Article

Stability result for a class of weakly dissipative second-order systems with infinite memory

Adel M. Al-Mahdi1,∗ and Mohammad M. Al-Gharabli2

1 The Preparatory Year Program, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia.
2 The Interdisciplinary Research Center in Construction and Building Materials, King Fahd University of Petroleum & Minerals, Dhahran 31261, Saudi Arabia.
* Correspondence: almahdi@kfupm.edu.sa

Academic Editor: Wulong Liu
Received: 1 June 2021; Accepted: 5 May 2022; Published: 22 June 2022.

Abstract: In this paper we consider the following abstract class of weakly dissipative second-order systems with infinite memory,
\[ u''(t) + Au(t) - \int_0^\infty g(s)A^\alpha u(t-s)ds = 0, \quad t > 0, \]
and establish a general stability result with a very general assumption on the behavior of \( g \) at infinity; that is \( g'(t) \leq -\xi(t)G(g(t)), \quad t \geq 0. \)
where \( \xi \) and \( G \) are two functions satisfying some specific conditions. Our result generalizes and improves many earlier results in the literature. Moreover, we obtain our result with imposing a weaker restrictive assumption on the boundedness of initial data used in many earlier papers in the literature such as the one in [1–5]. The proof is based on the energy method together with convexity arguments.

Keywords: Stability; Convex functions; Infinite memory; Viscoelasticity; Weakly dissipative system.

MSC: 35B40; 35L90; 34G10; 45K05; 93D20.

1. Introduction

Viscoelastic materials exhibit an instantaneous elasticity effect and creep characteristics at the same time. The importance of the viscoelastic properties of materials has been realized because of the rapid developments in rubber and plastics industry. The modeling of the dynamics of physical phenomena such as heat flow in conductors with memory, hereditary polarization in dielectrics, population dynamics, viscoelasticity can be described by an abstract integro-differential equation of the form

\[
\begin{cases}
  u''(t) + Au(t) - \int_0^t g(t-s)A^\alpha u(s)ds = 0, & t > 0, \\
  u(-t) = u_0(t), & t \geq 0, \quad u'(0) = u_1,
\end{cases}
\]  

(1)

where \( ' \) represents a derivative with respect to time \( t \), \( A : D(A) \subset H \rightarrow H \) is a positive definite self-adjoint operator on \( H \), \( g \) is the relaxation function (convolution kernel), \( \alpha \in [0, 1] \), \( u_0, u_1 \) are given history function and initial data respectively.

The study of viscoelastic problems has attracted the attention of many authors and several decay and blow up results have been established. We start with the pioneer work [6,7] where Dafermos considered a one-dimensional viscoelastic problem and established various existence results and then proved, for smooth monotone decreasing relaxation functions, that the solutions go to zero as \( t \) goes to infinity. After that, many results dealing with the existence, uniqueness, regularity and asymptotic behavior of many systems of the form (1) have been studied; see, for example, [1,8–11]. In the case of finite memory, that is, \( u_0(t) = 0 \) for \( t < 0 \), see [12–18]. In particular, Rivera et al., [15] considered the interpolating cases \( \alpha \in (0, 1) \) and a relaxation function \( g \) which decays exponentially to zero at infinity, that is,

\[-c_0g(s) \leq g'(s) \leq -c_1g(s) \quad \forall s \in \mathbb{R}_+.\]  

(2)
They showed that the energy decays polynomially at the rate of $\frac{1}{t}$. Recently, Hassan and Messaoudi [19] considered
\[
\begin{align*}
\begin{cases}
u''(t) + Au(t) - \int_0^t g(t-s)A^\alpha u(s)ds = 0, & t > 0, \\
u(0) = u_0(t), & u'(0) = u_1,
\end{cases}
\end{align*}
\]
and established a new general decay rate result for which the relaxation function $g$ satisfies condition
\[
g'(t) \leq -\xi(t)G(g(t)), & t \geq 0.
\]
For case of infinite memory, see [20–25]. In particular, Guesmia [1] considered
\[
u_{tt} + Au - \int_0^{+\infty} g(s)Bu(t-s)ds = 0 & \text{ for } t > 0,
\]
and introduced a new ingenuous approach for proving a more general decay result based on the properties of convex functions and the use of the generalized Young inequality. He used a larger class of infinite history kernels satisfies the following condition
\[
\int_0^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} ds + \sup_{s \in \mathbb{R}_+} \frac{g(s)}{G^{-1}(-g'(s))} < +\infty,
\]
such that
\[
G(0) = G'(0) = 0 & \text{ and } \lim_{t \to +\infty} G'(t) = +\infty,
\]
where $G : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing strictly convex function. Al-Mahdi and Al-Gharabli [2] considered the following viscoelastic problem
\[
\begin{align*}
\begin{cases}
u_{tt} - \Delta u + \int_0^{+\infty} g(s)\Delta u(t-s)ds + |\nu_t|^{m-2} u_t = 0, & \text{ in } \Omega \times (0,+\infty) \\
u(x,t) = 0, & \text{ on } \partial \Omega \times (0,+\infty) \\
u(x,-t) = u_0(x,t), & \nu_1(x,0) = u_1(x), & \text{ in } \Omega \times (0,+\infty),
\end{cases}
\end{align*}
\]
and they established decay results with using a relaxation function $g$, satisfying the condition
\[
g'(t) \leq -\xi(t)g^p(t), & 1 \leq p < \frac{3}{2}.
\]
Very recently, Guesmia [26] considered two models of wave equations with infinite memory and established an explicit and general decay rate results where the relaxation function satisfying the condition (4).

Motivated by the above works, we intend to study the following class of viscoelastic equations of the form
\[
\begin{align*}
\begin{cases}
u''(t) + Au(t) - \int_0^{+\infty} g(s)A^\alpha u(t-s)ds = 0, & t > 0, \\
u(-t) = u_0, & u'(0) = u_1,
\end{cases}
\end{align*}
\]
where $A : D(A) \subset H \to H$ is a positive definite self-adjoint operator on $H$ such that the embedding $D(A^{\beta}) \hookrightarrow D(A^\sigma)$ is compact for any $\beta > \sigma \geq 0$ and $\sigma \in (0, 1).

**Remark 1.** The assumption $D(A^{\beta}) \leftrightarrow D(A^\sigma)$ for any $\beta > \sigma \geq 0$ guarantees the existence of some constants $\omega$, $\omega_0$, $\omega_1$ such that
\[
\|v\|^2 \leq \omega \left\| A^{1/2}v \right\|^2 & \forall v \in D \left( A^{1/2} \right),
\]
\[
\left\| A^{\alpha/2}v \right\|^2 \leq \omega_0 \left\| A^{1/2}v \right\|^2 & \forall v \in D \left( A^{1/2} \right),
\]
and
\[
\left\| A^{1/2}v \right\|^2 \leq 1 \left\| A^{1-\alpha/2}v \right\|^2 & \forall v \in D \left( A^{1-\alpha/2} \right).
\]
2. Our main objectives

We intend to establish a two fold objective:

1. improve many earlier works such as the ones in [11,15,19] from finite memory to infinite memory;
2. prove a general decay estimate for the solution of Problem (10) with a wider class of relaxation functions than the ones considered in [1–5] by getting a better decay rate with imposing a weaker assumption on the boundedness of initial data than the one considered in the earlier papers such as the one in [1–5].

The paper is organized as follows: We present some assumptions and remarks in §3. We state and prove some technical lemmas in §4. The main result, its proof and some examples are presented in §5.

3. Assumptions

In this section, we state some assumptions needed in the proof of our main decay result. The strictly decreasing differentiable relaxation (kernel) function \( g : [0, \infty) \to (0, \infty) \) satisfies the following assumptions:

(A.1) \( g(0) > 0 \) and \( 1 - 0 \int_0^{+\infty} g(s) ds = l > 0 \).

(A.2) There exists a non-increasing differentiable function \( \xi : \mathbb{R}_+ \to (0, \infty) \) and a \( C^1 \) function \( G : [0, +\infty) \to [0, +\infty] \) which is linear or it is strictly increasing and strictly convex \( C^2 \) function on \((0, r]\), with \( G(0) = G'(0) = 0 \), such that

\[
g'(t) \leq -\xi(t) G(g(t)), \quad \forall t \geq 0,
\]

(14)

where \( \xi \) is satisfying \( \int_0^{+\infty} \xi(s) ds = +\infty \).

(A.3) We assume that

\[
\int_0^{+\infty} g(s)||A^{s/2}u_0(s)||^2 ds < +\infty,
\]

and

\[
\int_0^{+\infty} g(s)||A^{1/2}u_0(s)||^2 ds < +\infty.
\]

Remark 2. The class of relaxation functions satisfying (A.1) – (A.2) in the present paper is larger than the ones satisfying (6) and (7) used in some earlier papers such as the one in [1]. In fact, the boundedness of the sup in (6) use in [1], can be interpreted as the inequality in (A.2) in the present paper (with \( \xi = 1 \)). The conditions (6) and (7) used in [1] ask also the boundedness of the integral. So, it is better to consider the relaxation functions satisfy (A.1) – (A.2) used in the present paper than the one used in [1].

Remark 3. Hypothesis (A.3) is needed for proving the existence and stability results. For the stability, if (A.3) holds, then the functions \( l_0 \) and \( l_1 \) defined in Lemma 4 well be defined. Moreover, Hypothesis (A.3) is weaker than the one used in [1–5] that is, there exists a positive constant \( M \) such that

\[
||\nabla A^{s/2}u_0(s)||^2 \leq M,
\]

and

\[
||\nabla A^{1/2}u_0(s)||^2 \leq M.
\]

Remark 4. As is in Mustafa [14], if \( G \) is a strictly increasing and strictly convex \( C^2 \) function on \((0, r]\), with \( G(0) = G'(0) = 0 \), then there is a strictly convex and strictly increasing \( C^2 \) function \( \overline{G} : [0, +\infty) \to [0, +\infty) \) which is an extension of \( G \). For instance, we can define \( \overline{G} \), for any \( t > r \), by

\[
\overline{G}(t) := \frac{G''(r)}{2} t^2 + (G'(r) - G''(r)r)t + \left( G(r) + \frac{G''(r)}{2} r^2 - G'(r)r \right).
\]

We state the existence, regularity and uniqueness theorem whose proof is in [15].

Theorem 1 ([15]). Suppose that \((u_0(\cdot, 0), u_1) \in \mathcal{D}(A) \times \mathcal{D}(A^{1/2})\) and (A.1-A.3) hold. Then, Problem (10) has a unique global solution satisfying

\[
u \in C(\mathbb{R}_+; \mathcal{D}(A)) \cap C^1(\mathbb{R}_+; \mathcal{D}(A^{1/2})) \cap C^2(\mathbb{R}_+; H).
\]
Moreover, if \((u_0(\cdot, 0), u_1) \in D \left( A^{\sigma + 1/2} \right) \times D \left( A^\sigma \right) \) for \(\sigma \geq 0\), then the solution satisfies

\[ u \in C \left( \mathbb{R}^+; D \left( A^{\sigma + 1/2} \right) \right) \cap C^1 \left( \mathbb{R}^+; D \left( A^\sigma \right) \right) \cap C^2 \left( \mathbb{R}^+; D \left( A^{\sigma - 1/2} \right) \right). \]

The “modified” energy functionals associated to our problem are given by

\[
E(t) := \frac{1}{2} \left[ \left\| u'(t) \right\|^2 + \left\| A^{1/2} u(t) \right\|^2 - \left( \frac{1 - l}{\omega_0} \right) \left\| A^{2/2} u(t) \right\|^2 + \left( g \circ A^{2/2} u \right)(t) \right],
\]

\[
E(t) := \frac{1}{2} \left[ \left\| A^{(1-\alpha)/2} u'(t) \right\|^2 + \left\| A^{1-\alpha/2} u(t) \right\|^2 - \left( \frac{1 - l}{\omega_0} \right) \left\| A^{1/2} u(t) \right\|^2 + \left( g \circ A^{1/2} u \right)(t) \right],
\]

for any \(t \geq 0\), where for \(v \in L^2_{\text{loc}}(\mathbb{R}^+; H)\),

\[(g \circ v)(t) := \int_0^\infty g(s) \left\| v(t) - v(t - s) \right\|^2 ds.

**Remark 5.** The positiveness of the energy functionals comes from inequalities (12) and (13).

**Lemma 1** ([15]). For any initial data \((u_0, u_1) \in D(A) \times D \left( A^{1/2} \right)\), the energy functionals associated to Problem (10) satisfy, for any \(t \geq 0\), the identities

\[
E'(t) = \frac{1}{2} \left( g' \circ A^{1/2} u \right)(t) \leq 0,
\]

\[
E'(t) = \frac{1}{2} \left( g' \circ A^{1/2} u \right)(t) \leq 0.
\]

As in Jin et al., [27], we set, for any \(0 < \varepsilon < 1\),

\[C_\varepsilon := \int_0^\infty \frac{g^2(s)}{\varepsilon g(s) - g(s)} ds \quad \text{and} \quad h_\varepsilon(t) := \varepsilon g(t) - g'(t).

**Lemma 2** ([27]). Assume that the condition (A.1) holds. Then, for any \(v \in L^2_{\text{loc}}([0, +\infty); L^2(0, L))\), we have

\[
\int_0^L \left( \int_0^\infty g(s)(v(t) - v(t - s)) ds \right)^2 dx \leq C_\varepsilon(h_\varepsilon \circ v)(t), \quad \forall t \geq 0.
\]

**Lemma 3** (Jensen’s inequality). Let \(F : [a, b] \rightarrow \mathbb{R}\) be a convex function. Assume that the functions \(f : \Omega \rightarrow [a, b]\) and \(h : \Omega \rightarrow \mathbb{R}\) are integrable such that \(h(x) \geq 0\), for any \(x \in \Omega\) and \(\int \Omega h(x) dx = k > 0\). Then,

\[
F \left( \frac{1}{k} \int \Omega f(x)h(x) dx \right) \leq \frac{1}{k} \int \Omega F(f(x))h(x) dx.
\]

4. Technical lemmas

In this section, we state and prove some Lemmas that are useful in the proof of Theorem 2. Throughout this work we use \(c > 1\) to represent a generic constant, which is independent of \(t\) and the initial data.

**Lemma 4.** Assume that (A.1-A.3) hold. Then, there exist two positive constants \(M_0, M_1\) such that

\[
\int_1^{+\infty} g(s) \left\| A^{1/2} u(t) - A^{1/2} u(t - s) \right\|^2 ds \leq M_0 h_0(t),
\]

and

\[
\int_1^{+\infty} g(s) \left\| A^{1/2} u(t) - A^{1/2} u(t - s) \right\|^2 ds \leq M_1 h_1(t),
\]

where \(h_0(t) = \int_0^{+\infty} g(t + s) \left( 1 + \left\| A^{1/2} u_0(s) \right\|^2 \right) ds\), and \(h_1(t) = \int_0^{+\infty} g(t + s) \left( 1 + \left\| A^{1/2} u_0(s) \right\|^2 \right) ds\).
Proof. Indeed, we have
\[
\int_{t}^{t+\infty} g(s) \|A^{\alpha/2}u(t) - A^{1/2}u(t-s)\|^2 ds \\
\leq 2\|A^{\alpha/2}u(t)\|^2 \int_{t}^{t+\infty} g(s) ds + 2 \int_{t}^{t+\infty} g(s) \|A^{\alpha/2}u(t-s)\|^2 ds \\
\leq 2 \sup_{s \geq 0} \|A^{\alpha/2}u(s)\|^2 \int_{0}^{t} g(t+s) ds + 2 \int_{0}^{t} g(t+s) \|A^{\alpha/2}u(-s)\|^2 ds \\
\leq \frac{4\omega_0 \sup_{s \geq 0} E(s) E(t)}{1-I} \int_{0}^{t} g(t+s) ds + 2 \int_{0}^{t} g(t+s) \|A^{\alpha/2}u_0(s)\|^2 ds \\
\leq \frac{4\omega_0 E(0) E(t)}{1-I} \int_{0}^{t} g(t+s) ds + 2 \int_{0}^{t} g(t+s) \|A^{\alpha/2}u_0(s)\|^2 ds \\
\leq M_0 \int_{0}^{t} g(t+s) \left( 1 + \|A^{\alpha/2}u_0(s)\|^2 \right) ds.
\]
where \(M_0 = \max \{2, \frac{4\omega_0 E(0)}{1-I} \}\). □

The proof of (21) can be established similarly to the proof of (20).

**Lemma 5.** Assume that conditions (A.1-A.3) hold. Then, for any \(0 < \delta < 1\), the functional \(l_1\) defined by
\[
l_1(t) := - \left( u'(t), \int_{0}^{t} g(s)(u(t) - u(t-s)) ds \right)
\]
satisfies, along the solution of (10), the estimate
\[
l'_1(t) \leq - \left( \frac{1}{\omega_0} - \delta \right) \|u'(t)\|^2 + \delta \|A^{1/2}u(t)\|^2 + \frac{C_\delta}{\delta} (C_\epsilon + 1) \left( h_\epsilon \circ A^{1/2}u \right)(t), \quad \forall t \geq 0.
\]

**Proof.** Differentiating \(l_1\) and exploiting the differential equation in Problem (10), we get
\[
l'_1(t) = \left( A^{1/2}u(t), \int_{0}^{t} g(s)A^{1/2}(u(t) - u(t-s)) ds \right) \\
- \left( \int_{0}^{t} g(s)A^{\alpha/2}u(t-s) ds, \int_{0}^{t} g(s)A^{\alpha/2}(u(t) - u(t-s)) ds \right) \\
- \left( \frac{1}{\omega_0} \|u'(t)\|^2 - \left( u'(t), \int_{0}^{t} g'(s)(u(t) - u(t-s)) ds \right) \right).
\]

Next, we estimate the terms in the right-hand side of the above identity. Using the Cauchy-Schwarz, Young and Hölder inequalities, Lemma 2 and inequalities (11) and (12), it follows that, for any \(0 < \delta < 1\),
\[
\left( A^{1/2}u(t), \int_{0}^{t} g(s)A^{1/2}(u(t) - u(t-s)) ds \right) \\
\leq \frac{\delta}{2} \|A^{1/2}u(t)\|^2 + \frac{C_\delta}{\delta} h_\epsilon \circ A^{1/2}u \left( h_\epsilon \circ A^{1/2}u \right)(t) \\
- \left( \int_{0}^{t} g(s)A^{\alpha/2}u(t-s) ds, \int_{0}^{t} g(s)A^{\alpha/2}(u(t) - u(t-s)) ds \right) \\
= \left( \int_{0}^{t} g(s)A^{\alpha/2}(u(t) - u(t-s)) ds \right)^2 - \left( \frac{1}{\omega_0} \right) A^{\alpha/2}u(t) \int_{0}^{t} g(s)A^{\alpha/2}(u(t) - u(t-s)) ds \\
\leq \frac{\delta}{2} \left( A^{\alpha/2}u(t) \right)^2 + \frac{C_\delta}{\delta} h_\epsilon \circ A^{\alpha/2}u \left( h_\epsilon \circ A^{1/2}u \right)(t) \\
\leq \frac{\delta}{2} \left( A^{1/2}u(t) \right)^2 + \frac{C_\delta}{\delta} h_\epsilon \circ A^{1/2}u \left( h_\epsilon \circ A^{1/2}u \right)(t),
\]
and
\[
\left( u'(t), \int_{0}^{t} g'(s)(u(t) - u(t-s)) ds \right) \\
= \left( u'(t), \epsilon \int_{0}^{t} g(s)(u(t) - u(t-s)) ds \right) - \left( u'(t), \int_{0}^{t} h_\epsilon(s)(u(t) - u(t-s)) ds \right)
\]
Proof. For any \( t \) \( \geq 0 \), and \( p \) such that \( p(t) \leq p(0) = \frac{1-l}{o} \), we obtain, for any \( t \geq 0 \),

\[
\begin{align*}
& \leq \delta \|u'(t)\|^2 + \frac{2}{2l} \left( \int_0^\infty g(s) (u(t) - u(t-s)) ds \right)^2 + \delta \|u'(t)\|^2 + \frac{1}{2l} \left( \int_0^\infty h_i(s) (u(t) - u(t-s)) ds \right)^2 \\
& \leq \delta \|u'(t)\|^2 + \frac{\zeta}{\delta} (C_\varepsilon + 1) \left( h_\varepsilon \circ A^{1/2} u \right) (t).
\end{align*}
\]

Plugging the above estimates in (24), we obtain the desired estimate. \( \square \)

**Lemma 6.** Under the conditions (A.1-A.3), the functional \( l_2 \) defined by

\[ l_2(t) := \langle u'(t), u(t) \rangle \]

satisfies, along the solution of (10), the estimate

\[ l_2'(t) \leq \|u'(t)\|^2 - \frac{1}{2} \|A^{1/2} u(t)\|^2 + cC_\varepsilon \left( h_\varepsilon \circ A^{1/2} u \right) (t), \quad \forall \ t \geq 0. \tag{25} \]

**Proof.** Differentiating \( l_2 \), using the equation in (10), and repeating the above computations, we get

\[
l_2'(t) = \|u'(t)\|^2 - \|A^{1/2} u(t)\|^2 + \left( 1 - \frac{1}{\omega_0} \right) \|A^{\alpha/2}\|^2 + \left( \int_0^\infty g(s) A^{\alpha/2} (u(s) - u(t-s)) ds, A^{1/2} u(t) \right) \]

\[
\leq \|u'(t)\|^2 - l \|A^{1/2} u(t)\|^2 + \frac{1}{2\omega_0} \|A^{\alpha/2} u(t)\|^2 + \frac{\omega_0}{2l} \left( \int_0^\infty g(s) A^{\alpha/2} (u(t) - u(t-s)) ds \right)^2 \\
\leq \|u'(t)\|^2 - \frac{1}{2} \|A^{1/2} u(t)\|^2 + cC_\varepsilon \left( h_\varepsilon \circ A^{1/2} u \right) (t), \quad \forall \ t \geq 0. \]

\( \square \)

**Lemma 7.** Assume that (A.1-A.3) hold. Then, the functionals \( l_1 \) and \( l_2 \) defined by

\[ l_1(t) := \int_0^t p(t-s) \|A^{\alpha/2} u(s)\|^2 ds \]

and

\[ l_2(t) := \int_0^t p(t-s) \|A^{1/2} u(s)\|^2 ds \]

with \( p(t) := \int_t^\infty g(s)ds \) satisfy, along the solution of (10), the estimates

\[ l_1'(t) \leq 3(1-l) \|A^{1/2} u(t)\|^2 - \frac{1}{2} (g \circ A^{\alpha/2} u)(t) + \frac{1}{2} \int_t^\infty g(s) \|A^{\alpha/2} u(t) - A^{1/2} u(s)\|^2 ds, \]

and

\[ l_2'(t) \leq \frac{3}{2} (1-l) \|A^{1/2} u(t)\|^2 - \frac{1}{2} (g \circ A^{\alpha/2} u)(t) + \frac{1}{2} \int_t^\infty g(s) \|A^{1/2} u(t) - A^{1/2} u(s)\|^2 ds, \]

for any \( t \geq 0 \).

**Proof.** Exploiting Young’s inequality, (A.1-A.3), inequality (12) and the fact that \( p(t) \leq p(0) = \frac{1-l}{o} \), we obtain, for any \( t \geq 0 \),

\[
l_1'(t) = p(0) \|A^{\alpha/2} u(t)\|^2 - \int_0^t g(t-s) \|A^{\alpha/2} u(s)\|^2 ds \\
= p(t) \|A^{\alpha/2} u(t)\|^2 - \int_0^t g(t-s) \|A^{\alpha/2} (u(s) - u(t))\|^2 ds \\
- 2 \int_0^t g(t-s) \left( A^{\alpha/2} u(t), A^{1/2} (u(s) - u(t)) \right) ds \\
\leq p(0) \|A^{\alpha/2} u(t)\|^2 - \int_0^t g(t-s) \|A^{\alpha/2} u(t) - A^{\alpha/2} u(t-s)\|^2 ds \\
+ \frac{2}{2} (1-l) \|A^{\alpha/2} u(t)\|^2 + \frac{1}{2} \int_0^t g(t-s) \|A^{\alpha/2} u(t) - A^{\alpha/2} u(t-s)\|^2 ds
\]
Similarly, differentiating \( J_2 \) and repeating the above computations, we get, for any \( t \geq 0 \),

\[
J_2(t) = p(0) \left\| A^{1/2}u(t) \right\|^2 - \frac{1}{2} \int_0^t g(t-s) \left\| A^{1/2}u(s) \right\|^2 ds \\
= p(t) \left\| A^{1/2}u(t) \right\|^2 - \frac{1}{2} \int_0^t g(t-s) \left\| A^{1/2}(u(s) - u(t)) \right\|^2 ds \\
- 2 \int_0^t g(t-s) \left\langle A^{1/2}u(t), A^{1/2}(u(s) - u(t)) \right\rangle ds \\
\leq p(0) \left\| A^{1/2}u(t) \right\|^2 - \frac{1}{2} \int_0^t g(t-s) \left\| A^{1/2}(u(t) - A^{1/2}u(t-s)) \right\|^2 ds \\
+ \frac{2}{0} (1-l) \left\| A^{1/2}u(t) \right\|^2 + \frac{1}{2} \int_0^t g(t-s) \left\| A^{1/2}(u(t) - A^{1/2}u(t-s)) \right\|^2 ds \\
= 3 \left\| A^{1/2}u(t) \right\|^2 - \frac{1}{2} \int_0^t g(t-s) \left\| A^{1/2}(u(t) - A^{1/2}u(t-s)) \right\|^2 ds \\
= 3 \left\| A^{1/2}u(t) \right\|^2 - \frac{1}{2} \int_0^t g(t-s) \left\| A^{1/2}(u(t) - A^{1/2}u(t-s)) \right\|^2 ds \\
+ \frac{1}{2} \int_0^t g(t-s) \left\| A^{1/2}(u(t) - A^{1/2}u(t-s)) \right\|^2 ds. \tag{27}
\]

\[\square\]

**Lemma 8.** Assume (A.1 - A.3) hold. Then, the functional \( \mathcal{L} \) defined by

\[
\mathcal{L}(t) := N(E(t) + \mathcal{E}(t)) + \epsilon_1 I_1(t) + \epsilon_2 I_2(t)
\]

satisfies, for a suitable choice of \( N, \epsilon_1, \epsilon_2 > 0 \),

\[
\mathcal{L} \sim E + \mathcal{E}, \tag{28}
\]

and the estimate

\[
\mathcal{L}'(t) \leq -\frac{2}{7} (1-l) \left( 4 + \frac{3}{\omega_0} \right) \left\| u'(t) \right\|^2 - (1-l) \left( 4 + \frac{3}{\omega_0} \right) \left\| A^{1/2}u(t) \right\|^2 \\
+ \frac{1}{4} \left( g \circ A^{\alpha/2}u + g \circ A^{1/2}u \right)(t), \quad \forall t \geq 0, \tag{29}
\]

**Proof.** It is straightforward to establish the equivalence (28). To prove (29), we start by exploiting relations (17), (18), (23) and (25) to get

\[
\mathcal{L}'(t) \leq -\left( \frac{1-l}{\omega_0} - \delta \right) (\epsilon_1 - \epsilon_2) \left\| u'(t) \right\|^2 - \left( \frac{1}{2} \epsilon_2 - \delta \epsilon_1 \right) \left\| A^{1/2}u(t) \right\|^2 \\
- \left( N \frac{2}{\delta} (\epsilon_1 + \epsilon_2) - \frac{c}{\delta} C_0(\epsilon_1 + \epsilon_2) \right) \left( h_{\epsilon} \circ A^{1/2}u \right)(t) \\
- \left( N \frac{2}{\delta} (\epsilon_1 + \epsilon_2) - \frac{c}{\delta} C_0(\epsilon_1 + \epsilon_2) \right) \left( h_{\epsilon} \circ A^{\alpha/2}u \right)(t) \\
+ \frac{N \epsilon}{2} \left( g \circ A^{\alpha/2}u + g \circ A^{1/2}u \right)(t). \tag{30}
\]

Now, we set \( \beta := \frac{1-l}{\omega_0} \) and choose \( \delta \) small enough so that

\[
\xi \min \left\{ \frac{1}{2} \beta, \frac{1}{8} \beta \right\}. 
\]
Consequently, for $\epsilon_1 = \frac{16(1-l)}{tp} \left(4 + \frac{1}{2p}\right)$, we pick $\epsilon_2 = \frac{3}{8} \beta \epsilon_1$ satisfying
\[
\frac{1}{4} \beta \epsilon_1 < \epsilon_2 < \frac{1}{2} \beta \epsilon_1.
\]

Then,
\[
(\beta - \epsilon) \epsilon_1 - \epsilon_2 > \frac{1}{2} \beta \epsilon_1 - \epsilon_2 = \frac{1}{8} \beta \epsilon_1 = \frac{2}{l} \left(4 + \frac{3}{20}\right)
\]
and
\[
\frac{1}{2} \epsilon_2 - \delta \epsilon_1 > \frac{1}{2} \left(\epsilon_2 - \frac{1}{4} \beta \epsilon_1\right) = \frac{l}{16} \beta \epsilon_1 = (1 - l) \left(4 + \frac{3}{20}\right).
\]

From $\frac{\epsilon g^2(s)}{\epsilon g(s) - g'(s)} < g(s)$ and the Lebesgue Dominated Convergence Theorem, we deduce
\[
\lim_{\epsilon \to 0^+} \epsilon C_\epsilon = \lim_{\epsilon \to 0^+} \int_0^\infty \frac{\epsilon g^2(s)}{\epsilon g(s) - g'(s)} ds = 0.
\]

So there exists $0 < \epsilon_0 < 1$ such that if $\epsilon < \epsilon_0$, then
\[
\epsilon C_\epsilon < \frac{1}{8c} \epsilon (\epsilon_1 + \epsilon_2).
\]

Now, we choose $N$ large enough so that $L \sim E + \mathcal{E}$ and
\[
N > \max \left\{ \frac{4c}{\delta} (\epsilon_1 + \epsilon_2), \frac{1}{2\epsilon_0} \right\}.
\]

For $\epsilon = \frac{1}{2N}$, we have
\[
\frac{N}{4} - \frac{\epsilon}{\epsilon_1 + \epsilon_2} > 0 \quad \text{and} \quad \epsilon < \epsilon_0.
\]

This gives
\[
\frac{N}{2} - \frac{\epsilon}{\epsilon_1 + \epsilon_2} - \frac{\epsilon}{\epsilon_1 + \epsilon_2} > \frac{N}{2} - \frac{\epsilon}{\epsilon_1 + \epsilon_2} - \frac{1}{8\epsilon} = \frac{N}{4} - \frac{\epsilon}{\epsilon_1 + \epsilon_2} > 0.
\]

Thus estimate (30) becomes
\[
L'(t) \leq -\frac{2}{l} (1 - l) \left(4 + \frac{3}{20}\right) \| u'(t) \|^2 - (1 - l) \left(4 + \frac{3}{20}\right) \| A^{1/2} u(t) \|^2
+ \frac{1}{4} \left(\frac{g \circ A^{1/2} u + g \circ A^{1/2} u}{(t)} \right), \quad \forall t \geq 0.
\]

☐

**Lemma 9.** Assume that (A.1-A.3) hold. Then, the energy functional satisfies, for all $t \in \mathbb{R}^+$ and for some positive constant $\tilde{m}$, the following estimate
\[
\int_0^t E(s) ds < \tilde{m} f(t), \tag{31}
\]
where $f(t) = 1 + \int_0^t h(s) ds$ and $h = h_0 + h_1$ and $h_0, h_1$ are defined in (20) and (21).

**Proof.** Let $F(t) = L(t) + J_1(t) + J_2(t)$, then we obtain, for all $t \in \mathbb{R}_+$,
\[
F'(t) \leq -\frac{2}{l} (4 + \frac{3}{20}) \| u'(t) \|^2 - (1 - l) \| A^{1/2} u(t) \|^2 - \frac{1}{4} (g \circ A^{1/2} u)(t)
+ \frac{1}{2} \int_t^{+\infty} g(s) \| A^{1/2} u(t) - A^{1/2} u(t - s) \|^2 ds dx
+ \frac{1}{2} \int_t^{+\infty} g(s) \| A^{1/2} u(t) - A^{1/2} u(t - s) \|^2 ds dx. \tag{32}
\]
Estimates (18) and (32) yield, for some positive constant \( \lambda \) and for all \( t \in \mathbb{R}_+ \),

\[
F'(t) \leq -\lambda E(t) + \frac{1}{2} \int_t^{t+\infty} g(s) \left\| A^{\alpha/2} u(t) - A^{\alpha/2} u(t-s) \right\|^2 ds dx \\
+ \frac{1}{2} \int_t^{t+\infty} g(s) \left\| A^{1/2} u(t) - A^{1/2} u(t-s) \right\|^2 ds dx.
\]

Therefore, using (20) and (21) and integrating both sides of the last inequality, over \((0,t)\), we arrive at

\[
\lambda \int_0^t E(s) ds \leq F(0) - F(t) + \frac{M_0}{2} \int_0^t h(s) ds \leq F(0) + \frac{M_0}{2} \int_0^t h(s) ds.
\]

Hence, we get

\[
\int_0^t E(s) ds \leq \frac{F(0)}{\lambda} + \frac{M_0}{2\lambda} \int_0^t h(s) ds \leq \hat{m} \left( 1 + \int_0^t h(s) ds \right),
\]

where \( \hat{m} = \max \left\{ \frac{F(0)}{\lambda}, \frac{M_0}{2\lambda} \right\}. \) \( \square \)

**Corollary 1.** There exists \( 0 < q_0 < 1 \) such that, for all \( t \geq 0 \), we have the following estimate:

\[
\int_0^t g(s) \left( \left\| A^{\alpha/2} (u(t) - u(t-s)) \right\|^2 + \left\| A^{1/2} (u(t) - u(t-s)) \right\|^2 \right) ds \leq \frac{1}{q(t)} G^{-1} \left( \frac{q(t) \mu(t)}{\zeta(t)} \right)
\]

where

\[
\mu(t) := -\int_0^t g'(s) \left( \left\| A^{\alpha/2} (u(t) - u(t-s)) \right\|^2 + \left\| A^{1/2} (u(t) - u(t-s)) \right\|^2 \right) ds,
\]

\[
q(t) := \frac{q_0}{f(t)}.
\]

\( G \) is defined in Remark 4 and \( f(t) \) is defined in (31).

**Proof.** We introduce a functional \( \eta \) defined by

\[
\eta(t) := q(t) \int_0^t \left( \left\| A^{\alpha/2} (u(t) - u(t-s)) \right\|^2 + \left\| A^{1/2} (u(t) - u(t-s)) \right\|^2 \right) ds, \quad \forall \ t \geq 0,
\]

and observe, from inequality (12), that

\[
E(t) \geq \frac{1}{2} \left\| A^{1/2} u(t) \right\|^2 \quad \text{and} \quad E(t) \geq \frac{1}{2\alpha} \left\| A^{\alpha/2} u(t) \right\|^2, \quad \forall \ t \geq 0.
\]

Use of (15), (17) and (38) yields

\[
\eta(t) \leq 2q(t) \int_0^t \left( \left\| A^{\alpha/2} u(t) \right\|^2 + \left\| A^{\alpha/2} u(t-s) \right\|^2 + \left\| A^{1/2} u(t) \right\|^2 + \left\| A^{1/2} u(t-s) \right\|^2 \right) ds
\]

\[
\leq \frac{4q(t)}{t} \left( 1 + \frac{1}{\alpha} \right) \int_0^t (E(t) + E(t-s)) ds
\]

\[
\leq \frac{8q(t)}{t} \left( 1 + \frac{1}{\alpha} \right) \int_0^t E(s) ds, \quad \forall \ t \geq 0.
\]

Thanks to (31), we can pick \( 0 < q_0 < \min \left\{ 1, \frac{1}{8q(t)(1+\alpha)} \right\} \) so that

\[
\eta(t) < 1, \quad \forall \ t \geq 0.
\]

To prove (35), we define another functional \( \mu \) by

\[
\mu(t) := -\int_0^t g'(s) \left( \left\| A^{\alpha/2} (u(t) - u(t-s)) \right\|^2 + \left\| A^{1/2} (u(t) - u(t-s)) \right\|^2 \right) ds
\]

\[
\leq -c (E'(t) + E'(t)).
\]
Also, the strict convexity of $G$ and the fact that $G(0) = 0$ entail that
\[
G(s\tau) \leq sG(\tau), \quad \text{for} \quad 0 \leq s \leq 1 \quad \text{and} \quad \tau \in (0, r].
\]
Combining this with the hypothesis (A.2), Jensen’s inequality and (39), we obtain, for any $t \geq 0$,
\[
\mu(t) = -\frac{1}{\eta(t)} \int_0^t \eta(t) g'(s) \left( \|A^{a/2}(u(t) - u(t-s))\|^2 + \|A^{1/2}(u(t) - u(t-s))\|^2 \right) ds
\geq \frac{1}{\eta(t)} \int_0^t \eta(t) \xi(s) G(g(s)) \left( \|A^{a/2}(u(t) - u(t-s))\|^2 + \|A^{1/2}(u(t) - u(t-s))\|^2 \right) ds
\geq \frac{\xi(t)}{\eta(t)} \int_0^t G(\eta(t) g(s)) \left( \|A^{a/2}(u(t) - u(t-s))\|^2 + \|A^{1/2}(u(t) - u(t-s))\|^2 \right) ds
\geq \frac{\xi(t)}{\eta(t)} \frac{1}{G(0)} \left( \frac{q(t) \mu(t)}{\xi(t)} \right).
\]

\[\Box\]

5. **The main result**

In this section, we state and prove our decay result. We introduce the following functions:
\[
G_2(t) = tG'(\varepsilon_0 t), \quad G_3(t) = tG'^{-1}(t), \quad G_4(t) = \overline{G}_4(t).
\]
(41)
It is not difficult to show that the above functions are convex and increasing on $(0, r]$. Now we state our main result.

**Theorem 2.** Assume that hypotheses (A.1)–(A.3) hold and the initial data satisfy
\[
(u_0, u_1) \in \left[ D \left( A^{1-a}/2 \right) \times D \left( A^{(1-a)/2} \right) \right] \cap \left[ D \left( A^{1/2} \right) \times H \right].
\]
Then, for all $0 \leq s \leq t$ and for strictly positive constant $C$, we have the following decay results
\[
E(t) \leq C \left( \frac{E(0)}{q(t)} \right) G_2^{-1} \left[ C + \int_0^t \xi(s) G_4 \left( \frac{q(s) h(s)}{s} \right) ds \right],
\]
(42)
where $q$ is defined in (37), $h = h_0 + h_1$ where $h_0, h_1$ are defined in (20) and (21) and the functions $G_2(s)$ and $G_4(s)$ are defined in (41).

**Proof.** We start by combining (15), (20), (21), (29) and (35); then, for some $m > 0$ and for any $t \geq 0$, we have
\[
\mathcal{L}'(t) \leq -mE(t) + \frac{c}{q(t)} G^{-1} \left( \frac{q(t) \mu(t)}{\xi(t)} \right) + ch(t), \quad \forall t \geq 0.
\]
(43)
Let $0 < \varepsilon_0 < r$, then define a functional $\mathcal{F}$ by
\[
\mathcal{F}(t) := G' \left( \frac{\varepsilon_0 q(t) E(t)}{E(0)} \right) \mathcal{L}(t), \quad \forall t \geq 0.
\]
Using the facts that $E' \leq 0$, $G' > 0$ and $G'' > 0$, we get for any $t \geq 0$,
\[
\mathcal{F}'(t) = \frac{\varepsilon_0 q(t) E'(t)}{E(0)} G'' \left( \frac{\varepsilon_0 q(t) E(t)}{E(0)} \right) \mathcal{L}(t) + G' \left( \frac{\varepsilon_0 q(t) E(t)}{E(0)} \right) \mathcal{L}'(t)
\]
\[
\leq -mE(t)G' \left( \frac{\epsilon_0 q(t)E(t)}{E(0)} \right) + \frac{c}{q(t)}G' \left( \frac{\epsilon_0 q(t)(t)}{E(0)} \right) G^{-1} \left( \frac{q(t)\mu(t)}{\xi(t)} \right) + ch(t)G' \left( \frac{\epsilon_0 q(t)E(t)}{E(0)} \right). \tag{44}
\]

Let \( G^* \) be the convex conjugate of \( G \) in the sense of Young (see [28]), then
\[
G^*(s) = s(G')^{-1}(s) - G \left[ (G')^{-1}(s) \right], \quad \text{if } s \in (0, G'(r) \bigcup)
\]
and it satisfies the following generalized Young inequality
\[
AB \leq G^*(A) + G(B), \quad \text{if } A \in (0, G'(r)], \quad B \in (0, r]. \tag{46}
\]

So, with \( A = G' \left( \frac{E(t)q(t)}{E(0)} \right) \), \( B = G^{-1} \left( \frac{q(t)\mu(t)}{\xi(t)} \right) \), and using (17), (18), and (44)-(46), we arrive at
\[
\mathcal{F}'(t) \leq -mE(t)G' \left( \frac{E(t)q(t)}{E(0)} \right) + \frac{c}{q(t)}G^* \left( G' \left( \frac{E(t)q(t)}{E(0)} \right) \right) + c \left( \frac{\mu(t)q(t)}{\xi(t)} \right) + ch(t)G' \left( \frac{E(t)q(t)}{E(0)} \right).
\]

So, multiplying (47) by \( \xi(t) \) and using (40) and the fact that \( \epsilon_0 \frac{E(t)q(t)}{E(0)} < r \), we obtain
\[
\xi(t)\mathcal{F}'(t) \leq -m \xi(t)G(t)G' \left( \frac{E(t)q(t)}{E(0)} \right) + c\xi(t)E(t) \left( \frac{E(t)q(t)}{E(0)} \right) G' \left( \frac{E(t)q(t)}{E(0)} \right)
\]
\[
+ c\mu(t)q(t) + c\xi(t)h(t)G' \left( \frac{E(t)q(t)}{E(0)} \right)
\]
\[
\leq -\epsilon_0 \frac{mE(0)}{\epsilon_0} - c\xi(t)E(t) \left( \frac{E(t)q(t)}{E(0)} \right) - c(E'(t) + E'(t))(t) + c\xi(t)h(t)G' \left( \frac{E(t)q(t)}{E(0)} \right).
\]

Consequently, recalling the definition of \( G_2 \) and choosing \( \epsilon_0 \) so that \( k = \frac{mE(0)}{\epsilon_0} - c > 0 \), we obtain, for all \( t \in \mathbb{R}_+ \),
\[
\mathcal{F}'(t) \leq -k \xi(t) \left( \frac{E(t)}{E(0)} \right) G' \left( \frac{E(t)q(t)}{E(0)} \right) + c\xi(t)h(t)G' \left( \frac{E(t)q(t)}{E(0)} \right)
\]
\[
\leq -k \xi(t)G_2 \left( \frac{E(t)q(t)}{E(0)} \right) + c\xi(t)h(t)G' \left( \frac{E(t)q(t)}{E(0)} \right), \tag{48}
\]
where \( \mathcal{F}_1 = \xi \mathcal{F} + c(E + \epsilon) E(t) \). Since \( G'(t) = G'(t) + tG''(t) \), then, using the strict convexity of \( G \) on \( (0, r) \), we find that \( G'_2(t), G_2(t) > 0 \) on \( (0, r) \).

Using the general Young inequality (46) for the last term in (48) with \( A = G' \left( \frac{E(t)q(t)}{E(0)} \right) \) and \( B = [\xi h(t)] \), we have for any \( d > 0 \),
\[
ch(t)G' \left( \frac{E(t)q(t)}{E(0)} \right) = \frac{d}{q(t)} \left[ \frac{c}{d} \xi(t)h(t) \right] \left( G' \left( \frac{E(t)q(t)}{E(0)} \right) \right)
\]
\[
\leq \frac{d}{q(t)}G_3 \left( G' \left( \frac{E(t)q(t)}{E(0)} \right) \right) + \frac{d}{q(t)}G_4 \left[ \frac{c}{d} \xi(t)h(t) \right]
\]
\[
\leq \frac{d}{q(t)} \left( \frac{E(t)q(t)}{E(0)} \right) \left( G' \left( \frac{E(t)q(t)}{E(0)} \right) \right) + \frac{d}{q(t)}G_4 \left[ \frac{c}{d} \xi(t)h(t) \right]
\]
\[
\leq \frac{d}{q(t)}G_2 \left( \frac{E(t)q(t)}{E(0)} \right) + \frac{d}{q(t)}G_4 \left[ \frac{c}{d} \xi(t)h(t) \right]. \tag{49}
\]

where \( G_2, G_3 \) and \( G_4 \) are given in (41). Now, combining (48) and (49) and choosing \( d \) small enough so that \( k_1 = (k - d) > 0 \), we arrive at
\[
\mathcal{F}'(t) \leq -k_1 \xi(t)G_2 \left( \frac{E(t)q(t)}{E(0)} \right) + \frac{d_2}{q(t)}G_4 \left[ \frac{c}{d} \xi(t)h(t) \right]. \tag{50}
\]
Since \( E' < 0 \) and \( q' < 0 \), then \( (qE)(t) \) is decreasing function. Using this fact and since \( G_2 \) is increasing, we have, for \( 0 \leq t \leq T \),

\[
G_2 \left( \frac{\epsilon_0 E(T)q(T)}{E(0)} \right) \leq G_2 \left( \frac{\epsilon_0 E(t)q(t)}{E(0)} \right)
\]  

(51)

Combining (50) with (51) and multiplying by \( q \), we get

\[
q(t)\mathcal{F}_1'(t) + k_1 \xi(t)G_2 \left( \frac{\epsilon_0 E(T)q(T)}{E(0)} \right) \leq d\xi(t)G_4 \left( \frac{c}{d} q(t)h(t) \right).
\]  

(52)

Since \( q' < 0 \), then for all \( 0 \leq t \leq T \),

\[
\left( q(t)\mathcal{F}_1 \right)'(t) + k_1 \xi(t)G_2 \left( \frac{\epsilon_0 E(T)q(T)}{E(0)} \right) \leq d\xi(t)G_4 \left( \frac{c}{d} q(t)h(t) \right).
\]  

(53)

Integrating (53) over \([0, T]\) and using the fact \( q(0) = q_0 \), we have

\[
G_2 \left( \frac{\epsilon_0 E(T)q(T)}{E(0)} \right) \int_0^T \xi(t) dt \leq \frac{q_0 \mathcal{F}_1(0)}{k_1} + d \int_0^T \xi(t)G_4 \left( \frac{c}{d} q(t)h(t) \right) dt.
\]  

Hence,

\[
G_2 \left( \frac{\epsilon_0 E(T)q(T)}{E(0)} \right) \leq \left[ \frac{\mathcal{F}_1(0)}{c} + d \int_0^T \xi(t)G_4 \left( \frac{c}{d} q(t)h(t) \right) dt \right].
\]  

(55)

Thus

\[
\left( \frac{\epsilon_0 E(T)q(T)}{E(0)} \right) \leq G_2^{-1} \left[ \frac{\mathcal{F}_1(0)}{c} + d \int_0^T \xi(t)G_4 \left( \frac{c}{d} q(t)h(t) \right) dt \right],
\]  

(56)

which yields

\[
E(T) \leq C \left( \frac{E(0)}{q(T)} \right) G_2^{-1} \left[ C + \int_0^T \xi(t)G_4 \left( \frac{c}{d} q(t)h(t) \right) dt \right],
\]  

(57)

where \( C = \max \{1, \frac{\mathcal{F}_1(0)}{c}, \frac{1}{q(T)}\} \). \(\square\)

**Example 1.** Let \( g(t) = \frac{a}{(1+t)^{\nu}} \), where \( \nu > 1 \) and \( 0 < a < \nu - 1 \). In this case \( \xi(t) = \nu a^{-1} \) and \( G(t) = t^{\nu-1} \). Then \( G'(t) = a_1 t^\frac{\nu-1}{\nu} \). We will discuss two cases:

**Case 1:** if \( m_1 \leq 2 + ||A^{1/2}u_0 + A^{1/2}u_1||2 \leq m_1 \). Then we have the following:

\[
G_4(t) = a_1 t^{\frac{\nu-1}{\nu}}, \quad G_2(t) = a_2 t^{\frac{\nu-1}{\nu}},
\]

\[
a_3 (1 + t)^{-\nu-1} \leq h(t) \leq a_4 (1 + t)^{-\nu+1},
\]

\[
\int_0^T \xi(t)G_4 \left( \frac{c}{d} q(t)h(t) \right) dt < +\infty,
\]

\[
G_2^{-1} \left[ C + \int_0^T \xi(t)G_4 \left( \frac{c}{d} q(t)h(t) \right) dt \right] \leq a_5 T^{-\frac{\nu}{\nu+1}},
\]

\[
\frac{q_0}{q(T)} \leq a_6 \left\{ \begin{array}{ll}
1 + \ln(1 + T), & \nu = 2; \\
2, & \nu > 2; \\
(1 + T)^{-\nu + 2 + r}, & 1 < \nu < 2
\end{array} \right.
\]

(58)

Then

\[
E(T) \leq a_7 \left\{ \begin{array}{ll}
\left( 1 + \ln(1 + T) \right) t^{\frac{\nu-1}{\nu+1}}, & \nu = 2; \\
T^{\frac{\nu}{\nu+1}}, & \nu > 2; \\
(1 + T)^{-(\nu - 2 + \frac{\nu}{\nu+1})}, & 1 < \nu < 2
\end{array} \right.
\]

(60)

Thus for \( \nu \geq 2 \) or \( \sqrt{2} < \nu < 2 \) we have \( \lim_{T \to +\infty} E(T) = 0 \).
**Case 2:** If \( m_0(1 + t)^r \leq 2 + \| A^{\nu/2} u_0 + A^{1/2} u_0 \|^2 \leq m_1(1 + t)^{\nu - 1} \), then we have the following:

\[
\begin{align*}
a_3(1 + t)^{-\nu + 1 + r} & \leq h(t) \leq a_4(1 + t)^{-\nu + 1 + r}, \\
\int_0^t \xi(t) G_4 (\frac{\xi}{2} q(t) h(t)) \, dt & < +\infty,
\end{align*}
\]

Then,

\[
E(T) \leq a_7 \begin{cases} 
(1 + \ln(1 + T)) t^{-\nu + 1}, & v - r = 2; \\
T^{-\nu}, & v - r > 2; \\
(1 + T)^{-\nu + 1}, & 1 < v - r < 2.
\end{cases}
\]

Thus for \( v - r \geq 2 \) or \( 1 < v - r < 2 \) we have \( \lim_{T \to +\infty} E(T) = 0 \).

**Acknowledgments:** The authors thank University of King Fahd University of Petroleum and Minerals (KFUPM) and the referee for his/her very careful reading and valuable comments. This work is sponsored by KFUPM under Project No. SB201012.

**Author Contributions:** All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

**Conflicts of Interest:** “The authors declare no conflict of interest.”

**Data Availability:** All data required for this research is included within this paper.

**Funding Information:** No funding is available for this research.

**References**


