0)*.

[5] Liao, K., & Tuan-Seng, C. (1993). The descriptive definitions and properties of the AP integral and their application to the problem of controlled convergence. *Real Analysis Exchange, 19*(1), 81-97.

[6] Gordon, R. A. (1994). *The Integrals of Lebesgue, Denjoy, Perron, and Henstock* (No. 4). American Mathematical Society, USA.

[7] Yoon, J. H., Park, J. M., Kim, Y. K., & Kim, B. M. (2010). The AP-Henstock extension of the Dunford and Pettis integrals. *Journal of the Chungcheong Mathematical Society, 23*(4), 879-884.

[8] Yoon, J. H. (2018). On AP-Henstock-Stieltjes integrals for fuzzy number-valued functions. *Journal of the Chungcheong Mathematical Society, 31*(1), 151-160.

[9] Zhao, D., & Ye, G. (2006). On AP-Henstock-Stieltjes integral. *Journal of the Chungcheong Mathematical Society, 19*(2), 177-177.

[10] Bartle, R. G. (2001). *A Modern Theory of Integration* (Vol. 32). American Mathematical Soc.