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One-sided law of the iterated logarithm for dyadic martingale using sub-gaussian estimates

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Abstract: The martingale analogue of Kolmogorov's law of the iterated logarithm was obtained by W. Stout using probabilistic approach. In this paper, we give a new proof of one side of the same law of the iterated logarithm for dyadic martingale using subgaussian type estimates and Borel-Cantelli Lemma.

Keywords: Dyadic martingales; Square function; Borel-Cantelli Lemma.

MSC: 60G46; 60F99.

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1. Introduction

Kolmogorov's law of the iterated logarithm (LIL) for the sequence of independent random variables is in the words of K. L. Chung, "a crowning achievement in classical probability theory". We first begin with Kolmogorov's celebrated law of the iterated logarithm.

Theorem 1 (Kolmogorov [1]). *Let $S_m = \sum_{k=1}^m X_k$ where $\{X_k\}$ is a sequence of real valued independent random variables. Let s_m be the variance of S_m. Suppose $s_m \to \infty$ and $|X_m|^2 \leq \frac{K_m s_m^2}{\log\log(e^e + s_m^2)}$ for some sequence of constants $K_m \to 0$. Then, almost surely,*

$$\limsup_{m\to\infty} \frac{S_m(\omega)}{\sqrt{2s_m \log\log s_m^2}} = 1.$$

This beautiful law of the iterated logarithm result of Kolmogorov was first proved by Khintchine [2] for Bernoulli random variables. Khintchine obtained this result while improvising the efforts of Hausdorff (1913), Hardy and Littlewood (1914) and Steinhaus (1922) to obtain the exact rate of convergence in Borel's Theorem on normal numbers. Over the years, people have obtained the analog of the Kolmogorov's result in various settings in analysis. Some of the existing settings are lacunary trigonometric series, martingales, harmonic functions, Bloch functions etc. Readers are referred to a survey article by Bingham [3] which has more than 400 references on the law of the iterated logarithm. Salem and Zygmund [4] obtained the analogue of Kolmogorov's LIL in the context of lacunary trigonometric series and their result is the first LIL in analysis. Moreover, Salem and Zygmund [4] also introduced a law of the iterated logarithm for the tail sums of lacunary trigonometric series, known as tail LIL. The tail LIL of lacunary series was then completed by Ghimire and Moore [5].

In 1970, Stout [6] obtained a martingale version of Kolmogorov's LIL where he used the probabilistic approach. In this paper, we prove one side of the same law of the iterated logarithm for dyadic martingales using a different approach. Precisely, we use the harmonic analysis approach and easily obtain the upper bound. In the proof, we make the use of a subgaussian type estimate and Borel-Cantelli lemma. Our main result is:

Theorem 2. *If $\{f_n\}_{n=0}^\infty$ is a dyadic martingale on $[0,1)$, then*

$$\limsup_{n\to\infty} \frac{|f_n(x)|}{S_n f(x)\sqrt{2\log\log S_n f(x)}} \leq 1$$

almost everywhere on the set where $\{f_n\}_{n=0}^\infty$ is unbounded.

2. Preliminaries

We first fix some notations, give some definitions and state some lemmas which will be used in the course of the proof.

Let $\mathcal{D}_n$ denote the family of dyadic subintervals of the unit interval $[0,1)$ of the form $\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right)$, where $n = 0, 1, 2\cdots$ and $j = 0, 1, \cdots 2^n - 1$.

Definition 1 (Dyadic martingale)**.** A dyadic martingale is a sequence of integrable functions, $\{f_n\}_{n=0}^{\infty}$ with $f_n : [0,1) \to \mathbb{R}$ such that,

(i) for every n, f_n is $\mathfrak{F}_n-$ measurable where $\mathfrak{F}_n$ is the $\sigma-$algebra generated by dyadic intervals of the form $[\frac{j}{2^n}, \frac{j+1}{2^n})$, $j \in \{0, 1, 2, \cdots 2^n - 1\}$;
(ii) and the following conditional expectation condition holds

$$\mathbb{E}(f_{n+1}|\mathfrak{F}_n) = f_n,$$

where $\mathbb{E}(f_{n+1}|\mathfrak{F}_n)(x) = \frac{1}{|Q_n|}\int_{Q_n} f_{n+1}(y)dy$, for $Q_n \in \mathcal{D}_n$ and $x \in Q_n$.

Definition 2. For a dyadic martingale, $\{f_n\}_{n=0}^{\infty}$, we define

(i) the increments: $d_k = f_k - f_{k-1}$. So $f_n(x) = \sum_{k=1}^{n} d_k(x) + f_0$.
(ii) the quadratic characteristics or square function: $S_n^2 f(x) = \sum_{k=1}^{n} d_k^2(x)$.
(iii) the limit function: $S^2 f(x) = \lim\limits_{n\to\infty} S_n^2 f(x) = \sum_{k=1}^{\infty} d_k^2(x)$.

Next, we define Hardy-Littlewood maximal function:

Definition 3 (Hardy-Littlewood maximal function)**.** Let $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$. Let

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy.$$

Then Mf is called the Hardy-Littlewood maximal function of f. Here $|B(x,r)|$ denotes the measure of the ball centered at x and of radius r.

Let m denote the Lebesgue measure on $\mathbb{R}$.

Lemma 1 (Borel-Cantelli [7])**.** *Let $\{E_k\}_{k=1}^{\infty}$ be a countable collections of measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost all $x \in \mathbb{R}$ belong to at most finitely many of the sets E_k's.*

Next, we obtain an estimate for the sequence of dyadic martingales. This estimate will be used in the proof of a lemma. The estimate is stated as a lemma below:

Lemma 2. *For a dyadic martingale $\{f_n\}_{n=0}^{\infty}$, with $f_0 = 0$*

$$\int_0^1 \exp\left(f_n(x) - \frac{1}{2}S_n^2 f(x)\right) dx \le 1.$$

This estimate was originally obtained by Chang *et al.*, [8] using the probabilistic approach. Recently, S. Ghimire also obtained the same estimate using the analytic approach. Please refer [9] for the detail.

Remark 1. Note that if we rescale the sequence $\{f_n\}$ by λ, then Lemma 2 gives,

$$\int_0^1 \exp\left(\lambda f_n(x) - \frac{1}{2}\lambda^2 S_n^2 f(x)\right) dx \le 1.$$

This shows that the above inequality is inhomogeneous type. We will make the use of this form in the proof the lemma that follows.

With the help of Lemma 2, we now obtain a subgaussian type estimate related to dyadic martingales. This estimate plays the central role in the proof of our main result. The proof of the estimate can be found in [9]. We also revisit the same proof here. The estimate is given as a lemma below;

Lemma 3. *For a dyadic martingale $\{f_n\}$ and $\lambda > 0$, we have*

$$\left|\left\{x \in [0,1) : \sup_{m\geq 1} |f_m(x)| > \lambda\right\}\right| \leq 6\exp\left(\frac{-\lambda^2}{2||Sf||_\infty^2}\right).$$

Proof. Fix n. Let $\lambda > 0$, $\gamma > 0$. Then for every $m \leq n$,

$$f_m(x) = \frac{1}{|Q_m|}\int_{Q_m} f_n(y)dy, \quad x \in Q_m, \quad |Q_m| = \frac{1}{2^m}.$$

Fix x. Then $\sup_{1\leq m\leq n} |f_m(x)| \leq M|f_n|(x)$, where Mf_n is the Hardy-Littlewood maximal function of f_n. Then using Jensen's inequality, we have

$$\begin{aligned}
\exp(\gamma|f_m(x)|) &= \exp\left(\gamma\left|\int_{Q_m} f_n(y)d\left(\frac{y}{|Q_m|}\right)\right|\right)\\
&\leq \frac{1}{|Q_m|}\int_{Q_m}\exp(\gamma|f_n(y)|)dy\\
&\leq M(e^{\gamma|f_m(x)|})(x).
\end{aligned}$$

Using the Hardy-Littlewood maximal estimate, we have

$$\begin{aligned}
\left|\left\{x \in [0,1) : \sup_{1\leq m\leq n} |f_m(x)| > \lambda\right\}\right| &= \left|\left\{x \in [0,1) : \sup_{1\leq m\leq n} e^{\gamma|f_m(x)|} > e^{\gamma\lambda}\right\}\right|\\
&\leq \left|\left\{x \in [0,1) : M(e^{\gamma|f_m|})(x) > e^{\gamma\lambda}\right\}\right|\\
&\leq \frac{3}{e^{\gamma\lambda}}\int_0^1 \exp(\gamma|f_n(y)|)dy\\
&\leq \frac{3}{e^{\gamma\lambda}}\exp\left(\frac{\gamma^2}{2}||S_nf||_\infty^2\right)\int_0^1\exp\left(\gamma|f_n(y)| - \frac{\gamma^2}{2}S_n^2f(y)\right)dy.
\end{aligned}$$

Using Lemma 2, we have

$$\begin{aligned}
&\int_0^1 \exp\left(\gamma|f_n(y)| - \frac{\gamma^2}{2}S_n^2f\right)dy\\
&= \int_{\{y:f_n(y)\geq 0\}}\exp\left(\gamma f_n(y) - \frac{\gamma^2}{2}S_n^2f(y)\right)dy + \int_{\{y:f_n(y)<0\}}\exp\left(-\gamma f_n(y) - \frac{\gamma^2}{2}S_n^2f(y)\right)dy\\
&\leq \int_0^1\exp\left(\gamma f_n(y) - \frac{\gamma^2}{2}S_n^2f(y)\right)dy + \int_0^1\exp\left(-\gamma f_n(y) - \frac{\gamma^2}{2}S_n^2f(y)\right)dy \leq 2.
\end{aligned}$$

So,

$$\left|\left\{x \in [0,1) : \sup_{1\leq m\leq n} |f_m(x)| > \lambda\right\}\right| \leq \frac{6}{e^{\gamma\lambda}}\exp\left(\frac{\gamma^2}{2}||S_nf||_\infty^2\right).$$

Choose $\gamma = \dfrac{\lambda}{||S_nf||_\infty^2}$. With this γ, the above inequality becomes,

$$\left|\left\{x \in [0,1) : \sup_{1\leq m\leq n} |f_m(x)| > \lambda\right\}\right| \leq 6\exp\left(\frac{-\lambda^2}{2||S_nf||_\infty^2}\right).$$

Note that for the dyadic martingale $\{f_n\}$,

$$S_n^2f(x) = \sum_{k=1}^n d_k^2(x) \longrightarrow S^2f(x) = \sum_{k=1}^\infty d_k^2(x).$$

Consequently,

$$\frac{-1}{2||S_nf||_\infty^2} \le \frac{-1}{2||Sf||_\infty^2}.$$

Recall the continuity property of Lebesgue measure, if $\{E_n\}$ is a sequence of sets with $E_n \subset E_{n+1}$ for all n and $E = \bigcup_{n=1}^{\infty} E_n$, then $|E| = \lim_{n\to\infty} |E_n|$. Using this we get,

$$\left| \left\{ x \in [0,1) : \sup_{m\ge1} |f_m(x)| > \lambda \right\} \right| \le 6 \exp\left(\frac{-\lambda^2}{2||Sf||_\infty^2} \right).$$

This completes the proof of the lemma. □

Burkholder and Gundy [10] obtained the asymptotic behavior of dyadic martingale. They showed that the sets $\{x : Sf(x) < \infty\}$ and $\{x : \lim f_n \quad \text{exists}\}$ are equal almost everywhere (a.e.) where a.e. equal means that the measure of the set where they are not equal is zero. From the result of Burkholder and Gundy, we see that the sequence of dyadic martingales $\{f_n\}$ behave asymptotically well on the set $\{x : Sf(x) < \infty\}$. How does the dyadic martingale behave on the set $\{x : Sf(x) = \infty\}$, which is the complement of the set $\{x : Sf(x) < \infty\}$? The behavior of dyadic martingales is quite pathological on the set $\{x : Sf(x) = \infty\}$. Precisely, it is unbounded a.e. on this set. Even though, one can obtain the rate of growth of $|f_n|$ on the set $\{x : Sf(x) = \infty\}$. The rate of growth of $|f_n|$ can be obtained by the martingale analogue of Kolmogorov's law of the iterated logarithm. Stout [6] obtained the law of the iterated logarithm for dyadic martingales. Here we obtain the same upper bound in the law of the iterated logarithm for dyadic martingales using the estimates obtained in Lemma 3 and Borel-Cantelli Lemma (Lemma 1).

3. Proof of main result

Proof of Theorem 2. Let $\theta > 1$ and $\delta > 0$. We note that for every $x \in [0,1)$, we have either $S_nf(x) > \theta^k$ for some n or $S_nf(x) \le \theta^k$, for every n, and thus, $Sf(x) \le \theta^k$. We define stopping time as;

$$\gamma_k(x) = \begin{cases} \min\left(n : S_{n+1}f(x) > \theta^k \right); \\ \infty, \quad if \quad Sf(x) \le \theta^k. \end{cases}$$

So by stopping time, γ_k is the smallest index such that $S_{\gamma_k+1}f(x) > \theta^k$. This means $S_{\gamma_k}f(x) \le \theta^k$. Define,

$$\tilde{f}_n(x) = f_{n\wedge\gamma_k}(x) = \begin{cases} f_1(x), f_2(x), \dots, f_{\gamma_k}(x), f_{\gamma_k}(x), \dots, & \text{for } \gamma_k \ne \infty, \\ f_1(x), f_2(x), f_3(x), \dots, & \text{if } \gamma_k = \infty. \end{cases}$$

We first show that $S\tilde{f} \le \theta^k$. So for $n < \gamma_k(x)$, we have $S\tilde{f}_n(x) = Sf_n(x) \le Sf_{\gamma_k}(x) \le \theta^k$. Again if $n \ge \gamma_k(x)$, then $S\tilde{f}_n(x) = Sf_{\gamma_k}(x) \le \theta^k$. Thus, $\forall n\ S\tilde{f}_n(x) \le \theta^k$. Then, $\lim_{n\to\infty} S\tilde{f}_n(x) \le \theta^k$. So we have $S\tilde{f} \le \theta^k$. Choose $\lambda = (1+\delta)\theta^k\sqrt{2\log\log\theta^k}$. Then using Lemma 3 for the dyadic martingale $\{\tilde{f}_n\}$ with the chosen λ, we get

$$\begin{aligned}\left| \left\{ x \in [0,1) : \sup_{n\ge1} |\tilde{f}_n(x)| > (1+\delta)\theta^k\sqrt{2\log\log\theta^k} \right\} \right| &\le 6\exp\left(\frac{-(1+\delta)^2\theta^{2k}2\log\log\theta^k}{2||Sf||_\infty^2} \right) \\ &\le 6\exp\left(\frac{-(1+\delta)^2\theta^{2k}2\log\log\theta^k}{2\theta^{2k}} \right) \\ &= \frac{6}{(k\log\theta)^{(1+\delta)^2}}.\end{aligned}$$

Summing over all k, we have

$$\sum_{k=1}^{\infty} \left| \left\{ x \in [0,1) : \sup_{n\ge1} |\tilde{f}_n(x)| > (1+\delta)\theta^k\sqrt{2\log\log\theta^k} \right\} \right| \le \frac{6}{(\log\theta)^{(1+\delta)^2}} \sum_{k=1}^{\infty} \frac{1}{k^{(1+\delta)^2}} < \infty.$$

Then by Borel-Cantelli Lemma 2, we have for a.e. x,

$$\sup_{n\geq 1} |\tilde{f}_n(x)| \leq (1+\delta)\theta^k \sqrt{2\log\log\theta^k}$$

for sufficiently large k, say, $k \geq M$, M depends on x. Thus for a.e. x, we have,

$$\sup_{n\geq 1} |f_{n\wedge\gamma_k}(x)(x)| \leq (1+\delta)\theta^k \sqrt{2\log\log\theta^k}$$

for sufficiently large $k \geq M$. We choose x such that $f_n(x)$ is unbounded. Then from [10] we have,

$$\{x : Sf(x) < \infty\} \overset{a.e.}{=} \{x : f_n(x) \quad \text{converges}\}.$$

So we have $Sf(x) = \infty$. Then $\gamma_1(x) \leq \gamma_2(x) \leq \gamma_3(x) \leq \dots$ i.e. for every i, $\gamma_i(x) < \infty$.

Let $n \geq \gamma_M$. Then choose k such that $\gamma_k(x) < n \leq \gamma_{k+1}(x)$. Here, $\gamma_k(x) < n$ gives $\gamma_k(x) \leq n-1$. Thus, $S_n f(x) = S_{n-1+1} f(x) > \theta^k$. Using this, we have

$$\begin{aligned}
|f_n(x)| &\leq \sup_{1\leq m\leq\gamma_{k+1}} |f_{m\wedge\gamma_{k+1}}(x)| \\
&\leq \sup_{m\geq 1} |f_{m\wedge\gamma_{k+1}}(x)| \\
&\leq (1+\delta)\theta^{k+1}\sqrt{2\log\log\theta^{k+1}} \\
&< (1+\delta)S_n f(x)\theta\sqrt{2\log(\log S_n f(x) + \log\theta)}.
\end{aligned}$$

So,

$$\limsup_{n\to\infty} \frac{|f_n(x)|}{S_n f(x)\sqrt{2\log(\log S_n f(x))}} < (1+\delta)\theta \limsup_{n\to\infty} \sqrt{\frac{2\log(\log S_n f(x) + \log\theta)}{2\log(\log S_n f(x))}}.$$

Clearly,

$$\limsup_{n\to\infty} \sqrt{\frac{\log(\log S_n f(x) + \log\theta)}{\log(\log S_n f(x))}} = 1.$$

Therefore for a.e. x,

$$\limsup_{n\to\infty} \frac{|f_n(x)|}{S_n f(x)\sqrt{2\log\log S_n f(x)}} < (1+\delta)\theta.$$

Letting $\theta \searrow 1$ we get,

$$\limsup_{n\to\infty} \frac{|f_n(x)|}{S_n f(x)\sqrt{2\log\log S_n f(x)}} \leq 1+\delta.$$

This can be done for every $\delta > 0$. Hence we have for a.e. x,

$$\limsup_{n\to\infty} \frac{|f_n(x)|}{S_n f(x)\sqrt{2\log\log S_n f(x)}} \leq 1.$$

□

4. Conclusion

The upper bound of the law of the iterated logarithm in the context of dyadic martingale using analytic approach has been obtained where we made the use of some subgaussian type estimates and Borel-Cantelli Lemma. We look forward to obtain the lower bound result using the similar approach.

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