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Stability result for a class of weakly dissipative second-order systems with infinite memory

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Abstract: In this paper we consider the following abstract class of weakly dissipative second-order systems with infinite memory, $u''(t) + Au(t) - \int_0^\infty g(s)A^\alpha u(t-s)ds = 0,\ t > 0$, and establish a general stability result with a very general assumption on the behavior of g at infinity; that is $g'(t) \leq -\xi(t)G\left(g(t)\right),\ \ t \geq 0$. where ξ and G are two functions satisfying some specific conditions. Our result generalizes and improves many earlier results in the literature. Moreover, we obtain our result with imposing a weaker restrictive assumption on the boundedness of initial data used in many earlier papers in the literature such as the one in [1–5]. The proof is based on the energy method together with convexity arguments.

Keywords: Stability; Convex functions; Infinite memory; Viscoelasticity; Weakly dissipative system.

MSC: 35B40; 35L90; 34G10; 45K05; 93D20.

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1. Introduction

Viscoelastic materials exhibit an instantaneous elasticity effect and creep characteristics at the same time. The importance of the viscoelastic properties of materials has been realized because of the rapid developments in rubber and plastics industry. The modeling of the dynamics of physical phenomena such as heat flow in conductors with memory, hereditary polarization in dielectrics, population dynamics, viscolasticity can be described by an abstract integro-differential equation of the form

$$\begin{cases} u''(t) + Au(t) - \int_{-\infty}^{t} g(t-s)A^\alpha u(s)ds = 0, & t > 0, \\ u(-t) = u_0(t), & t \geq 0,\ \ u'(0) = u_1, \end{cases} \tag{1}$$

where $'$ represents a derivative with respect to time t, $A : \mathcal{D}(A) \subset H \longrightarrow H$ is a positive definite self-adjoint operator on H, g is the relaxation function (convolution kernel), $\alpha \in [0,1]$, u_0, u_1 are given history function and initial data respectively.

The study of viscoelastic problems has attracted the attention of many authors and several decay and blow up results have been established. We start with the pioneer work [6,7] where Dafermos considered a one-dimensional viscoelastic problem and established various existence results and then proved, for smooth monotone decreasing relaxation functions, that the solutions go to zero as t goes to infinity. After that, many results dealing with the existence, uniqueness, regularity and asymptotic behavior of many systems of the form (1) have been studied; see, for example, [1,8–11]. In the case of finite memory, that is, $u_0(t) = 0$ for $t < 0$, see [12–18]. In particular, Rivera *et al.*, [15] considered the interpolating cases $\alpha \in (0,1)$ and a relaxation function g which decays exponentially to zero at infinity, that is,

$$-c_0 g(s) \leq g'(s) \leq -c_1 g(s) \qquad \forall\, s \in \mathbb{R}_+. \tag{2}$$

They showed that the energy decays polynomially at the rate of $\frac{1}{t}$. Recently, Hassan and Messaoudi [19] considered

$$\begin{cases} u''(t) + Au(t) - \int_0^t g(t-s)A^{\alpha}u(s)ds = 0, & t > 0, \\ u(0) = u_0(t), & u'(0) = u_1, \end{cases} \tag{3}$$

and established a new general decay rate result for which the relaxation function g satisfies condition

$$g'(t) \leq -\xi(t)G(g(t)), \ \ t \geq 0. \tag{4}$$

For case of infinite memory, see [20–25]. In particular, Guesmia [1] considered

$$u_{tt} + Au - \int_0^{+\infty} g(s)Bu(t-s)ds = 0 \ \text{ for } \ t > 0, \tag{5}$$

and introduced a new ingenuous approach for proving a more general decay result based on the properties of convex functions and the use of the generalized Young inequality. He used a larger class of infinite history kernels satisfies the following condition

$$\int_0^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} ds + \sup_{s \in \mathbb{R}_+} \frac{g(s)}{G^{-1}(-g'(s))} < +\infty, \tag{6}$$

such that

$$G(0) = G'(0) = 0 \text{ and } \lim_{t \to +\infty} G'(t) = +\infty, \tag{7}$$

where $G : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing strictly convex function. Al-Mahdi and Al-Gharabli [2] considered the following viscoelastic problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^{+\infty} g(s)\Delta u(t-s)ds + |u_t|^{m-2}u_t = 0, & \text{in } \Omega \times (0, +\infty) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0, +\infty) \\ u(x,-t) = u_0(x,t), \ u_t(x,0) = u_1(x), & \text{in } \Omega \times (0, +\infty), \end{cases} \tag{8}$$

and they established decay results with using a relaxation function g, satisfying the condition

$$g'(t) \leq -\xi(t)g^p(t), \ \ 1 \leq p < \frac{3}{2}. \tag{9}$$

Very recently, Guesmia [26] considered two models of wave equations with infinite memory and established an explicit and general decay rate results where the relaxation function satisfying the condition (4).

Motivated by the above works, we intend to study the following class of viscoelastic equations of the form

$$\begin{cases} u''(t) + Au(t) - \int_0^{\infty} g(s)A^{\alpha}u(t-s)ds = 0, & t > 0, \\ u(-t) = u_0, & u'(0) = u_1, \end{cases} \tag{10}$$

where $A : \mathcal{D}(A) \subset H \longrightarrow H$ is a positive definite self-adjoint operator on H such that the embedding $\mathcal{D}(A^{\beta}) \hookrightarrow \mathcal{D}(A^{\sigma})$ is compact for any $\beta > \sigma \geq 0$ and $\alpha \in (0,1)$.

Remark 1. The assumption $\mathcal{D}(A^{\beta}) \hookrightarrow\hookrightarrow \mathcal{D}(A^{\sigma})$ for any $\beta > \sigma \geq 0$ guarantees the existence of some constants ω, ω_0, ω_1 such that

$$\|v\|^2 \leq \omega \left\|A^{1/2}v\right\|^2 \qquad \forall v \in \mathcal{D}\left(A^{1/2}\right), \tag{11}$$

$$\left\|A^{\alpha/2}v\right\|^2 \leq \omega_0 \left\|A^{1/2}v\right\|^2 \qquad \forall v \in \mathcal{D}\left(A^{1/2}\right), \tag{12}$$

and

$$\left\|A^{1/2}v\right\|^2 \leq {}_1 \left\|A^{1-\alpha/2}v\right\|^2 \qquad \forall v \in \mathcal{D}\left(A^{1-\alpha/2}\right). \tag{13}$$

2. Our main objectives

We intend to establish a two fold objective:

1. improve many earlier works such as the ones in [11,15,19] from finite memory to infinite memory;
2. prove a general decay estimate for the solution of Problem (10) with a wider class of relaxation functions than the ones considered in [1–5] by getting a better decay rate with imposing a weaker assumption on the boundedness of initial data than the one considered in the earlier papers such as the one in [1–5].

The paper is organized as follows: We present some assumptions and remarks in §3. We state and prove some technical lemmas in §4. The main result, its proof and some examples are presented in §5.

3. Assumptions

In this section, we state some assumptions needed in the proof of our main decay result. The strictly decreasing differentiable relaxation (kernel) function $g : [0, \infty) \longrightarrow (0, \infty)$ satisfies the following assumptions:

(A.1) $$g(0) > 0 \quad \text{and} \quad 1 - {}_0\int_0^{+\infty} g(s)ds = l > 0.$$

(A.2) There exists a non-increasing differentiable function $\xi : \mathbb{R}_+ \longrightarrow (0, \infty)$ and a $\boldsymbol{C}^1$ function $G : [0, +\infty) \longrightarrow [0, +\infty)$ which is linear or it is strictly increasing and strictly convex C^2 function on $(0, r]$, with $G(0) = G'(0) = 0$, such that
$$g'(t) \leq -\xi(t)G(g(t)), \qquad \forall\, t \geq 0, \tag{14}$$
where ξ is satisfying $\int_0^{+\infty} \xi(s)ds = +\infty$.

(A.3) We assume that
$$\int_0^{+\infty} g(s)||A^{\alpha/2}u_0(s)||^2 ds < +\infty,$$
and
$$\int_0^{+\infty} g(s)||A^{1/2}u_0(s)||^2 ds < +\infty.$$

Remark 2. The class of relaxation functions satisfying (**A.1**) − (**A.2**) in the present paper is larger than the ones satisfying (6) and (7) used in some earlier papers such as the one in [1]. In fact, the boundedness of the sup in (6) use in [1], can be interpreted as the inequality in (**A.2**) in the present paper (with $\xi = 1$). The conditions (6) and (7) used in [1] ask also the boundedness of the integral. So, it is better to consider the relaxation functions satisfy (**A.1**) − (**A.2**) used in the present paper than the one used in [1].

Remark 3. Hypothesis (**A.3**) is needed for proving the existence and stability results. For the stability, if (**A.3**) holds, then the functions h_0 and h_1 defined in Lemma 4 well be defined. Moreover, Hypothesis (**A.3**) is weaker than the one used in [1–5] that is, there exists a positive constant M such that
$$||\nabla A^{\alpha/2}u_0(s)||^2 \leq M,$$
and
$$||\nabla A^{1/2}u_0(s)||^2 \leq M.$$

Remark 4. As is in Mustafa [14], if G is a strictly increasing and strictly convex $\boldsymbol{C}^2$ function on $(0, r]$, with $G(0) = G'(0) = 0$, then there is a strictly convex and strictly increasing $\boldsymbol{C}^2$ function $\overline{G} : [0, +\infty) \longrightarrow [0, +\infty)$ which is an extension of G. For instance, we can define $\overline{G}$, for any $t > r$, by
$$\overline{G}(t) := \frac{G''(r)}{2}t^2 + (G'(r) - G''(r)r)t + \left(G(r) + \frac{G''(r)}{2}r^2 - G'(r)r\right).$$

We state the existence, regularity and uniqueness theorem whose proof is in [15].

Theorem 1 ([15])**.** *Suppose that $(u_0(\cdot, 0), u_1) \in \mathcal{D}(A) \times \mathcal{D}(A^{1/2})$ and (**A.1-A.3**) hold. Then, Problem (10) has a unique global solution satisfying*
$$u \in \boldsymbol{C}(\mathbb{R}_+; \mathcal{D}(A)) \cap \boldsymbol{C}^1\left(\mathbb{R}_+; \mathcal{D}\left(A^{1/2}\right)\right) \cap \boldsymbol{C}^2(\mathbb{R}_+; H).$$

Moreover, if $(u_0(\cdot,0),u_1) \in \mathcal{D}\left(A^{\sigma+1/2}\right) \times \mathcal{D}\left(A^{\sigma}\right)$ *for* $\sigma \geq 0$, *then the solution satisfies*

$$u \in \boldsymbol{C}\left(\mathbb{R}_+;\mathcal{D}\left(A^{\sigma+1/2}\right)\right) \cap \boldsymbol{C}^1\left(\mathbb{R}_+;\mathcal{D}\left(A^{\sigma}\right)\right) \cap \boldsymbol{C}^2\left(\mathbb{R}_+;\mathcal{D}\left(A^{\sigma-1/2}\right)\right).$$

The "modified" energy functionals associated to our problem are given by

$$E(t) := \frac{1}{2}\left[\|u'(t)\|^2 + \left\|A^{1/2}u(t)\right\|^2 - \left(\frac{1-l}{\omega_0}\right)\left\|A^{\alpha/2}u(t)\right\|^2 + \left(g \circ A^{\alpha/2}u\right)(t)\right], \tag{15}$$

$$\mathcal{E}(t) := \frac{1}{2}\left[\|A^{(1-\alpha)/2}u'(t)\|^2 + \left\|A^{1-\alpha/2}u(t)\right\|^2 - \left(\frac{1-l}{\omega_0}\right)\left\|A^{1/2}u(t)\right\|^2 + \left(g \circ A^{1/2}u\right)(t)\right], \tag{16}$$

for any $t \geq 0$, where for $v \in L^2_{loc}(\mathbb{R}_+;H)$,

$$(g \circ v)(t) := \int_0^\infty g(s)\|v(t) - v(t-s)\|^2 ds.$$

Remark 5. The positiveness of the energy functionals comes from inequalities (12) and (13).

Lemma 1 ([15])**.** *For any initial data* $(u_0,u_1) \in \mathcal{D}(A) \times \mathcal{D}\left(A^{1/2}\right)$, *the energy functionals associated to Problem* (10) *satisfy, for any* $t \geq 0$, *the identities*

$$E'(t) = \frac{1}{2}\left(g' \circ A^{\alpha/2}u\right)(t) \leq 0, \tag{17}$$

$$\mathcal{E}'(t) = \frac{1}{2}\left(g' \circ A^{1/2}u\right)(t) \leq 0. \tag{18}$$

As in Jin *et al.*, [27], we set, for any $0 < \varepsilon < 1$,

$$C_\varepsilon := \int_0^\infty \frac{g^2(s)}{\varepsilon g(s) - g'(s)} ds \qquad \text{and} \qquad h_\varepsilon(t) := \varepsilon g(t) - g'(t).$$

Lemma 2 ([27])**.** *Assume that the condition* (**A.1**) *holds. Then, for any* $v \in L^2_{loc}([0,+\infty);L^2(0,L))$, *we have*

$$\int_0^L \left(\int_0^\infty g(s)(v(t) - v(t-s))ds\right)^2 dx \leq C_\varepsilon (h_\varepsilon \circ v)(t), \qquad \forall\, t \geq 0. \tag{19}$$

Lemma 3 (Jensen's inequality)**.** *Let* $F : [a,b] \longrightarrow \mathbb{R}$ *be a convex function. Assume that the functions* $f : \longrightarrow [a,b]$ *and* $h : \Omega \longrightarrow \mathbb{R}$ *are integrable such that* $h(x) \geq 0$, *for any* $x \in \Omega$ *and* $\int_\Omega h(x)dx = k > 0$. *Then,*

$$F\left(\frac{1}{k}\int_\Omega f(x)h(x)dx\right) \leq \frac{1}{k}\int_\Omega F(f(x))h(x)dx.$$

4. Technical lemmas

In this section, we state and prove some Lemmas that are useful in the proof of Theorem 2. Through out this work we use $c > 1$ to represent a generic constant, which is independent of t and the initial data.

Lemma 4. *Assume that* (**A.1-A.3**) *hold. Then, there exist two positive constants* M_0, M_1 *such that*

$$\int_t^{+\infty} g(s)||A^{\alpha/2}u(t) - A^{\alpha/2}u(t-s)||^2 ds \leq M_0 h_0(t), \tag{20}$$

and

$$\int_t^{+\infty} g(s)||A^{1/2}u(t) - A^{1/2}u(t-s)||^2 ds \leq M_1 h_1(t), \tag{21}$$

where $h_0(t) = \int_0^{+\infty} g(t+s)\left(1 + ||A^{\alpha/2}u_0(s)||^2\right) ds$, *and* $h_1(t) = \int_0^{+\infty} g(t+s)\left(1 + ||A^{1/2}u_0(s)||^2\right) ds.$

Proof. Indeed, we have

$$\begin{aligned}
&\int_t^{+\infty} g(s)||A^{\alpha/2}u(t) - A^{1/2}u(t-s)||^2 ds \\
&\leq 2||A^{\alpha/2}u(t)||^2 \int_t^{+\infty} g(s)ds + 2\int_t^{+\infty} g(s)||A^{\alpha/2}u(t-s)||^2 ds \\
&\leq 2\sup_{s\geq 0}||A^{\alpha/2}u(s)||^2 \int_0^{+\infty} g(t+s)ds + 2\int_0^{+\infty} g(t+s)||A^{\alpha/2}u(-s)||^2 ds \\
&\leq \frac{4\omega_0 \sup_{s\geq 0} E(s)}{1-l}\int_0^{+\infty} g(t+s)ds + 2\int_0^{+\infty} g(t+s)||A^{\alpha/2}u_0(s)||^2 ds \\
&\leq \frac{4\omega_0 E(0)}{1-l}\int_0^{+\infty} g(t+s)ds + 2\int_0^{+\infty} g(t+s)||A^{\alpha/2}u_0(s)||^2 ds \\
&\leq M_0 \int_0^{+\infty} g(t+s)\left(1+||A^{\alpha/2}u_0(s)||^2\right) ds.
\end{aligned} \tag{22}$$

where $M_0 = \max\left\{2, \frac{4\omega_0 E(0)}{1-l}\right\}$. □

The proof of (21) can be established similarly to the proof of (20).

Lemma 5. *Assume that conditions (**A.1**-**A.3**) hold. Then, for any $0<\delta<1$, the functional I_1 defined by*

$$I_1(t) := -\left\langle u'(t), \int_0^\infty g(s)(u(t)-u(t-s))ds\right\rangle$$

satisfies, along the solution of (10), the estimate

$$I_1'(t) \leq -\left(\frac{1-l}{\omega_0} - \delta\right)\|u'(t)\|^2 + \delta\left\|A^{1/2}u(t)\right\|^2 + \frac{c}{\delta}(C_\varepsilon + 1)\left(h_\varepsilon \circ A^{1/2}u\right)(t), \qquad \forall t\geq 0. \tag{23}$$

Proof. Differentiating I_1 and exploiting the differential equation in Problem (10), we get

$$\begin{aligned}
I_1'(t) = &\left\langle A^{1/2}u(t), \int_0^\infty g(s)A^{1/2}(u(t)-u(t-s))ds\right\rangle \\
&-\left\langle \int_0^\infty g(s)A^{\alpha/2}u(t-s)ds, \int_0^\infty g(s)A^{\alpha/2}(u(t)-u(t-s))ds\right\rangle \\
&-\left(\frac{1-l}{\omega_0}\right)\|u'(t)\|^2 - \left\langle u'(t), \int_0^\infty g'(s)(u(t)-u(t-s))ds\right\rangle.
\end{aligned} \tag{24}$$

Next, we estimate the terms in the right-hand side of the above identity. Using the Cauchy-Schwarz, Young and Hölder inequalities, Lemma 2 and inequalities (11) and (12), it follows that, for any $0<\delta<1$,

$$\begin{aligned}
&\left\langle A^{1/2}u(t), \int_0^\infty g(s)A^{1/2}(u(t)-u(t-s))ds\right\rangle \\
&\leq \frac{\delta}{2}\|A^{1/2}u(t)\|^2 + \frac{c}{\delta}C_\varepsilon\left(h_\varepsilon \circ A^{1/2}u\right)(t), -\left\langle \int_0^\infty g(s)A^{\alpha/2}u(t-s)ds,\ \int_0^\infty g(s)A^{\alpha/2}(u(t)-u(t-s))ds\right\rangle \\
&= \left\|\int_0^\infty g(s)A^{\alpha/2}(u(t)-u(t-s))ds\right\|^2 - \left\langle\left(\frac{1-l}{\omega_0}\right)A^{\alpha/2}u(t), \int_0^\infty g(s)A^{\alpha/2}(u(t)-u(t-s))ds\right\rangle \\
&\leq \frac{\delta}{2\omega_0}\left\|A^{\alpha/2}u(t)\right\|^2 + \frac{c}{\delta}C_\varepsilon\left(h_\varepsilon\circ A^{\alpha/2}u\right)(t) \\
&\leq \frac{\delta}{2}\left\|A^{1/2}u(t)\right\|^2 + \frac{c}{\delta}C_\varepsilon\left(h_\varepsilon\circ A^{1/2}u\right)(t),
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle u'(t), \int_0^\infty g'(s)(u(t)-u(t-s))ds\right\rangle \\
&= \left\langle u'(t), \varepsilon\int_0^\infty g(s)(u(t)-u(t-s))ds\right\rangle - \left\langle u'(t), \int_0^\infty h_\varepsilon(s)(u(t)-u(t-s))ds\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\delta}{2}\|u'(t)\|^2 + \frac{\varepsilon^2}{2\delta}\left\|\int_0^\infty g(s)(u(t)-u(t-s))ds\right\|^2 + \frac{\delta}{2}\|u'(t)\|^2 + \frac{1}{2\delta}\left\|\int_0^\infty h_\varepsilon(s)(u(t)-u(t-s))ds\right\|^2 \\
&\leq \delta\|u'(t)\|^2 + \frac{c}{\delta}(C_\varepsilon+1)\left(h_\varepsilon \circ A^{1/2}u\right)(t).
\end{aligned}
$$

Plugging the above estimates in (24), we obtain the desired estimate. □

Lemma 6. *Under the conditions (**A.1-A.3**), the functional I_2 defined by*

$$I_2(t) := \langle u'(t), u(t)\rangle$$

satisfies, along the solution of (10), *the estimate*

$$I_2'(t) \leq \|u'(t)\|^2 - \frac{l}{2}\|A^{1/2}u(t)\|^2 + cC_\varepsilon\left(h_\varepsilon \circ A^{\alpha/2}u\right)(t), \qquad \forall\, t \geq 0. \tag{25}$$

Proof. Differentiating I_2, using the equation in (10), and repeating the above computations, we get

$$
\begin{aligned}
I_2'(t) &= \|u'(t)\|^2 - \left\|A^{1/2}u(t)\right\|^2 + \left(\frac{1-l}{\omega_0}\right)\left\|A^{\alpha/2}\right\|^2 + \left\langle \int_0^\infty g(s)A^{\alpha/2}(u(s)-u(t-s))ds, A^{\alpha/2}u(t)\right\rangle \\
&\leq \|u'(t)\|^2 - l\left\|A^{1/2}u(t)\right\|^2 + \frac{l}{2\omega_0}\left\|A^{\alpha/2}u(t)\right\|^2 + \frac{\omega_0}{2l}\left\|\int_0^\infty g(s)A^{\alpha/2}((u(t)-u(t-s))ds\right\|^2 \\
&\leq \|u'(t)\|^2 - \frac{l}{2}\left\|A^{1/2}u(t)\right\|^2 + cC_\varepsilon\left(h_\varepsilon \circ A^{1/2}u\right)(t), \qquad \forall\, t \geq 0.
\end{aligned}
$$

□

Lemma 7. *Assume that (**A.1-A.3**) hold. Then, the functionals J_1 and J_2 defined by*

$$J_1(t) := \int_0^t p(t-s)\left\|A^{\alpha/2}u(s)\right\|^2 ds$$

and

$$J_2(t) := \int_0^t p(t-s)\left\|A^{1/2}u(s)\right\|^2 ds$$

with $p(t) := \int_t^\infty g(s)ds$ satisfy, along the solution of (10), *the estimates*

$$J_1'(t) \leq 3(1-l)\|A^{1/2}u(t)\|^2 - \frac{1}{2}(g\circ A^{\alpha/2}u)(t) + \frac{1}{2}\int_t^\infty g(s)\|A^{\alpha/2}u(t) - A^{\alpha/2}u(s)\|_2^2 ds,$$

and

$$J_2'(t) \leq \frac{3}{_0}(1-l)\|A^{1/2}u(t)\|^2 - \frac{1}{2}(g\circ A^{1/2}u)(t) + \frac{1}{2}\int_t^\infty g(s)\|A^{1/2}u(t) - A^{1/2}u(s)\|_2^2 ds,$$

for any $t \geq 0$.

Proof. Exploiting Young's inequality, (**A.1-A.3**), inequality (12) and the fact that $p(t) \leq p(0) = \frac{1-l}{_0}$, we obtain, for any $t \geq 0$,

$$
\begin{aligned}
J_1'(t) =& p(0)\left\|A^{\alpha/2}u(t)\right\|^2 - \int_0^t g(t-s)\left\|A^{\alpha/2}u(s)\right\|^2 ds \\
=& p(t)\left\|A^{\alpha/2}u(t)\right\|^2 - \int_0^t g(t-s)\left\|A^{\alpha/2}(u(s)-u(t))\right\|^2 ds \\
& - 2\int_0^t g(t-s)\left\langle A^{\alpha/2}u(t), A^{1/2}(u(s)-u(t))\right\rangle ds \\
\leq& p(0)\left\|A^{\alpha/2}u(t)\right\|^2 - \int_0^t g(t-s)\|A^{\alpha/2}u(t) - A^{\alpha/2}u(t-s)\|^2 ds \\
& + \frac{2}{_0}(1-l)\left\|A^{\alpha/2}u(t)\right\|^2 + \frac{1}{2}\int_0^t g(t-s)\|A^{\alpha/2}u(t) - A^{\alpha/2}u(t-s)\|^2 ds
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{3}{0}(1-l)\left\|A^{\alpha/2}u(t)\right\|^2-\frac{1}{2}\int_0^t g(t-s)\|A^{\alpha/2}u(t)-A^{\alpha/2}u(t-s)\|^2 ds\\
&\leq 3(1-l)\left\|A^{1/2}u(t)\right\|^2-\frac{1}{2}\int_0^t g(t-s)\|A^{\alpha/2}u(t)-A^{\alpha/2}u(t-s)\|^2 ds\\
&\leq 3(1-l)\left\|A^{1/2}u(t)\right\|^2-\frac{1}{2}\int_0^\infty g(t-s)\|A^{\alpha/2}u(t)-A^{\alpha/2}u(t-s)\|^2 ds\\
&\quad+\frac{1}{2}\int_t^\infty g(t-s)\|A^{\alpha/2}u(t)-A^{\alpha/2}u(t-s)\|^2 ds. \qquad (26)
\end{aligned}
$$

Similarly, differentiating J_2 and repeating the above computations, we get, for any $t\geq 0$,

$$
\begin{aligned}
J_2'(t) =&p(0)\left\|A^{1/2}u(t)\right\|^2-\int_0^t g(t-s)\left\|A^{1/2}u(s)\right\|^2 ds\\
=&p(t)\left\|A^{1/2}u(t)\right\|^2-\int_0^t g(t-s)\left\|A^{1/2}(u(s)-u(t))\right\|^2 ds\\
&-2\int_0^t g(t-s)\left\langle A^{1/2}u(t),A^{1/2}(u(s)-u(t))\right\rangle ds\\
\leq&p(0)\left\|A^{1/2}u(t)\right\|^2-\int_0^\infty g(t-s)\|A^{1/2}u(t)-A^{1/2}u(t-s)\|^2 ds\\
&+\frac{2}{0}(1-l)\left\|A^{1/2}u(t)\right\|^2+\frac{1}{2}\int_0^\infty g(t-s)\|A^{1/2}u(t)-A^{1/2}u(t-s)\|^2 ds\\
=&\frac{3}{0}(1-l)\left\|A^{1/2}u(t)\right\|^2-\frac{1}{2}\int_0^\infty g(t-s)\|A^{1/2}u(t)-A^{1/2}u(t-s)\|^2 ds\\
=&\frac{3}{0}(1-l)\left\|A^{1/2}u(t)\right\|^2-\frac{1}{2}\int_0^\infty g(t-s)\|A^{1/2}u(t)-A^{1/2}u(t-s)\|^2 ds\\
&+\frac{1}{2}\int_t^\infty g(t-s)\|A^{1/2}u(t)-A^{1/2}u(t-s)\|^2 ds. \qquad (27)
\end{aligned}
$$

□

Lemma 8. *Assume (**A.1**- **A.3**) hold. Then, the functional $\mathcal{L}$ defined by*

$$\mathcal{L}(t):=N(E(t)+\mathcal{E}(t))+\varepsilon_1 I_1(t)+\varepsilon_2 I_2(t)$$

satisfies, for a suitable choice of $N,\varepsilon_1,\varepsilon_2>0$,

$$\mathcal{L}\sim E+\mathcal{E}, \qquad (28)$$

and the estimate

$$
\begin{aligned}
\mathcal{L}'(t)\leq&-\frac{2}{l}(1-l)\left(4+\frac{3}{2\omega_0}\right)\|u'(t)\|^2-(1-l)\left(4+\frac{3}{2\omega_0}\right)\|A^{1/2}u(t)\|^2\\
&+\frac{1}{4}\left(g\circ A^{\alpha/2}u+g\circ A^{1/2}u\right)(t),\qquad \forall\, t\geq 0,
\end{aligned} \qquad (29)
$$

Proof. It is straightforward to establish the equivalence (28). To prove (29), we start by exploiting relations (17), (18), (23) and (25) to get

$$
\begin{aligned}
\mathcal{L}'(t)\quad\leq\quad&-\left[\left(\frac{1-l}{\omega_0}-\delta\right)\varepsilon_1-\varepsilon_2\right]\|u'(t)\|^2-\left(\frac{l}{2}\varepsilon_2-\delta\varepsilon_1\right)\left\|A^{1/2}u(t)\right\|^2\\
&-\left(\frac{N}{2}-\frac{c}{\delta}(\varepsilon_1+\varepsilon_2)-\frac{c}{\delta}C_\varepsilon(\varepsilon_1+\varepsilon_2)\right)\left(h_\varepsilon\circ A^{1/2}u\right)(t)\\
&-\left(\frac{N}{2}-\frac{c}{\delta}(\varepsilon_1+\varepsilon_2)-\frac{c}{\delta}C_\varepsilon(\varepsilon_1+\varepsilon_2)\right)\left(h_\varepsilon\circ A^{\alpha/2}u\right)(t)\\
&+\frac{N\varepsilon}{2}\left(g\circ A^{\alpha/2}u+g\circ A^{1/2}u\right)(t). \qquad (30)
\end{aligned}
$$

Now, we set $\beta:=\frac{1-l}{\omega_0}$ and choose δ small enough so that

$$<\min\left\{\frac{1}{2}\beta,\,\frac{l}{8}\beta\right\}.$$

Consequently, for $\varepsilon_1 = \frac{16(1-l)}{l\beta}\left(4+\frac{3}{2_0}\right)$, we pick $\varepsilon_2 = \frac{3}{8}\beta\varepsilon_1$ satisfying

$$\frac{1}{4}\beta\varepsilon_1 < \varepsilon_2 < \frac{1}{2}\beta\varepsilon_1.$$

Then,

$$(\beta -)\varepsilon_1 - \varepsilon_2 > \frac{1}{2}\beta\varepsilon_1 - \varepsilon_2 = \frac{1}{8}\beta\varepsilon_1 = \frac{2}{l}(1-l)\left(4+\frac{3}{2_0}\right)$$

and

$$\frac{l}{2}\varepsilon_2 - \delta\varepsilon_1 > \frac{l}{2}\left(\varepsilon_2 - \frac{1}{4}\beta\varepsilon_1\right) = \frac{l}{16}\beta\varepsilon_1 = (1-l)\left(4+\frac{3}{2_0}\right).$$

From $\dfrac{\varepsilon g^2(s)}{\varepsilon g(s) - g'(s)} < g(s)$ and the Lebesgue Dominated Convergence Theorem, we deduce

$$\lim_{\varepsilon\to 0^+} \varepsilon C_\varepsilon = \lim_{\varepsilon\to 0^+} \int_0^\infty \frac{\varepsilon g^2(s)}{\varepsilon g(s) - g'(s)} ds = 0.$$

So there exists $0 < \varepsilon_0 < 1$ such that if $\varepsilon < \varepsilon_0$, then

$$\varepsilon C_\varepsilon < \frac{1}{\frac{8c}{\delta}(\varepsilon_1+\varepsilon_2)}.$$

Now, we choose N large enough so that $\mathcal{L} \sim E + \mathcal{E}$ and

$$N > \max\left\{\frac{4c}{\delta}(\varepsilon_1+\varepsilon_2),\ \frac{1}{2\varepsilon_0}\right\}.$$

For $\varepsilon = \frac{1}{2N}$, we have

$$\frac{N}{4} - \frac{c}{\delta}(\varepsilon_1+\varepsilon_2) > 0 \qquad \text{and} \qquad \varepsilon < \varepsilon_0.$$

This gives

$$\frac{N}{2} - \frac{c}{\delta}(\varepsilon_1+\varepsilon_2) - \frac{c}{\delta}C_\varepsilon(\varepsilon_1+\varepsilon_2) > \frac{N}{2} - \frac{c}{\delta}(\varepsilon_1+\varepsilon_2) - \frac{1}{8\varepsilon} = \frac{N}{4} - \frac{c}{\delta}(\varepsilon_1+\varepsilon_2) > 0.$$

Thus estimate (30) becomes

$$\begin{aligned}\mathcal{L}'(t) &\leq -\frac{2}{l}(1-l)\left(4+\frac{3}{2_0}\right)\|u'(t)\|^2 - (1-l)\left(4+\frac{3}{2_0}\right)\|A^{1/2}u(t)\|^2 \\ &+\frac{1}{4}\left(g\circ A^{\alpha/2}u + g\circ A^{1/2}u\right)(t), \qquad \forall\, t \geq 0.\end{aligned}$$

□

Lemma 9. *Assume that (**A.1-A.3**) hold. Then, the energy functional satisfies, for all $t \in \mathbb{R}^+$ and for some positive constant $\tilde{m}$, the following estimate*

$$\int_0^t E(s)ds < \tilde{m} f(t), \tag{31}$$

where $f(t) = 1 + \int_0^t h(s)ds$ and $h = h_0 + h_1$ and h_0, h_1 are defined in (20) and (21).

Proof. Let $F(t) = \mathcal{L}(t) + J_1(t) + \frac{1}{2}J_2(t)$, then we obtain, for all $t \in \mathbb{R}_+$,

$$\begin{aligned}F'(t) \leq &-\frac{2}{l}(4+\frac{3}{2\omega_0})||u'(t)||^2 - (1-l)||A^1/2u(t)||^2 - \frac{1}{4}(g\circ A^1/2u)(t) \\ &+\frac{1}{2}\int_t^{+\infty} g(s)\|A^{\alpha/2}u(t) - A^{\alpha/2}u(t-s)\|^2 dsdx \\ &+\frac{1}{2}\int_t^{+\infty} g(s)\|A^{1/2}u(t) - A^{1/2}u(t-s)\|^2 dsdx.\end{aligned} \tag{32}$$

Estimates (18) and (32) yield, for some positive constant λ and for all $t \in \mathbb{R}_+$,

$$\begin{aligned} F'(t) \leq & -\lambda E(t) + \frac{1}{2}\int_t^{+\infty} g(s)\|A^{\alpha/2}u(t) - A^{\alpha/2}u(t-s)\|^2 ds dx \\ & + \frac{1}{2}\int_t^{+\infty} g(s)\|A^{1/2}u(t) - A^{1/2}u(t-s)\|^2 ds dx. \end{aligned}$$

Therefore, using (20) and (21) and integrating both sides of the last inequality, over $(0,t)$, we arrive at

$$\lambda \int_0^t E(s)ds \leq F(0) - F(t) + \frac{M_0}{2}\int_0^t h(s)ds \leq F(0) + \frac{M_0}{2}\int_0^t h(s)ds. \tag{33}$$

Hence, we get

$$\int_0^t E(s)ds \leq \frac{F(0)}{\lambda} + \frac{M_0}{2\lambda}\int_0^t h(s)ds \leq \tilde{m}\Big(1 + \int_0^t h(s)ds\Big), \tag{34}$$

where $\tilde{m} = \max\left\{\frac{F(0)}{\lambda}, \frac{M_0}{2\lambda}\right\}$. $\square$

Corollary 1. *There exists $0 < q_0 < 1$ such that, for all $t \geq 0$, we have the following estimate:*

$$\int_0^t g(s)\left(\left\|A^{\alpha/2}(u(t) - u(t-s))\right\|^2 + \left\|A^{1/2}(u(t) - u(t-s))\right\|^2\right) ds \leq \frac{1}{q(t)} G^{-1}\left(\frac{q(t)\mu(t)}{\xi(t)}\right) \tag{35}$$

where

$$\mu(t) := -\int_0^t g'(s)\left(\left\|A^{\alpha/2}(u(t) - u(t-s))\right\|^2 + \left\|A^{1/2}(u(t) - u(t-s))\right\|^2\right) ds, \tag{36}$$

$$q(t) := \frac{q_0}{f(t)}, \tag{37}$$

G is defined in Remark 4 and $f(t)$ is defined in (31).

Proof. We introduce a functional η defined by

$$\eta(t) := q(t)\int_0^t \left(\left\|A^{\alpha/2}(u(t) - u(t-s))\right\|^2 + \left\|A^{1/2}(u(t) - u(t-s))\right\|^2\right) ds, \quad \forall\, t \geq 0,$$

and observe, from inequality (12), that

$$E(t) \geq \frac{l}{2}\left\|A^{1/2}u(t)\right\|^2 \qquad \text{and} \qquad E(t) \geq \frac{l}{2_0}\left\|A^{\alpha/2}u(t)\right\|^2, \qquad \forall\, t \geq 0. \tag{38}$$

Use of (15), (17) and (38) yields

$$\begin{aligned} \eta(t) &\leq 2q(t)\int_0^t \left(\left\|A^{\alpha/2}u(t)\right\|^2 + \left\|A^{\alpha/2}u(t-s)\right\|^2 + \left\|A^{1/2}u(t)\right\|^2 + \left\|A^{1/2}u(t-s)\right\|^2\right) ds \\ &\leq \frac{4q(t)}{l}(1 + {}_0)\int_0^t (E(t) + E(t-s))ds \\ &\leq \frac{8q(t)}{l}(1 + {}_0)\int_0^t E(s)ds, \qquad \forall\, t \geq 0. \end{aligned}$$

Thanks to (31), we can pick $0 < q_0 < \min\left\{1, \frac{l}{8\tilde{m}(1+\omega_0)}\right\}$ so that

$$\eta(t) < 1, \qquad \forall\, t \geq 0. \tag{39}$$

To prove (35), we define another functional μ by

$$\begin{aligned} \mu(t) &:= -\int_0^t g'(s)\left(\left\|A^{\alpha/2}(u(t) - u(t-s))\right\|^2 + \left\|A^{1/2}(u(t) - u(t-s))\right\|^2\right) ds \\ &\leq -c\big(E'(t) + \mathcal{E}'(t)\big). \end{aligned} \tag{40}$$

Also, the strict convexity of G and the fact that $G(0) = 0$ entail that

$$G(s\tau) \leq sG(\tau), \qquad \text{for} \quad 0 \leq s \leq 1 \quad \text{and} \quad \tau \in (0, r].$$

Combining this with the hypothesis (**A.2**), Jensen's inequality and (39), we obtain, for any $t \geq 0$,

$$\begin{aligned}
\mu(t) &= -\frac{1}{\eta(t)} \int_0^t \eta(t) g'(s) \left(\left\| A^{\alpha/2}(u(t) - u(t-s)) \right\|^2 + \left\| A^{1/2}(u(t) - u(t-s)) \right\|^2 \right) ds \\
&\geq \frac{1}{\eta(t)} \int_0^t \eta(t) \xi(s) G(g(s)) \left(\left\| A^{\alpha/2}(u(t) - u(t-s)) \right\|^2 + \left\| A^{1/2}(u(t) - u(t-s)) \right\|^2 \right) ds \\
&\geq \frac{\xi(t)}{\eta(t)} \int_0^t G(\eta(t) g(s)) \left(\left\| A^{\alpha/2}(u(t) - u(t-s)) \right\|^2 + \left\| A^{1/2}(u(t) - u(t-s)) \right\|^2 \right) ds \\
&\geq \frac{\xi(t)}{q(t)} \overline{G} \left(q(t) \int_0^t g(s) \left(\left\| A^{\alpha/2}(u(t) - u(t-s)) \right\|^2 + \left\| A^{1/2}(u(t) - u(t-s)) \right\|^2 \right) ds \right),
\end{aligned}$$

where $\overline{G}$ is a C^2 extension of G which is strictly increasing and strictly convex on $(0, \infty)$. For simplicity, in the rest of this paper, we use G instead of $\overline{G}$. Then we have for any $t \geq 0$,

$$\int_0^t g(s) \left(\left\| A^{\alpha/2}(u(t) - u(t-s)) \right\|^2 + \left\| A^{1/2}(u(t) - u(t-s)) \right\|^2 \right) ds \leq \frac{1}{q(t)} G^{-1} \left(\frac{q(t)\mu(t)}{\xi(t)} \right).$$

□

5. The main result

In this section, we state and prove our decay result. We introduce the following functions:

$$G_2(t) = tG'(\varepsilon_0 t), \quad G_3(t) = tG'^{-1}(t), \quad G_4(t) = \overline{G}_3^*(t). \tag{41}$$

It is not difficult to show that the above functions are convex and increasing on $(0, r]$. Now we state our main result.

Theorem 2. *Assume that hypotheses (**A.1**)–(**A.3**) hold and the initial data satisfy*

$$(u_0, u_1) \in \left[\mathcal{D}\left(A^{1-\alpha/2}\right) \times \mathcal{D}\left(A^{(1-\alpha)/2}\right) \right] \cap \left[\mathcal{D}\left(A^{1/2}\right) \times H \right].$$

Then, for all $0 \leq s \leq t$ and for strictly positive constant C, we have the following decay results

$$E(t) \leq C \left(\frac{E(0)}{q(t)} \right) G_2^{-1} \left[\frac{C + \int_0^t \xi(s) G_4 \left[\frac{c}{d} q(s) h(s) \right] ds,}{\int_0^t \xi(s) ds} \right], \tag{42}$$

where q is defined in (37), $h = h_0 + h_1$ where h_0, h_1 are defined in (20) and (21) and the functions $G_2(s)$ and $G_4(s)$ are defined in (41).

Proof. We start by combining (15), (20), (21), (29) and (35); then, for some $m > 0$ and for any $t \geq 0$, we have

$$\mathcal{L}'(t) \leq -mE(t) + \frac{c}{q(t)} G^{-1} \left(\frac{q(t)\mu(t)}{\xi(t)} \right) + ch(t), \qquad \forall t \geq 0. \tag{43}$$

Let $0 < \varepsilon_0 < r$, then define a functional $\mathcal{F}$ by

$$\mathcal{F}(t) := G' \left(\frac{\varepsilon_0 q(t) E(t)}{E(0)} \right) \mathcal{L}(t), \qquad \forall t \geq 0.$$

Using the facts that $E' \leq 0$, $G' > 0$ and $G'' > 0$, we get for any $t \geq 0$,

$$\mathcal{F}'(t) = \frac{\varepsilon_0 q(t) E'(t)}{E(0)} G'' \left(\frac{\varepsilon_0 q(t) E(t)}{E(0)} \right) \mathcal{L}(t) + G' \left(\frac{\varepsilon_0 q(t) E(t)}{E(0)} \right) \mathcal{L}'(t)$$

$$\leq -mE(t)G'\left(\frac{\varepsilon_0 q(t)E(t)}{E(0)}\right)+\frac{c}{q(t)}G'\left(\frac{\varepsilon_0 q(t)E(t)}{E(0)}\right)G^{-1}\left(\frac{q(t)\mu(t)}{\xi(t)}\right)+ch(t)G'\left(\frac{\varepsilon_0 q(t)E(t)}{E(0)}\right). \tag{44}$$

Let G^* be the convex conjugate of G in the sense of Young (see [28]), then

$$G^*(s)=s(G')^{-1}(s)-G\left[(G')^{-1}(s)\right], \quad \text{if } s\in(0,G'(r)] \tag{45}$$

and it satisfies the following generalized Young inequality

$$AB\leq G^*(A)+G(B), \quad \text{if } A\in(0,G'(r)],\ B\in(0,r]. \tag{46}$$

So, with $A=G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)$, $B=G^{-1}\left(\frac{q(t)\mu(t)}{\xi(t)}\right)$, and using (17), (18), and (44)-(46), we arrive at

$$\begin{aligned}\mathcal{F}'(t)\leq&-mE(t)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)+\frac{c}{q(t)}G^*\left(G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\right)+c\left(\frac{\mu(t)q(t)}{\xi(t)}\right)+ch(t)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\\ \leq&-mE(t)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)+c\varepsilon_0\frac{E(t)}{E(0)}G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)+c\left(\frac{\mu(t)q(t)}{\xi(t)}\right)+ch(t)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right).\end{aligned} \tag{47}$$

So, multiplying (47) by $\xi(t)$ and using (40) and the fact that $\varepsilon_0\frac{E(t)q(t)}{E(0)}<r$, we obtain

$$\begin{aligned}\xi(t)\mathcal{F}'(t)\leq&-m\xi(t)E(t)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)+c\xi(t)\varepsilon_0\frac{E(t)}{E(0)}G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\\&+c\mu(t)q(t)+c\xi(t)h(t)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\\ \leq&-\varepsilon_0(\frac{mE(0)}{\varepsilon_0}-c)\xi(t)\frac{E(t)}{E(0)}G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)-c\big(E'(t)+\mathcal{E}'(t)\big)(t)\\&+c\xi(t)h(t)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right).\end{aligned}$$

Consequently, recalling the definition of G_2 and choosing ε_0 so that $k=(\frac{mE(0)}{\varepsilon_0}-c)>0$, we obtain, for all $t\in\mathbb{R}_+$,

$$\begin{aligned}\mathcal{F}_1'(t)\leq&-k\xi(t)\left(\frac{E(t)}{E(0)}\right)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)+c\xi(t)h(t)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\\ \leq&-k\frac{\xi(t)}{q(t)}G_2\left(\frac{E(t)q(t)}{E(0)}\right)+c\xi(t)h(t)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right),\end{aligned} \tag{48}$$

where $\mathcal{F}_1=\xi\mathcal{F}+c(E+\mathcal{E})$. Since $G_2'(t)=G'(t)+tG''(t)$, then, using the strict convexity of G on $(0,r]$, we find that $G_2'(t),G_2(t)>0$ on $(0,r]$.

Using the general Young inequality (46) for the last term in (48) with $A=G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)$ and $B=[\frac{c}{d}h(t)]$, we have for any $d>0$,

$$\begin{aligned}ch(t)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)=&\frac{d}{q(t)}\left[\frac{c}{d}q(t)h(t)\right]\left(G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\right)\\ \leq&\frac{d}{q(t)}G_3\left(G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\right)+\frac{d}{q(t)}G_3^*\left[\frac{c}{d}q(t)h(t)\right]\\ \leq&\frac{d}{q(t)}\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\left(G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\right)+\frac{d}{q(t)}G_4\left[\frac{c}{d}q(t)h(t)\right]\\ \leq&\frac{d}{q(t)}G_2\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)+\frac{d}{q(t)}G_4\left[\frac{c}{d}q(t)h(t)\right],\end{aligned} \tag{49}$$

where G_2, G_3 and G_4 are given in (41). Now, combining (48) and (49) and choosing d small enough so that $k_1=(k-d)>0$, we arrive at

$$\mathcal{F}_1'(t)\leq-k_1\frac{\xi(t)}{q(t)}G_2\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)+\frac{d\xi(t)}{q(t)}G_4\left[\frac{c}{d}q(t)h(t)\right]. \tag{50}$$

Since $E' < 0$ and $q' < 0$, then $(qE)(t)$ is decreasing function. Using this fact and since G_2 is increasing, we have, for $0 \le t \le T$,

$$G_2\left(\varepsilon_0 \frac{E(T)q(T)}{E(0)}\right) \le G_2\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) \tag{51}$$

Combining (50) with (51) and multiplying by $q(t)$, we get

$$q(t)\mathcal{F}_1'(t) + k_1\xi(t)G_2\left(\varepsilon_0 \frac{E(T)q(T)}{E(0)}\right) \le d\xi(t)G_4\left(\frac{c}{d}q(t)h(t)\right). \tag{52}$$

Since $q' < 0$, then for all $0 \le t \le T$,

$$\left(q(t)\mathcal{F}_1\right)'(t) + k_1\xi(t)G_2\left(\varepsilon_0 \frac{E(T)q(T)}{E(0)}\right) \le d\xi(t)G_4\left(\frac{c}{d}q(t)h(t)\right). \tag{53}$$

Integrating (53) over $[0,T]$ and using the fact $q(0) = q_0$, we have

$$G_2\left(\varepsilon_0 \frac{E(T)q(T)}{E(0)}\right)\int_0^T \xi(t)dt \le \frac{q_0\mathcal{F}_1(0)}{k_1} + d\int_0^T \xi(t)G_4\left(\frac{c}{d}q(t)h(t)\right)dt. \tag{54}$$

Hence,

$$G_2\left(\varepsilon_0 \frac{E(T)q(T)}{E(0)}\right) \le \left[\frac{\frac{\mathcal{F}_1(0)}{c} + d\int_0^T \xi(t)G_4\left(\frac{c}{d}q(t)h(t)\right)dt,}{\int_0^T \xi(t)dt}\right]. \tag{55}$$

Thus

$$\left(\varepsilon_0 \frac{E(T)q(T)}{E(0)}\right) \le G_2^{-1}\left[\frac{\frac{\mathcal{F}_1(0)}{c} + d\int_0^T \xi(t)G_4\left(\frac{c}{d}q(t)h(t)\right)dt,}{\int_0^T \xi(t)dt}\right], \tag{56}$$

which yields

$$E(T) \le C\left(\frac{E(0)}{q(T)}\right)G_2^{-1}\left[\frac{C + \int_0^T \xi(t)G_4\left(\frac{c}{d}q(t)h(t)\right)dt,}{\int_0^T \xi(t)dt}\right], \tag{57}$$

where $C = \max\left\{1, \frac{\mathcal{F}_1(0)}{c}, \frac{c}{d}, \frac{1}{\varepsilon_0}\right\}$. □

Example 1. Let $g(t) = \frac{a}{(1+t)^\nu}$, where $\nu > 1$ and $0 < a < \nu - 1$. In this case $\xi(t) = \nu a^{\frac{-1}{\nu}}$ and $G(t) = t^{\frac{\nu+1}{\nu}}$. Then $G'(t) = a_0 t^{\frac{1}{\nu}}$. We will discuss two cases:
Case 1: if $m_0 \le 2 + ||A^{\alpha/2}u_0 + A^{1/2}u_0||2 \le m_1$. Then we have the following:

$$\begin{cases} G_4(t) = a_1 t^{\frac{\nu+1}{\nu}}, \quad G_2(t) = a_2 t^{\frac{\nu+1}{\nu}}, \\ a_3(1+t)^{-\nu+1} \le h(t) \le a_4(1+t)^{-\nu+1}, \\ \int_0^T \xi(t)G_4\left(\frac{c_1}{d}q(t)h(t)\right)dt < +\infty, \\ G_2^{-1}\left[\frac{C+\int_0^T \xi(t)G_4\left(\frac{c_1}{d}q(t)h(t)\right)dt,}{\int_0^T \xi(t)dt}\right] \le a_5 T^{-\left(\frac{\nu}{\nu+1}\right)}, \end{cases} \tag{58}$$

$$\frac{q_0}{q(T)} \le a_6 \begin{cases} 1 + \ln(1+T), & \nu = 2; \\ 2, & \nu > 2; \\ (1+T)^{-\nu+2+r}, & 1 < \nu < 2. \end{cases} \tag{59}$$

Then

$$E(T) \le a_7 \begin{cases} \Big(1 + \ln(1+T)\Big)t^{-\left(\frac{\nu}{\nu+1}\right)}, & \nu = 2; \\ T^{-\left(\frac{\nu}{\nu+1}\right)}, & \nu > 2; \\ (1+T)^{-\left(\nu-2+\frac{\nu}{\nu+1}\right)}, & 1 < \nu < 2. \end{cases} \tag{60}$$

Thus for $\nu \ge 2$ or $\sqrt{2} < \nu < 2$ we have $\lim_{T\to+\infty} E(T) = 0$.

Case 2: if $m_0(1+t)^r \leq 2 + ||A^{\alpha/2}u_0 + A^{1/2}u_0||^2 \leq m_1(1+t)^r$, where $0 < r < \nu - 1$, then we have the following:

$$\begin{cases} a_3(1+t)^{-\nu+1+r} \leq h(t) \leq a_4(1+t)^{-\nu+1+r}, \\ \\ \int_0^T \xi(t) G_4\left(\frac{c_1}{d} q(t) h(t)\right) dt < +\infty, \end{cases} \tag{61}$$

$$\frac{q_0}{q(T)} \leq a_6 \begin{cases} 1+\ln(1+T), & \nu - r = 2; \\ 2, & \nu - r > 2; \\ (1+T)^{-\nu+2+r}, & 1 < \nu - r < 2\,. \end{cases} \tag{62}$$

Then,

$$E(T) \leq a_7 \begin{cases} \Big(1+\ln(1+T)\Big) t^{-\left(\frac{\nu}{\nu+1}\right)}, & \nu - r = 2; \\ T^{-\left(\frac{\nu}{\nu+1}\right)}, & \nu - r > 2; \\ (1+T)^{-\left(\nu-2-r+\frac{\nu}{\nu+1}\right)}, & 1 < \nu - r < 2\,. \end{cases} \tag{63}$$

Thus for $\nu - r \geq 2$ or $\frac{1}{2}\left(r + \sqrt{r^2+4r+8}\right) < \nu < r+2$ we have $\lim_{T\to+\infty} E(T) = 0$.

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References

[1] Guesmia, A. (2011). Asymptotic stability of abstract dissipative systems with infinite memory. *Journal of Mathematical Analysis and Applications, 382*(2), 748-760.

[2] Al-Mahdi, A. M., & Al-Gharabli, M. M. (2019). New general decay results in an infinite memory viscoelastic problem with nonlinear damping. *Boundary Value Problems, 2019*, Article No 140. https://doi.org/10.1186/s13661-019-1253-6.

[3] Guesmia, A., & Messaoudi, S. A. (2014). A new approach to the stability of an abstract system in the presence of infinite history. *Journal of Mathematical Analysis and Applications, 416*(1), 212-228.

[4] Youkana, A. (2018). Stability of an abstract system with infinite history. *arXiv e-prints, arXiv-1805.*

[5] Guesmia, A., & Tatar, N. E. (2015). Some well-posedness and stability results for abstract hyperbolic equations with infinite memory and distributed time delay. *Communications on Pure and Applied Analysis, 14*(2), 457-491.

[6] Dafermos, C. M. (1970). An abstract Volterra equation with applications to linear viscoelasticity. *Journal of Differential Equations, 7*(3), 554-569.

[7] Dafermos, C. M. (1970). Asymptotic stability in viscoelasticity. *Archive for Rational Mechanics and Analysis, 37*(4), 297-308.

[8] Fabrizio, M., Giorgi, C., & Pata, V. (2010). A new approach to equations with memory. *Archive for Rational Mechanics and Analysis, 198*(1), 189-232.

[9] Guesmia, A., & Messaoudi, S. A. (2014). A new approach to the stability of an abstract system in the presence of infinite history. *Journal of Mathematical Analysis and Applications, 416*(1), 212-228.

[10] Rivera, J. E. M., & Naso, M. G. (2007). Asymptotic stability of semigroups associated with linear weak dissipative systems with memory. *Journal of Mathematical Analysis and Applications, 326*(1), 691-707.

[11] Al-Mahdi, A. M., Al-Gharabli, M. M., & Kafini, M. (2019). A new general decay result for abstract evolution equation with time-dependent nonlinear dissipation. *Annali Dell'Universita'Di Ferrara, 65*(2), 201-230.

[12] Messaoudi, S. A. (2008). General decay of solutions of a viscoelastic equation. *Journal of Mathematical Analysis and Applications, 341*(2), 1457-1467.

[13] Messaoudi, S. A. (2008). General decay of the solution energy in a viscoelastic equation with a nonlinear source. *Nonlinear Analysis: Theory, Methods and Applications, 69*(8), 2589-2598.

[14] Mustafa, M. I. (2018). General decay result for nonlinear viscoelastic equations. *Journal of Mathematical Analysis and Applications, 457*(1), 134-152.

[15] Rivera, J. E. M., Naso, M. G., & Vegni, F. M. (2003). Asymptotic behavior of the energy for a class of weakly dissipative second-order systems with memory. *Journal of Mathematical Analysis and Applications, 286*(2), 692-704.

[16] Al-Gharabli, M. M., Al-Mahdi, A. M., & Messaoudi, S. A. (2019). General and optimal decay result for a viscoelastic problem with nonlinear boundary feedback. *Journal of Dynamical and Control Systems, 25*(4), 551-572.

[17] Al-Mahdi, A. M. (2019). Optimal decay result for Kirchhoff plate equations with nonlinear damping and very general type of relaxation functions. *Boundary Value Problems, 2019*, Article No 82. https://doi.org/10.1186/s13661-019-1196-y.

[18] Al-Gharabli, M. M., Al-Mahdi, A. M., & Messaoudi, S. A. (2020). Decay results for a viscoelastic problem with nonlinear boundary feedback and logarithmic source term. *Journal of Dynamical and Control Systems, 28*, 71-89.

[19] Hassan, J. H., & Messaoudi, S. A. (2019). General decay rate for a class of weakly dissipative second-order systems with memory. *Mathematical Methods in the Applied Sciences, 42*(8), 2842-2853.

[20] Appleby, J. A., Fabrizio, M., Lazzari, B., & Reynolds, D. W. (2006). On exponential asymptotic stability in linear viscoelasticity. *Mathematical Models and Methods in Applied Sciences, 16*(10), 1677-1694.

[21] Pata, V. (2009). Stability and exponential stability in linear viscoelasticity. *Milan Journal of Mathematics, 77*(1), 333-360.

[22] Mustafa, M. I. (2019). Energy decay in a quasilinear system with finite and infinite memories. In *Mathematical Methods in Engineering* (pp. 235-256). Springer, Cham.

[23] Al-Mahdi, A. M. (2020). Stability result of a viscoelastic plate equation with past history and a logarithmic nonlinearity. *Boundary Value Problems, 2020*, Article No 84. https://doi.org/10.1186/s13661-020-01382-9.

[24] Al-Mahdi, A. M. (2020). General stability result for a viscoelastic plate equation with past history and general kernel. *Journal of Mathematical Analysis and Applications, 490*(1), Article ID: 124216. https://doi.org/10.1016/j.jmaa.2020.124216.

[25] Al-Mahdi, A. M., Al-Gharabli, M. M., & Messaoudi, S. A. (2021). New general decay result for a system of viscoelastic wave equations with past history. *Communications on Pure and Applied Analysis, 20*(1), 389–404.

[26] Guesmia, A. (2020). New general decay rates of solutions for two viscoelastic wave equations with infinite memory. *Mathematical Modelling and Analysis, 25*(3), 351-373.

[27] Jin, K. P., Liang, J., & Xiao, T. J. (2014). Coupled second order evolution equations with fading memory: optimal energy decay rate. *Journal of Differential Equations, 257*(5), 1501-1528.

[28] Arnol'd, V. I. (2013). *Mathematical Methods of Classical Mechanics* (Vol. 60). Springer Science & Business Media.