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Article

Strong and periodic solutions of Navier-Stokes equations, in 2D, with non-local viscosity

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Abstract: In this article we study the existence of periodic and strong solutions of Navier-Stokes equations, in two dimensions, with non-local viscosity.

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1. Introduction

Consider the following initial-boundary value problem for the Navier-Stokes equations in two dimensions with non-local viscosity. It means, find a vector function

$$\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^2,$$

and a scalar function

$$p : \Omega \times [0, T] \to \mathbb{R},$$

satisfying

$$\frac{d\boldsymbol{u}(x,t)}{dt} - c(l(u_1(x,t)), l(u_2(x,t)))\Delta \boldsymbol{u}(x,t) + (\boldsymbol{u}(x,t)\cdot\nabla)\boldsymbol{u}(x,t) + \nabla p(x) = \boldsymbol{f}(x,t) \quad \text{in } \Omega \times (0,T), \tag{1}$$

$$div(\boldsymbol{u}(x)) = 0 \quad \text{on } \Omega, \tag{2}$$

$$\boldsymbol{u}(x,t) = \boldsymbol{g} \quad \text{on } \partial\Omega, \tag{3}$$

$$\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x) \quad \text{in } \Omega, \tag{4}$$

where Ω is a domain sufficiently regular, $\partial\Omega$ its boundary well regular, and we have that $c(l(u_1(x,t)), l(u_2(x,t)))$ satisfies these hypotheses: Given $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$,

(A1) $0 < c_- \le c(x_1, x_2) \le c_+$,
(A2) $|c(x) - c(y)| \le A_1|x_1 - y_1| + A_2|x_2 - y_2|$, for some $A_1, A_2 > 0$,

and $l : L^2(\Omega) \to \mathbb{R}$ is a continuous linear functional, defined by $u \mapsto \int_\Omega u d\Omega$.

We mentioned that the existence, uniqueness and exponential decay of the solution to the problem (1)-(4) were studied by Ferreira, Shahrouzi, Andrade and Panni in [9].

The motivation to study this kind of problem is we can describe motion of fluids which viscosity depends of time and satisfies the hypotheses $(A1) - (A2)$, and, when $c(l(u_1(x,t)), l(u_2(x,t))) = \mu$, constant, we obtain the regular Navier-Stokes equations in two dimensions. This non-local term was introduced by Chipot [1], and it arrives naturally when we study the growth of a bacteria population, one kind of this problem was suggested by Ladyzhenskaya [2] where $c(t) := \mu_0 + \mu_1\|\boldsymbol{u}(t)\|^2$ when μ_0 and μ_1 are positive constants.

The rest of the paper is organized as follows. In §2, we recall some notations, the weak formulation, lemmas and theorems. In §3, we study the existence of strong solutions to the problem (1)-(4). In §4, we introduce the existence of periodic solutions using Brouwer's fixed point theorem. The conclusions of the paper are presented in §5.

2. Preliminaries

In this section we introduce some notations, achieve weak formulation and enunciate some important results.

2.1. Notations

Let $\Omega \subset \mathbb{R}^2$ be a regular domain with $\partial\Omega$ a well regular boundary. We denote the inner product in $H_0^1(\Omega)$ by $((\cdot,\cdot))$ and $(\cdot,\cdot)$ in $L^2(\Omega)$, and norms respectively by $\|\cdot\|$ and $|\cdot|$. By $\boldsymbol{H}_0^1(\Omega)$ we denote $(H_0^1(\Omega))^2$ and, $\boldsymbol{L}^2(\Omega)$ by $(L(\Omega))^2$. The set $\mathfrak{V}$ is the set of all distributions $\boldsymbol{u} : (\mathcal{D}(\Omega))^2 \mapsto \mathbb{R}^2$, which its divergent is null, in other words, $\mathfrak{V}(\Omega) := \{\boldsymbol{u} \in (\mathcal{D}(\Omega))^2; div(\boldsymbol{u}) = 0\}$. Also, we denote the closure of $\mathfrak{V}(\Omega)$ in $\boldsymbol{H}_0^1(\Omega)$ by $\boldsymbol{V}$ and the closure of $\mathfrak{V}(\Omega)$ in $(L(\Omega))^2$ by H.

A well known propriety of non-local term, see [3–7], is that this term commutes with spatial integral sign

$$\int_\Omega c(l(u_1), l(u_2))\boldsymbol{u} d\Omega = c(l(u_1), l(u_2)) \int_\Omega \boldsymbol{u} d\Omega.$$

2.2. Weak formulation

Consider $\boldsymbol{v} \in \boldsymbol{V}$. Doing inner-product in $L^2(\Omega)$ with Eq. (1) we get,

$$\left(\frac{d\boldsymbol{u}(x,t)}{dt}, \boldsymbol{v}(x)\right) - c(l(u_1), l(u_2))(\nabla \boldsymbol{u}(x,t), \boldsymbol{v}(x)) + ((\boldsymbol{u}(x,t)\cdot\nabla)\boldsymbol{u}(x,t), \boldsymbol{v}(x)) + (\nabla p(x), \boldsymbol{v}(x)), = (\boldsymbol{f}(x,t), \boldsymbol{v}).$$

By green first identity and integration by parts,

$$\frac{d}{dt}(\boldsymbol{u}(x,t), \boldsymbol{v}(x)) + c(l(u_1), l(u_2))((\boldsymbol{u}(x,t), \boldsymbol{v}(x))) + ((\boldsymbol{u}(x,t)\cdot\nabla)\boldsymbol{u}(x,t), \boldsymbol{v}(x)) = (\boldsymbol{f}(x,t), \boldsymbol{v}(x)).$$

Now we define a bilinear form $a(\boldsymbol{u}, \boldsymbol{v}) := ((\boldsymbol{u}, \boldsymbol{v}))$ and a trilinear form $b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) := ((\boldsymbol{u}\cdot\nabla)\boldsymbol{v}, \boldsymbol{w})$, and then we obtain the weak form of Eq. (1),

$$\frac{d}{dt}(\boldsymbol{u}, \boldsymbol{v}) + c(l(u_1), l(u_2))a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}).$$

2.3. Some results

Lemma 1. [8] *Let $\boldsymbol{u} \in \boldsymbol{L}^2(0, T; \boldsymbol{V})$, then the function $B\boldsymbol{u}$ defined by,*

$$\langle B\boldsymbol{u}(t), \boldsymbol{v}\rangle := b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}), \forall \boldsymbol{v} \in \boldsymbol{V}, \text{for a.e. } t \in [0, T],$$

belongs to $\boldsymbol{L}^1(0, T; \boldsymbol{V}')$.

Lemma 2. *Let $\Omega \subset \mathbb{R}^n$ be an bounded Lipschitz open set in $\mathbb{R}^2$.*

1. *If a distribution p has all its first-order derivatives $D_i p$ in $L^2(\Omega)$, then $p \in L^2(\Omega)$ and*

$$\|p\|_{L^2(\Omega)\backslash\mathbb{R}} \le c(\Omega)|\nabla p|_{L^2(\Omega)}.$$

2. *If a distribution p has all its first-order derivatives in $H^{-1}(\Omega)$. Then $p \in L^2(\Omega)$ and*

$$\|p\|_{L^2(\Omega)\backslash\mathbb{R}} \le c\|\nabla p\|_{H^{-1}(\Omega)},$$

where $L^2(\Omega)\backslash\mathbb{R} := \left\{p \in L^2(\Omega) \middle| \int_\Omega p(x)dx = 0\right\}$.

Problem 1. For $\boldsymbol{f}$ and $\boldsymbol{u}_0$ given, with

$$\boldsymbol{f} \in L^2(0,T;\boldsymbol{V}'), \tag{5}$$
$$\boldsymbol{u}_0 \in H, \tag{6}$$

to find $\boldsymbol{u}$ satisfying,

$$\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}), \boldsymbol{u}' \in L^1(0,T;\boldsymbol{V}'),$$

$$\frac{d}{dt}(\boldsymbol{u}(x,t),\boldsymbol{v}(x)) + c(l(u_1),l(u_2))((\boldsymbol{u}(x,t),\boldsymbol{v}(x))) + ((\boldsymbol{u}(x,t)\cdot\nabla)\boldsymbol{u}(x,t),\boldsymbol{v}(x)) = (\boldsymbol{f}(x,t),\boldsymbol{v}(x)),$$

for any $\boldsymbol{v} \in \boldsymbol{V}$.

3. Existence of strong solutions

Suppose the existence of weak solutions to the problem (1)-(4). Our goal in this section is recover the pressure and prove the existence of strong solutions.

Theorem 2. *Given $\boldsymbol{f}$ and $\boldsymbol{u}_0$ satisfying (5) and (6). Suppose that $\boldsymbol{u}$ is a solution of the Problem 1 and*

$$\boldsymbol{f} - c(l(u_1),(u_2))A\boldsymbol{u} - B\boldsymbol{u} - \boldsymbol{u}' \in L^2(0,T;V'),$$

then the solution $\boldsymbol{u}$ is also strong.

Proof. Let,

$$\boldsymbol{U}(t) := \int_0^t \boldsymbol{u}(s)ds, \boldsymbol{F}(t) := \int_0^t \boldsymbol{f}(s)ds \quad \text{and } \beta(t) := \int_0^t B(\boldsymbol{u}(s),\boldsymbol{u}(s))ds \in V'.$$

Since $\boldsymbol{u}, \boldsymbol{f}, B\boldsymbol{u} \in L^2(0,T;V')$ then,

$$\boldsymbol{U}, \boldsymbol{F} \quad \text{and } \beta \in C^0(0,T;V') \text{ these are absolute continuous.} \tag{7}$$

Integrating $c(l(u_1),(u_2))A\boldsymbol{u} + B\boldsymbol{u} + \boldsymbol{u}' = \boldsymbol{f}$, and, by (7), we get

$$\boldsymbol{u}(t) - \boldsymbol{u}(0) + c(l(u_1),l(u_2))\int_0^t A\boldsymbol{u}(s)ds + \int_0^t B\boldsymbol{u}(s)ds = \int_0^t \boldsymbol{f}(s)ds \text{ in V}'.$$

Then,

$$\boldsymbol{u}(t) - \boldsymbol{u}_0 + c(l(u_1),l(u_2))A\boldsymbol{U}(t) + \beta(t) = \boldsymbol{F}(t) \text{ in } V', \forall t \in [0,T].$$

So, for each $\phi \in \mathfrak{V}$,

$$\langle \boldsymbol{u}(t) - \boldsymbol{u}(0) + c(l(u_1),l(u_2))A\boldsymbol{U}(t) + \beta(t) - \boldsymbol{F}(t), \phi\rangle = 0. \tag{8}$$

Define,

$$\boldsymbol{S}(t) := \boldsymbol{u}(t) - \boldsymbol{u}_0 + c(l(u_1),l(u_2))A\boldsymbol{U}(t) + \beta(t) - \boldsymbol{F}(t) \in V'. \tag{9}$$

For each $t \in [0,T]$ it is possible to extend $\boldsymbol{S}(t)$ on a functional $\boldsymbol{T}(t) \in H^{-1}(\Omega)$ such as,

$$\langle \boldsymbol{T}(t), \boldsymbol{v}\rangle = \langle \boldsymbol{S}(t), \boldsymbol{v}\rangle, \forall \boldsymbol{v} \in V. \tag{10}$$

But, from (8) and (10) we can conclude that,

$$\langle \boldsymbol{T}(t), \boldsymbol{\phi}\rangle = 0, \forall \boldsymbol{\phi} \in \mathfrak{V}.$$

From Lemma 2 results that $\exists P(t) \in L^2(\Omega)$ satisfying,

$$\boldsymbol{T}(t) = \nabla P(t) \text{ in } H^{-1}(\Omega). \tag{11}$$

So, from (10) and (11) we get,

$$\nabla P(t)\big|_V \equiv \boldsymbol{S}(t) \text{ in } V', \forall t \in [0,T]. \tag{12}$$

Replacing (12) in (9),

$$\boldsymbol{u}(t) - \boldsymbol{u}_0 + c(l(u_1), l(u_2))A\boldsymbol{U}(t) + \beta(t) - \boldsymbol{F}(t) = \nabla P(t) \text{ in } V', \forall v \in [0,T].$$

As the expression on the left belongs to the space $C^0(0,T;V')$ we have $\nabla P \in C^0(0,T;V')$, and hence we can derive the above equation in the sense of distributions, with this:

$$\boldsymbol{u}' + c(l(u_1), l(u_2))A\boldsymbol{u} - \boldsymbol{f} + B\boldsymbol{u} = \nabla \frac{\partial P}{\partial t} \text{ in } L^2(0,T;V').$$

Therefore is possible to say that equality above is given a.e. in $(0,T)$. Setting $p(x,t) = -\dfrac{\partial P}{\partial t}$, results in,

$$\boldsymbol{u}' + c(l(u_1), l(u_2))A\boldsymbol{u} + B\boldsymbol{u} = \boldsymbol{f} - \nabla p \in L^2(0,T;V').$$

□

4. Existence of periodic solutions

The purpose of this section is to prove the existence of periodic solutions to the Navier-Stokes equations.

Theorem 3. *Let $\Omega \subset \mathbb{R}^2$ a bounded open set with boundary $\partial\Omega$ well regular and $Q := [0,T] \times \Omega$. Consider the following problem,*

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - c(l(u_1), l(u_2))\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\Delta)\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } Q, \\ div(\boldsymbol{u}) = 0 & \text{in } Q, \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}(x,T), \forall x \in \Omega, \end{cases} \tag{13}$$

where $\boldsymbol{f} \in L^2(0,T;V')$. This problem admits weak solution in $\boldsymbol{u} : Q \to \mathbb{R}^2$, $\boldsymbol{u} \in L^2(0,T;H) \cap L^\infty(0,T;H)$ and $\boldsymbol{u}' \in L^2(0,T;V')$.

Proof. The weak formulation of (13) is given by,

$$\begin{cases} \langle \boldsymbol{u}'(t), \boldsymbol{v}\rangle + c(l(u_1), l(u_2))((\boldsymbol{u}(t), \boldsymbol{v})) + b(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{v}) = \langle \boldsymbol{f}(t), \boldsymbol{v}\rangle \text{ in } \mathcal{D}'(0,T), \forall \boldsymbol{v} \in V, \\ \boldsymbol{u}(0) = \boldsymbol{u}(T). \end{cases} \tag{14}$$

Consider $\{\boldsymbol{w}_1, \cdots, \boldsymbol{w}_n, \cdots\}$ a base of V. We truncate the series in m-th term, which leads to the approximate solution space V_m. Setting $\boldsymbol{u}_m(t) := g_{im}(t)\boldsymbol{w}_i$,

$$\begin{cases} (\boldsymbol{u}_m'(t), \boldsymbol{w}_j) + c(l(u_1), l(u_2))((\boldsymbol{u}_m(t), \boldsymbol{w}_j)) + b(\boldsymbol{u}_m(t), \boldsymbol{u}_m(t), \boldsymbol{w}_j) = \langle \boldsymbol{f}(t), \boldsymbol{w}_j\rangle, \\ \boldsymbol{u}_m(0) = \boldsymbol{v} \in V_m, \end{cases} \tag{15}$$

where $j = 1, \cdots, m$.

The approximate system above has a global solution, since by similar procedure to the case of the existence of solutions [9], we obtain the following inequality,

$$|\boldsymbol{u}(t)|^2 + \int_0^t \|\boldsymbol{u}_m(s)\|^2 ds \le |\boldsymbol{v}| + \frac{1}{c_-}\|\boldsymbol{f}\|_{L^2(0,T;V')} \le c(m),$$

as m is fixed, we can extend $\boldsymbol{u}(t)$ in $[0, T]$. Our goal is to show that, among all solutions of the approximate equation, there is at least one $\boldsymbol{u}_m$ solution that satisfies periodicity,

$$\boldsymbol{u}_m(0) = \boldsymbol{u}_m(T).$$

To do this, just prove that for every $m \in \mathbb{N}$, the application,

$$\begin{aligned} \boldsymbol{\tau}_m &: V_m \to V_m \\ \boldsymbol{v} &\mapsto \boldsymbol{\tau}_m(\boldsymbol{v}) = \boldsymbol{u}_m(T), \end{aligned}$$

has a single fixed point, because in this case there will be a single function $\boldsymbol{v} \in V_m$ such that

$$\boldsymbol{u}_m(T) = \boldsymbol{\tau}_m(\boldsymbol{v}) = \boldsymbol{v} = \boldsymbol{u}_m(0), \quad \forall m \in \mathbb{N}. \tag{16}$$

Thus (16) we have a $(\boldsymbol{u}_m)$ sequence of approximate solutions such that they all satisfy the periodicity condition.

Lemma 3. *Exists $\rho_0 > 0$ such as $\boldsymbol{\tau}_m\overline{(B_{\rho_0}(0))} \subset \overline{B_{\rho_0}(0)}$.*

Proof. Using the H induced topology in V_m, it suffices to prove that

$$\exists \quad \rho_0 > 0 \text{ such that } |\boldsymbol{\tau}_m(\boldsymbol{v})|_H \leq \rho_0; \forall v \in V_m, \text{ where } |\boldsymbol{v}|_H \leq \rho_0.$$

Applying the energy method,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}_m(t)|^2 + c_-\|\boldsymbol{u}_m(t)\|^2 + b(\boldsymbol{u}_m(t), \boldsymbol{u}_m(t), \boldsymbol{u}_m(t)) \\ &\qquad \leq \frac{1}{2}\frac{d}{dt}|\boldsymbol{u}_m(t)|^2 + c(l(u_1), l(u_2))\|\boldsymbol{u}_m(t)\|^2 + b(\boldsymbol{u}_m(t), \boldsymbol{u}_m(t), \boldsymbol{u}_m(t)) \\ &\qquad = \langle \boldsymbol{f}(t), \boldsymbol{u}_m(t) \rangle \\ &\qquad \leq \|\boldsymbol{f}(t)\|_{V'}|\boldsymbol{u}_m(t)|, \end{aligned}$$

implies that,

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}_m(t)|^2 + c_-\|\boldsymbol{u}_m(t)\|^2 \leq \frac{1}{2c_-}\|\boldsymbol{f}(t)\|_{V'}^2 + \frac{c_-}{2}\|\boldsymbol{u}_m(t)\|^2,$$

then,

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}_m(t)|^2 + \frac{c_-}{2}\|\boldsymbol{u}_m(t)\|^2 \leq \frac{1}{c_-}\|\boldsymbol{f}(t)\|_{V'}^2. \tag{17}$$

As $V \hookrightarrow H$, exists $c_0 > 0$ such as,

$$c_0^2|\boldsymbol{u}_m(t)|^2 \leq \|\boldsymbol{u}_m(t)\|^2. \tag{18}$$

Thus from (17) and (18) we get,

$$\frac{d}{dt}|\boldsymbol{u}_m(t)|^2 + c_0^2 c_-|\boldsymbol{u}_m(t)|^2 \leq \frac{1}{c_-}\|\boldsymbol{f}(t)\|_{V'}^2.$$

Multiplying both sides by $e^{c_0^2 c_- t}$:

$$\frac{d}{dt}(|\boldsymbol{u}_m(t)|^2 e^{c_0^2 c_- t}) \leq \frac{1}{c_-}\|\boldsymbol{f}(t)\|_{V'}^2 e^{c_0^2 c_- t}.$$

Integrating from 0 to T we get,

$$|\boldsymbol{u}_m(t)|^2 e^{c_0^2 c_- t} \leq |\boldsymbol{u}_m(0)|^2 + \frac{1}{c_-}\int_0^T \|\boldsymbol{f}(t)\|_{V'}^2 e^{c_0^2 c_- t} dt,$$

which means,

$$|\boldsymbol{u}_m(t)|^2 \leq e^{-c_0^2 c_- T}|\boldsymbol{u}_m(0)|^2 + \frac{1}{c_-}\int_0^T \|\boldsymbol{f}(t)\|_{V'}^2 dt,$$

then,

$$|\boldsymbol{u}_m(t)|^2 \leq e^{-c_0^2 c_- T}|\boldsymbol{u}_m(0)| + \frac{1}{c_-}\|\boldsymbol{f}\|_{L^2(0,T;V')}^2.$$

Denoting $\theta = e^{-c_0^2 c_- T}$ and $c = \frac{1}{c_-}\|\boldsymbol{f}\|_{L^2(0,T;V')}^2$, we can write

$$|\boldsymbol{u}_m(t)|^2 \leq \theta|\boldsymbol{u}_m(0)|^2 + c,$$

so,

$$|\boldsymbol{\tau}_m(\boldsymbol{v})|^2 \leq \theta|\boldsymbol{v}|^2 + c, \quad \forall v \in V_m.$$

Now, how $0 < \theta < 1$ then $0 < 1-\theta < 1$. That way there is a $\rho_0 > 0$, big enough that $c < (1-\theta)\rho_0^2$. So if $|\boldsymbol{v}| < \rho_0$ then,

$$\theta|\boldsymbol{v}|^2 + c \leq \theta\rho_0^2 + (1-\theta)\rho_0^2 = \rho_0^2,$$

where,

$$|\boldsymbol{\tau}_m(\boldsymbol{v})|^2 \leq \rho_0^2, \quad \forall m \in \mathbb{N},$$

which proves this lemma. □

Lemma 4. *The application $\boldsymbol{\tau}_m : V_m \mapsto V_m$ defined in (15) is continuous.*

Proof. Let $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V_m$ and $\boldsymbol{u}_m, \boldsymbol{z}_m$ solutions of the approximate problem with initial data $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, respectively. Our goal is to show that the solutions are Lipschitz-continuous, $|\boldsymbol{\tau}_m(\boldsymbol{v}_1) - \boldsymbol{\tau}_m(\boldsymbol{v}_2)| \leq c_m|\boldsymbol{v}_1 - \boldsymbol{v}_2|$ for some $c_m > 0$.

$$\begin{aligned}(\boldsymbol{u}_m'(t), \boldsymbol{w}_j) + c(l(u_1), l(u_2))((\boldsymbol{u}_m(t), \boldsymbol{w}_j)) + b(\boldsymbol{u}_m(t), \boldsymbol{u}_m(t), \boldsymbol{w}_j) = \langle \boldsymbol{f}(t), \boldsymbol{w}_j\rangle,\\ (\boldsymbol{z}_m'(t), \boldsymbol{w}_j) + c(l(z_1), l(z_2))((\boldsymbol{z}_m(t), \boldsymbol{w}_j)) + b(\boldsymbol{z}_m(t), \boldsymbol{z}_m(t), \boldsymbol{w}_j) = \langle \boldsymbol{f}(t), \boldsymbol{w}_j\rangle.\end{aligned}$$

Doing the difference between these equations and defining $\boldsymbol{\eta}_m = \boldsymbol{z}_m - \boldsymbol{u}_m$,

$$\begin{aligned}(\boldsymbol{\eta}_m, \boldsymbol{w}_j) + c(l(u_1), l(u_2))((\boldsymbol{u}_m(t), \boldsymbol{w}_j)) - a(l(z_1), l(z_2))((\boldsymbol{z}_m(t), \boldsymbol{w}_j))\\ + b(\boldsymbol{u}_m(t), \boldsymbol{u}_m(t), \boldsymbol{w}_j) - b(\boldsymbol{z}_m(t), \boldsymbol{z}_m(t), \boldsymbol{w}_j) = 0,\end{aligned}$$

we proceed as in [9]

$$\frac{d|\boldsymbol{\eta}_m|^2}{dt} - |\boldsymbol{\eta}_m|^2\left(\frac{2}{c_-}\|u_{2m}(t)\|^2 + \frac{K^2}{c_-}\|\boldsymbol{z}_m\|^2\right) \leq 0.$$

Defining $\theta_m(t) = \left(\frac{2}{c_-}\|u_{2m}(t)\|^2 + \frac{K^2}{c_-}\|\boldsymbol{z}_m\|^2\right)$, we get,

$$\frac{d|\boldsymbol{\eta}_m|^2}{dt} - |\boldsymbol{\eta}_m|^2\theta_m(t) \leq 0.$$

Multiplying both sides of inequality $e^{-\int_0^t \theta_m(s)ds}$,

$$\frac{d}{dt}\left(|\boldsymbol{\eta}_m(t)|^2 e^{-\int_0^t \theta_m(s)ds}\right) \leq 0.$$

Integrating the inequality from 0 to T,

$$|\boldsymbol{\eta}_m(T)|^2 e^{-\int_0^t \theta_m(s)ds} - |\boldsymbol{\eta}_m(0)|^2 \leq 0.$$

Defining $c_m = e^{-\int_0^t \theta_m(s)ds}$,

$$|\boldsymbol{\eta}_m(T)|^2 \leq c_m|\boldsymbol{\eta}_m(0)|^2.$$

By other hand,

$$\boldsymbol{\eta}_m(s) = \boldsymbol{u}_m(s) - \boldsymbol{z}_m(s),$$

so,

$$|\boldsymbol{u}_m(T) - \boldsymbol{z}_m(T)|^2 \leq c_m |\boldsymbol{u}_m(0) - \boldsymbol{z}_m(0)|^2.$$

Then,

$$|\boldsymbol{\tau}_m(\boldsymbol{v}_1) - \boldsymbol{\tau}_m(\boldsymbol{v}_2)| \leq c_m |\boldsymbol{v}_1 - \boldsymbol{v}_2|,$$

which is what we want to prove. □

The hypotheses of Brouwer's fixed point theorem are satisfied by virtue of Lemmas 3 and 4, so we have

$$\boldsymbol{\tau}_m : \overline{B_{\rho_0}(0)} \to \overline{B_{\rho_0}(0)},$$

admits a fixed point, which means, there is a $\boldsymbol{v} \in \overline{B_{\rho_0}(0)}$ such as $\boldsymbol{\tau}_m(\boldsymbol{v}) = \boldsymbol{v}$, so, $\boldsymbol{u}_m(0) = \boldsymbol{u}_m(T)$.

Then, for each $m \in \mathbb{N}$, there is a least one $\boldsymbol{u}_m(t)$ such as $\boldsymbol{u}_m(0) \in \overline{B_{\rho_0}(0)}$ and, $\forall j = 1, \cdots, m$,

$$\begin{cases} (\boldsymbol{u}_m'(t), \boldsymbol{w}_j) + c(l(u_1), l(u_2))((\boldsymbol{u}_m(t), \boldsymbol{w}_j)) + b(\boldsymbol{u}_m(t), \boldsymbol{u}_m(t), \boldsymbol{w}_j) = \langle \boldsymbol{f}(t), \boldsymbol{w}_j \rangle, \\ \boldsymbol{u}_m(0) = \boldsymbol{u}_m(t). \end{cases}$$

From the fact that $\boldsymbol{u}_m(0) \in \overline{B_{\rho_0}(0)}$ we can repeat the estimates getting a subsequence $(\boldsymbol{u}_\nu)$ of $(\boldsymbol{u}_m)$ such as

$$\boldsymbol{u}_\nu \overset{\star}{\rightharpoonup} \boldsymbol{u} \text{ in } L^\infty(0, T; H), \tag{19}$$

$$\boldsymbol{u}_\nu \to \boldsymbol{u} \text{ in } L^2(0, T; V), \tag{20}$$

$$\boldsymbol{u}_\nu' \to \boldsymbol{u}' \text{ in } L^2(0, T; V'). \tag{21}$$

From the convergence results (19) - (21), by passing the limit in the approximate equation desired in (14). Similarly to the proof of the initial condition in the previous case, we prove that $\boldsymbol{u}(0) = \boldsymbol{u}(T)$, which concludes the statement. □

5. Conclusions

We studied the Navier-Stokes equations with non-local viscosity, considering a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega$. Using Faedo-Galerkin's method and Brouwer's fixed point theorem, we proved the strong solutions and periodic solutions.

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