Open Journal of
Mathematical Analysis

PSR Press

Article

Norm-attainable operators on involutive stereotype tubes with algebraically connected component of the identity

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Academic Editor: Wei-Shih Du
Received: 14 February 2022; Accepted: 18 June 2022; Published: 22 June 2022.

Abstract: This work is an in-depth study of the class of norm-attainable operators in a general Banach space setting. We give characterizations of norm-attainable operators on involutive stereotype tubes with algebraically connected component of the identity. In particular, we prove reflexivity, boundedness and compactness properties when the set of these operators contains unit balls with involution for the tubes when they are of stereotype category.

Keywords: Norm-attainable operator; Stereotype tube; Involution.

MSC: 32A10; 46E40; 47B35.

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1. Introduction

Studies on norm-attainable operators have been considered by several mathematicians (see [1–5]). Of interest has been the norm-attainability conditions for these operators in different algebras. For instance, the author in [4] considered these conditions in Banach spaces while others worked in Hilbert spaces particularly. However, norm and structural characterizations of these operators have not been done in detail, particularly in other classes of algebras [2].

Spectraloid cones of these operators have not been given attention, particularly in general Banach space setting [3]. In [6], the author studied cones on involutive stereotype tubes and showed that the unit balls of these tubes could be perturbed by the identity when calculating the distance between the identity and the commutant of these operators. Moreover, when the identity has an algebraically connected component, then [7] established that these operators become spectraloid if and only if the spectral radius is algebraically stereotyped with a converging sequence of eigenvalues to an algebraic multiplicity of degree n.

This research considers the class of norm-attainable operators on Involutive Stereotype Tubes (IST) with Algebraically Connected Component of the Identity (ACCI). Certain interesting algebraic features are exhibited by the algebra of norm-attainable operators on involutive stereotype tubes with an algebraically connected component of the identity when perturbed by infinitesimal summands of orthogonal isometries idempotents, orthogonal projections, co-isometries among other classes of operators [8]. However, in this paper, we restrict ourselves to IST with ACCI and study them in a general Banach space setting. We outline new characterizations of the set of norm-attainable operators on involutive stereotype tubes with algebraically connected components of the identity in terms of eigenvalues and the corresponding eigenvectors in (IST). In particular, we prove reflexivity, boundedness, and compactness properties when the cones contain unit balls with involution for the tubes when they are of the stereotype category.

2. Preliminaries

We give some definitions and make some important remarks. Consider H as a complex Hilbert space and $B(H)$ be the algebra oh all bounded linear operators on H. We state the following definition;

Definition 1. ([9], Definition 2.3) An operator $A \in B(H)$ is called a scalar operator of order m if it possesses a spectral distribution of order m, i.e., if there exists a continuous unital morphism $\phi : C_0^m(\mathbb{C}) \to B(H)$ such that $\phi(z) = A$, where z stands for the identity function on $\mathcal{C}$ and $C_0^m(\mathbb{C})$ for the space of compactly supported functions on $\mathbb{C}$ continuously differentiable of order m, $0 \leq m \leq \infty$. An operator $A_0 \in B(H)$ is called subscalar if it is similar to the restriction of a scalar operator to an invariant subspace of H.

Definition 2. ([1], Definition 1.1) An operator $A \in B(H)$ is said to be norm-attainable if there exists a unit vector $x_0 \in H$ such that $\|Ax_0\| = \|A\|$. The set of all norm-attainable operators on a Hilbert space H is denoted by $NA(H)$.

Remark 1. Consider ISTs with ACCI denoted by X and Y, and let $\mathcal{L}(X,Y)$ be the set of all norm-attainable operators $T : X \longrightarrow Y$ endowed with the usual operator norm $\|T\|_{\mathcal{L}(X,Y)} = \sup_{\|x\|_X=1} \|Tx\|_Y = \sup_{\|x\|_X=1, \|y^\star\|_{Y^\star}=1} |\langle Tx, y^\star \rangle_{Y,Y^\star}|$, in which $T \in \mathcal{L}(X,Y)$, then $\mathcal{L}(X,Y)$ is an IST with ACCI. In general, we denote the set of all norm attaining operators by $NA(H)$ and the spectraloid cone by $H^\infty(\mathbb{B}_n, X)$. Let $b : \mathbb{B}_n \longrightarrow \mathcal{L}(\overline{X}, Y)$ and consider $b \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$. The norm-attainable operator with operator-valued symbol b, given as h_b is well defined for $z \in \mathbb{B}_n$ as $h_b f(z) := \int_{\mathbb{B}_n} \frac{b(w)\overline{f(w)}}{(1-\langle z,w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w)$, $f \in H^\infty(\mathbb{B}_n, X)$. We note that b satisfies the condition $\int_{\mathbb{B}_n} \frac{\|b(w)\|_{\mathcal{L}(\overline{X},Y)}}{|1-\langle z,w\rangle|^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w) < \infty$, for every $z \in \mathbb{B}_n$, unless stated otherwise in the sequel. The next section forms the key part of this work in which new characterizations of the set of norm-attainable operators on IST with ACCI are unveiled.

3. Main results

Here we give the main results of this work. We start by some auxilliary proposition.

Proposition 1. *Consider an orthonormal sequence $\{a_k\}$ of non-negative scalars. Let $M_k := (a_0 I + N) \circ (a_1 I + N) \circ \ldots \circ (a_{k-1} I + N)$ be a norm-attainable operator for some constant $k > 0$. Then a function f is in $\Gamma_\gamma(\mathbb{B}_n, X)$ if and only if $k > \gamma$ is such that* $\sup_{z\in\mathbb{B}_n}(1-|z|^2)^{k-\gamma}\|M_k f(z)\|_X < \infty$.

Proof. Without loss of generality let $f \in \Gamma_\gamma(\mathbb{B}_n, X)$. From the statement of the proposition consider $k > \gamma$ and constant $C > 0$ such that $\|N^k f(z)\|_X \leq C(1-|z|^2)^{\gamma-k}$, for all $z \in \mathbb{B}_n$. A simple and straightforward manipulation together with some substitutions give $\phi'(r) = \frac{1}{r^{a+1}} \int_0^r s^a \psi'(s) \mathrm{d}s$. Since $k > \gamma$, we get the desired result. □

For ISTs with ACCI, we consider at this point two classes of norm-attainable operators D_k, and L_k, having an equivalence relation with $a_j = n + \alpha + j + 1$. Now for simplicity we denote $H^\infty(\mathbb{B}_n, X)$ and $H^\infty(\mathbb{B}_n, Y^\star)$ by $\mathfrak{S}_X$ and $\mathfrak{S}_Y$ respectively.

Proposition 2. *Let $\mathfrak{T}$ be an IST with ACCI and let $f \in \mathfrak{S}_X$ and $g \in \mathfrak{S}_{Y^*}$. Given that $b \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$, then we have* $\langle h_b f, g \rangle_{\alpha,Y} = \int_{\mathbb{B}_n} \langle b(z)\overline{f(z)}, g(z) \rangle_{Y,Y^\star} \mathrm{d}\nu_\alpha(z)$.

Proof. Let $\mathfrak{T}$ be an IST with ACCI and consider $f \in \mathfrak{S}_X$ and $g \in \mathfrak{S}_{Y^*}$. From Hahn-Banach theorem for norm-attainable operators, the inner product property $\langle \cdot, \cdot \rangle_{\alpha,Y}$, Proposition 1 and properties of reproducing kernels, we obtain

$$\langle h_b(f), g \rangle_{\alpha,Y} = \int_{\mathbb{B}_n} \langle b(w)\overline{f(w)}, g(w) \rangle_{Y,Y^\star} \mathrm{d}\nu_\alpha(w).$$

Using an analogy of Hahn-Banch theorem and Parseval's equality, we have double integral $\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \left| \frac{g(z)\left(b(w)(\overline{f(w)})\right)}{(1-\langle z,w\rangle)^{n+1+\alpha}} \right| \mathrm{d}\nu_\alpha(w)\mathrm{d}\nu_\alpha(z)$ giving the finite strict inequality $\int_{\mathbb{B}_n} \|b(w)\|_{\mathcal{L}(\overline{X},Y)} \log\left(\frac{1}{1-|w|^2}\right) \mathrm{d}\nu_\alpha(w) < \infty$. This completes the proof. □

Lemma 1. *Let $\mathfrak{T}$ be an IST with ACCI and let $f \in \mathfrak{S}_X$ and $z \in \mathbb{B}_n$. For some $b \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ we have the map $g_z(w)$ in $\mathfrak{S}_X$ and the condition $h_b(f)(z)$ is true for any integer $k > 0$ and $C_k > 0$.*

Proof. Since $g_z \in \mathfrak{S}_X$, then by the reproducing kernel property and norm-attainability property we have

$$\begin{aligned} h_b(f)(z) &= c_k^{-1} \int_{\mathbb{B}_n} L_k \left(\int_{\mathbb{B}_n} \frac{b(w)(\overline{g_z(\zeta)})}{(1-\langle \zeta, w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w) \right) \mathrm{d}\nu_{\alpha+k}(\zeta) \\ &= c_k^{-1} \int_{\mathbb{B}_n} L_k \left(b(\zeta)(\overline{g_z(\zeta)}) \right) \mathrm{d}\nu_{\alpha+k}(\zeta). \end{aligned}$$

Clearly, the conditions for Tonelli's theorem are satisfied. In fact, by Proposition 2 we have that

$$\int_{\mathbb{B}_n} \left\| \int_{\mathbb{B}_n} \frac{b(w)(\overline{g_z(\zeta)})}{(1-\langle \zeta, w\rangle)^{n+1+\alpha+k}} \mathrm{d}\nu_\alpha(w) \right\|_Y \mathrm{d}\nu_{\alpha+k}(\zeta) < \infty.$$

This completes the proof of this lemma. □

At this juncture, we make some assumptions on symbol b by postulating that $\int_{\mathbb{B}_n} \|b(z)\|_{\mathcal{L}(\overline{X},Y)} \log \left(\frac{1}{1-|z|^2} \right) \mathrm{d}\nu_\alpha(z) < \infty$ holds. We use this postulate in the next theorem.

Theorem 1. *Let $\mathfrak{T}$ be an IST with ACCI and let $f \in \mathfrak{S}_X$ and $g \in \mathfrak{S}_{X^*}$. Let z_n be an orthonormal sequence of a complex Hilbert space H converging to z as $n \to \infty$ and consider $0 < p \le 1$. The class $(A^p_\alpha(\mathfrak{S}_X))^\star$ has equivalence relation with $\Gamma_\gamma(\mathfrak{S}^\star_X)$ having $\gamma = (n+1+\alpha)\left(\frac{1}{p}-1\right)$ and for any class D_k which is dense, reflexive we have the coupling $\langle f, g\rangle_{\alpha,X} = c_k \int_{\mathbb{B}_n} \langle f(z), D_k g(z)\rangle_{X,X^\star}(1-|z|^2)^k \mathrm{d}\nu_\alpha(z)$, Furthermore, $\|g\|_{\Gamma_\gamma(\mathfrak{S}_{X^\star})} \simeq \sup_{\|f\|_{A^p_\alpha(\mathfrak{S}_X)}=1} |\langle f, g\rangle_{\alpha,X}|$.*

Proof. Consider $g \in \Gamma_\gamma(\mathfrak{S}^\star_{Y^*})$ having $\gamma = (n+1+\alpha)\left(\frac{1}{p}-1\right)$. For $\alpha > 0$, let $\wedge_g : A^p_\alpha(\mathfrak{S}_X) \longrightarrow \mathbb{C}$, $f \mapsto \wedge_g(f) = c_k \int_{\mathfrak{S}_X} \langle f(z), D_k g(z)\rangle_{X,X^\star}(1-|z|^2)^k \mathrm{d}\nu_\alpha(z)$ be a be positive linear functional where $k > \gamma$, and $c_k > 0$ a constant. We show that $\wedge_g$ is well defined. To see this, let $f \in A^p_\alpha(\mathfrak{S}_X)$. By Proposition 1, we have $|\wedge_g(f)| \lesssim \|g\|_{\Gamma_\gamma(\mathfrak{S}^\star_X)} \|f\|_{p,\alpha,X}$. Hence, the boundedness property of $\wedge_g$ on $A^p_\alpha(\mathfrak{S}_X)$ is satisfied.

Conversely, consider $\wedge$ as above on $A^p_\alpha(\mathfrak{S}_X)$. It suffices show the existence of the function $g \in \Gamma_\gamma(\mathfrak{S}^\star_X)$, with $\gamma = (n+1+\alpha)\left(\frac{1}{p}-1\right)$ in which $\wedge = \wedge_g$. By straightforward computation, by Lemma 1 and a simplified manipulation we have $\wedge(f) = \frac{c_\alpha c_k}{c_{\alpha+k}}(1-|w|^2)^{k-\gamma}\langle x, D_k g(w)\rangle_{X,X^\star}$. Now, $f \in A^p_\alpha(\mathfrak{S}_X, X)$. By reflexivity of $\mathfrak{S}_X$, $\|D_k g(w)\|_{X^\star} \lesssim \frac{\|\wedge\|}{(1-|w|^2)^{k-\gamma}}$. Applying separability of the IST with ACCI and the density of $\mathfrak{S}_X$ the proof is complete. □

Theorem 2. *Let $\mathfrak{T}$ be an IST with ACCI and let $f \in \mathfrak{S}_X$ and $g \in \mathfrak{S}_{X^*}$. Let z_n be an orthonormal sequence of a complex Hilbert space H converging to z as $n \to \infty$. Then the norm-attainable operator $h_b : A^p_\alpha(\mathfrak{S}_X, X) \longrightarrow A^{1,\infty}_\alpha(\mathfrak{S}_X, Y)$ is continuous, linear and bounded.*

Proof. The proofs for continuity and linearity are trivial so we only need to prove boundedness. Using Lemma 2 and the reproducing kernel property gives $|\langle h_b(f), g\rangle_{\alpha,Y}| = \frac{(1-|w|^2)^{k-\gamma}}{c_k} \left| \left\langle L_k\left(b(w)\left(\overline{x}\right)\right), y^\star \right\rangle_{Y,Y^\star} \right|$.

Now, we use Lemma 1 to obtain $\|h_b f(z)\|_Y = \|b\|_{\Gamma_\gamma(\mathfrak{S}_X, \mathcal{L}(\overline{X},Y))} P^+_\alpha g(z)$, but by parallelogram law the property of the reproducing kernel holds and we have that $P^+_\alpha g(z) = \int_{\mathfrak{S}_X} \frac{(1-|w|^2)^\gamma \|f(w)\|_X}{|1-\langle z, w\rangle|^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w)$ is norm-attainable. We obtain the set $\nu_\alpha(\{z \in \mathfrak{S}_X : \|h_b f(z)\|_Y > \lambda\}) \le \nu_\alpha(\{z \in \mathfrak{S}_X : c_k \|b\|_{\Gamma_\gamma(\mathfrak{S}_X, \mathcal{L}(\overline{X},Y))} P^+_\alpha g(z) > \lambda\})$. By positivity of P^+_α, it is bounded and this completes the proof. □

The following consequence comes immediately;

Corollary 1. *Let $\mathfrak{T}$ be an IST with ACCI and let $f \in \mathfrak{S}_X$ and $g \in \mathfrak{S}_{X^*}$. Let z_n be an orthonormal sequence of a complex Hilbert space H converging to z as $n \to \infty$ and consider $0 < p \le 1$, and $\alpha > -1$. The norm-attainable operator h_b has an extension via $A^p_\alpha(\mathfrak{S}_X, X)$ into $A^q_\alpha(\mathfrak{S}_X, Y)$.*

Proof. This is direct analogously from the proof of Theorem 2 and the fact that the sets $NA(H)$ and $\mathfrak{S}_X$ are desnse and satisfy the duality condition for reflexive spaces. □

Theorem 3. *Let $\mathfrak{T}$ be an IST with ACCI and let $f \in \mathfrak{S}_X$ and $g \in \mathfrak{S}_{X^*}$. Let z_n be an orthonormal sequence of a complex Hilbert space H converging to z as $n \to \infty$. and consider $0 < p \leq 1$, $\alpha > -1$ and $\gamma = (n+1+\alpha)\left(\frac{1}{p}-1\right)$. Then the norm-attainable operator is extendable and we have $\|N^k b(w)\|_{\mathcal{L}(\overline{X},Y)} \leq \frac{C}{(1-|w|^2)^{k-\gamma}}\left(\log\frac{1}{1-|w|^2}\right)^{-1}$.*

Proof. Utilizing Corollary 1, using Lemma 1 for every $f \in H^\infty(\mathfrak{S}_X, X)$, and invoking the conditions of Lemma 1 and Proposition 1 we obtain $\|h_b f\|_{A^1_\alpha(\mathfrak{S}_X,Y)} \lesssim \|f\|_{p,\alpha,X}$. Conversely, let h_b be extendable via $A^p_\alpha(\mathfrak{S}_X, X)$ to $A^1_\alpha(\mathfrak{S}_X, Y)$. With some manipulations involving f and g we get $\langle h_b f, g\rangle_{\alpha,Y} = (1-|w|^2)^{k-\gamma}\log(1-|w|^2)\langle L_k(b(w)(\overline{x})), y^\star\rangle_{Y,Y^\star} + \langle\int_{\mathfrak{S}_X} b(z)(\overline{\varphi(z)})\mathrm{d}\nu_\alpha(z), y^\star\rangle_{Y,Y^\star}$. Now I_2 is estimated as $|I_2| \leq \|h_b\|\|\varphi\|_{p,\alpha,X}\|y^\star\|_{Y^\star} \lesssim \|h_b\|\|x\|_X\|y^\star\|_{Y^\star}$. But by the fact that $I_1 = \langle h_b f, g\rangle_{\alpha,Y} - I_2$, Proposition 1 and prior estimation on I_2 gives $|I_1| \leq |\langle h_b f, g\rangle_{\alpha,Y}| + |I_2| \lesssim \|h_b\|\|x\|_X\|y^\star\|_{Y^\star}$. But $x \in X$, $y^\star \in Y^\star$ are not fixed and so invoking Corollary 1 completes the proof. □

Next, we consider reflexivity and characterize symbols b for compact norm-attainable operators with regard to IST with ACCI. We state the following proposition.

Proposition 3. *Let $\mathfrak{T}$ be an IST with ACCI and let $f \in \mathfrak{S}_X$ and $g \in \mathfrak{S}_{X^*}$. Let z_n be an orthonormal sequence of a complex Hilbert space H converging to z as $n \to \infty$. For an integer $N > 0$ and α we have an integro-differential norm-attainable operator (IDNAO) of order N with polynomial coefficients defined by $R^{\alpha,N}f(z) = \sum_{m\in\mathbb{N}^n, |m|\leq N} p_m(z)\frac{\partial^{|m|}f}{\partial z^m}(z)$, for some polynomial p_m.*

Proof. Let $x \in X$ and $w \in \mathfrak{S}_X$. By integrability of multinomial formula for IDNAO we have $\langle z, w\rangle^k = \sum_{|m|=k}\frac{k!}{m!}z^m\overline{w}^m$, and a simple calculation follows immediately that $R^{\alpha,N} = \sum_{k=0}^{N}\sum_{|m|=k} c_{mk}z^m\frac{\partial^k}{\partial z^m}$. This completes the proof. □

Remark 2. Next, we characterize integro-differential norm-attainable operator (IDNAO) of order N with polynomial coefficients defined in the infinite dimensional case by $R^{\alpha,N}f(z) = \sum_{m\in\mathbb{N}^n, |m|\leq N} p_m(z)\frac{\partial^{|m|}f}{\partial z^m}(z)$, for some polynomial p_m in a general setting for spectraloid cones of $NA(H)$. Moreover, we consider monomiality for IDNAO. We characterize integro-differential norm-attainable operator of order N with existing and unique polynomial coefficients well defined by $R^{\alpha,N}f(z) = \sum_{m\in\mathbb{N}^n, |m|\leq N} p_m(z)\frac{\partial^{|m|}f}{\partial z^m}(z)$, for some polynomial p_m in a general setting for spctraloid cones of $NA(H)$. We take into consideration the duality and reflexivity of the Banach spaces here. We state the following proposition.

Proposition 4. *Let $\mathfrak{T}$ be an IST with ACCI and let $f \in \mathfrak{S}_X$ and $g \in \mathfrak{S}_{X^*}$. Let z_n be an orthonormal sequence of a complex Hilbert space H converging to z as $n \to \infty$. Let $1 < p < \infty$ and for all $x^\star \in X^\star$ and $z \in \mathfrak{S}_X$, suppose that $e_{z,x^\star}(w) = \frac{x^\star}{(1-\langle w,z\rangle)^{n+1+\alpha}}$, $w \in \mathfrak{S}_X$. Then $e_{z,x^\star} \in A^{p'}_\alpha(\mathfrak{S}_X, X^\star)$ and $e_{z,x^\star}$ generates a separable subspectraloid cube in $A^{p'}_\alpha(\mathfrak{S}_X, X^\star)$.*

Proof. Consider $\phi \in A^p_\alpha(\mathfrak{S}_X, X)$ with $\langle\phi, e_{z,x^\star}\rangle_{\alpha,X} = 0$, for every $z \in \mathfrak{S}_X$ and $x^\star \in X^\star$. Suppose that $f^\star \in A^{p'}_\alpha(\mathfrak{S}_X, X^\star)$. By Radon-Nikodym theorem, it is enough to show that $\langle\phi, f^\star\rangle_{\alpha,X} = 0$. By the reproducing kernel formula for IDNAO, we have

$$
\begin{aligned}
0 &= \langle\phi, e_{z,x^\star}\rangle_{\alpha,X} = \int_{\mathfrak{S}_X}\langle\phi(w), e_{z,x^\star}(w)\rangle_{X,X^\star}\mathrm{d}\nu_\alpha(w)\\
&= \int_{\mathfrak{S}_X}\langle\phi(w), \frac{x^\star}{(1-\langle w,z\rangle)^{n+1+\alpha}}\rangle_{X,X^\star}\mathrm{d}\nu_\alpha(w)\\
&= \int_{\mathfrak{S}_X}\langle\frac{\phi(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}}, x^\star\rangle_{X,X^\star}\mathrm{d}\nu_\alpha(w)\\
&= \langle\phi(z), x^\star\rangle_{X,X^\star}.
\end{aligned}
$$

Hence, for every $x^\star \in X^\star$, we get $\langle \phi(z), x^\star \rangle_{X,X^\star} = 0$. Clearly, $f^\star \in A_\alpha^{p'}(\mathfrak{S}_X, X^\star)$, and so $\langle \phi, f^\star \rangle_{\alpha,X} = \int_{\mathfrak{S}_X} \langle \phi(z), f^\star(z) \rangle_{X,X^\star} \mathrm{d}\nu_\alpha(z) = 0$. $\square$

Lemma 2. *Let $\mathfrak{T}$ be an IST with ACCI and let $f \in \mathfrak{S}_X$ and $g \in \mathfrak{S}_{X^*}$. Let z_n be an orthonormal sequence of a complex Hilbert space H converging to z as $n \to \infty$ and let $\beta_0 \in \mathbb{N}^n$ and $\{f_j\}$ converge strongly to 0. Then $\{x_j\}$ converges strongly to 0 in $A_\alpha^p(\mathfrak{S}_X, X)$.*

Proof. We have that $\{f_j\}$ is strongly bounded in X because $f_j \to 0$ strongly in X as $j \to \infty$. Now, for all $g \in A_\alpha^{p'}(\mathfrak{S}_X, X^\star)$, we get

$$\begin{aligned} \langle x_j, g \rangle_{\alpha,X} &= \int_{\mathfrak{S}_X} \langle x_j(z), g(z) \rangle_{X,X^\star} \mathrm{d}\nu_\alpha(z) \\ &= \int_{\mathfrak{S}_X} \langle z^{\beta_0} f_j, g(z) \rangle_{X,X^\star} \mathrm{d}\nu_\alpha(z) \\ &= \int_{\mathfrak{S}_X} z^{\beta_0} \langle f_j, g(z) \rangle_{X,X^\star} \mathrm{d}\nu_\alpha(z), \end{aligned}$$

for

$$\begin{aligned} \left| z^{\beta_0} \langle f_j, g(z) \rangle_{X,X^\star} \right| &\leq |z^{\beta_0} \langle f_j, g(z) \rangle_{X,X^\star}| \\ &\leq \|f_j\|_X \|g(z)\|_{X^\star} \\ &\leq C \|g(z)\|_{X^\star}, \end{aligned}$$

and

$$\int_{\mathfrak{S}_X} \|g(z)\|_{X^\star} \mathrm{d}\nu_\alpha(z) \leq \left(\int_{\mathfrak{S}_X} \|g(z)\|_{X^\star}^{p'} \mathrm{d}\nu_\alpha(z) \right)^{1/p'} < \infty.$$

Applying Fatou's lemma and Lebegue's Dominated convergence theorem gives

$$\limsup_{j \longrightarrow \infty} \langle x_j, g \rangle_{\alpha,X} = \int_{\mathfrak{S}_X} z^{\beta_0} \lim_{j \longrightarrow \infty} \langle f_j, g(z) \rangle_{X,X^\star} \mathrm{d}\nu_\alpha(z) = 0.$$

This completes the proof as required. $\square$

Proposition 5. *Let $\mathfrak{T}$ be an IST with ACCI and let $f \in \mathfrak{S}_X$ and $g \in \mathfrak{S}_{X^*}$. Let z_n be an orthonormal sequence of a complex Hilbert space H converging to z as $n \to \infty$ and let $1 < p \leq q < \infty$, $0 \leq r < 1$ and $\gamma \in \mathbb{N}^n$. If $a_\gamma \in \mathcal{K}(\overline{X}, Y)$, then the nor-attainable $h_{g_r^\gamma} : A_\alpha^p(\mathfrak{S}_X, X) \to A_\alpha^q(\mathfrak{S}_X, Y)$ is compact, in which $g_r^\gamma(z) = a_\gamma (rz)^\gamma$ for all $z \in \mathfrak{S}_X$.*

Proof. A simple manipulation and by Theorem 2 gives the desired result. $\square$

Theorem 4. *Let $\mathfrak{T}$ be an IST with ACCI and let $f \in \mathfrak{S}_X$ and $g \in \mathfrak{S}_{X^*}$. Let z_n be an orthonormal sequence of a complex Hilbert space H converging to z as $n \to \infty$ and $1 < p \leq q < \infty$. The norm-attainable operator $h_b : A_\alpha^p(\mathfrak{S}_X, X) \to A_\alpha^q(\mathfrak{S}_X, Y)$ is bounded if and only if $b \in \Lambda_{\gamma_0}(\mathfrak{S}_X, \mathcal{L}(\overline{X}, Y))$, in which $\gamma_0 = (n + 1 + \alpha)\left(\frac{1}{p} - \frac{1}{q}\right)$. Furthermore, $\|h_b\|_{A_\alpha^p(\mathfrak{S}_X,X) \to A_\alpha^q(\mathfrak{S}_X,Y)} \simeq \|b\|_{\Lambda_{\gamma_0}(\mathfrak{S}_X,\mathcal{L}(\overline{X},Y))}$.*

Proof. Since h_b is norm-attainable from $A_\alpha^p(\mathfrak{S}_X, X)$ to $A_\alpha^q(\mathfrak{S}_X, Y)$ and having the norm $\|h_b\| = \|h_b\|_{A_\alpha^p(\mathfrak{S}_X,X) \to A_\alpha^q(\mathfrak{S}_X,Y)}$. Let $z \in \mathfrak{S}_X$ then for the function $f(w) = \dfrac{x}{(1 - \langle w, z \rangle)^k}$, $w \in \mathfrak{S}_X$, we have that $h_b f(z) = R^{\alpha,k} b(z)(\overline{x})$. and a straightforward calculation gives $\|R^{\alpha,k} b(z)(\overline{x})\|_Y = \|h_b f(z)\|_Y$. Now for all $x \in X$ and $\|x\|_X = \|\overline{x}\|_{\overline{X}}$ we obtain $\|R^{\alpha,k} b(z)\|_{\mathcal{L}(\overline{X},Y)} \lesssim \dfrac{\|h_b\|}{(1 - |z|^2)^{k - \gamma_0}}$. Therefore, $\sup_{z \in \mathfrak{S}_X} (1 - |z|^2)^{k - \gamma_0} \|R^{\alpha,k} b(z)\|_{\mathcal{L}(\overline{X},Y)} \lesssim \|h_b\|$. This implies that the symbol $b \in \Lambda_{\gamma_0}(\mathfrak{S}_X, \mathcal{L}(\overline{X}, Y))$ and $\|b\|_{\Lambda_{\gamma_0}(\mathfrak{S}_X,\mathcal{L}(\overline{X},Y))} \lesssim \|h_b\|$. For the reverse inclusion suppose that $b \in \Lambda_{\gamma_0}(\mathfrak{S}_X, \mathcal{L}(\overline{X}, Y))$. Let $f \in A_\alpha^p(\mathfrak{S}_X, X)$, $g \in A_\alpha^{q'}(\mathfrak{S}_X, Y^\star)$ and $k > \gamma_0$. Then we obtain $b \in \Lambda_{\gamma_0}(\mathfrak{S}_X, \mathcal{L}(\overline{X}, Y)) \subset A_\alpha^{p'}(\mathfrak{S}_X, \mathcal{L}(\overline{X}, Y))$. The rest is clear from Proposition 1 and Theorem 3. $\square$

4. Conclusion

In this work, we have studied the class of norm-attainable operators on involutive stereotype tubes with an algebraic connected component of the identity. We give characterizations of spectraloid cones of norm-attainable operators on involutive stereotype tubes with algebraic connected components of identity in terms of eigenvalues and the corresponding eigenvectors in involutive stereotype tubes. In particular, we have proven reflexivity, boundedness, and compactness properties when the cones contain unit balls with involution for the tubes when they are of the stereotype category.

Acknowledgments: The first author is grateful for the Chebyshev grant (ICM No.: 15729326) to attend the International Congress of Mathematicians (ICM-2022) in Saint Petersburg, Russia.

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: "The authors declare no conflict of interest."

Data Availability: All data required for this research is included within this paper.

Funding Information: No funding is available for this research.

References

[1] Okelo, B. N. (2013). The norm attainability of some elementary operators. *Applied Mathematics E-Notes, 13*, 1–7.
[2] Okelo, B. (2019). α-supraposinormality of operators in dense norm-attainable classes. *Universal Journal of Mathematics and Applications*, *2*(1), 42-43.
[3] Okelo, N. B., Agure, J. O., & Oleche, P. O. (2013). Various notions of orthogonality in normed spaces. *Acta Mathematica Scientia*, *33*(5), 1387-1397.
[4] Okelo, N. B. (2020). On norm-attainable operators in Banach spaces. *Journal of Function Spaces, 2020*, Article ID: 8926214. https://doi.org/10.1155/2020/8926214.
[5] Okelo, B., & Oburu, J. (2022). On monopole bundle systems of complex hypermanifolds for composition operators. *arXiv preprint arXiv:2202.04440.*
[6] Coifman, R. R., Rochberg, R., & Weiss, G. (1976). Factorization theorems for Hardy spaces in several variables. *Annals of Mathematics*, *103*(3), 611-635.
[7] Diestel, J., & Faires, B. (1974). On vector measures. *Transactions of the American Mathematical Society, 198*, 253-271.
[8] Zhu, K. (2005). *Spaces of Holomorphic Functions in the Unit Ball* (Vol. 226, pp. x+-271). New York: Springer.
[9] Grafakos, L. (2008). *Classical Fourier Analysis* (Vol. 2). New York: Springer.