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The q-Legendre inversions and balanced q-series identities

Xiaojing Chen[1], and Wenchang Chu[2,3,*]

[1] School of Statistics, Qufu Normal University, Qufu (Shandong), China.
[2] School of Mathematics and Statistics, Zhoukou Normal University (Henan), China.
[3] Department of Mathematics and Physics, University of Salento, Lecce 73100, Italy.
* Correspondence: chu.wenchang@unisalento.it

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Abstract: Two terminating balanced ${}_4\phi_3$-series identities are established by applying the bilateral q-Legendre inversions. Four variants of them are obtained by means of contiguous relations. According to the polynomial argument, four "dual" formulae for balanced ${}_4\phi_3$-series are deduced, that lead also to four non–terminating ${}_2\phi_2$-series identities.

Keywords: The q-Legendre inversions, balanced q-series, well–poised q-series, Bailey's identity for ${}_6\psi_6$-series..

MSC: 33D15, 05A30.

1. Introduction

Let $\mathbb{N}$ be the set of natural numbers with $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. For an indeterminate x, the shifted factorial of order $n \in \mathbb{N}_0$ with the base q is defined by $(x;q)_0 = 1$ and

$$(x;q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x) \quad \text{for} \quad n \in \mathbb{N}.$$

Then the Gaussian binomial coefficient can be expressed as

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{(q;q)_m}{(q;q)_n (q;q)_{m-n}} = \frac{(q^{m-n+1};q)_n}{(q;q)_n} \quad \text{where} \quad m, n \in \mathbb{N}.$$

Throughout the paper, the product and quotient of shifted factorials will be abbreviated respectively to

$$\begin{aligned} [\alpha, \beta, \cdots, \gamma; q]_n &= (\alpha;q)_n (\beta;q)_n \cdots (\gamma;q)_n, \\ \left[\begin{matrix} \alpha, \beta, \cdots, \gamma \\ A, B, \cdots, C \end{matrix} \,\middle|\, q\right]_n &= \frac{(\alpha;q)_n (\beta;q)_n \cdots (\gamma;q)_n}{(A;q)_n (B;q)_n \cdots (C;q)_n}. \end{aligned}$$

Following Gasper and Rahman [1], the unilateral and bilateral basic hypergeometric series (shortly as q-series) are defined, respectively, by

$$\begin{aligned} {}_{1+r}\phi_s \left[\begin{matrix} a_0, a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} \,\middle|\, q; z\right] &= \sum_{k=0}^{\infty} z^k \left[\begin{matrix} a_0, a_1, \cdots, a_r \\ q, b_1, \cdots, b_s \end{matrix} \,\middle|\, q\right]_k \left\{(-1)^k q^{\binom{k}{2}}\right\}^{s-r}, \\ {}_r\psi_s \left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix} \,\middle|\, q; z\right] &= \sum_{k=-\infty}^{\infty} z^k \left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix} \,\middle|\, q\right]_k \left\{(-1)^k q^{\binom{k}{2}}\right\}^{s-r}. \end{aligned}$$

When $r = s$, the ${}_{1+r}\phi_r$-series will be called balanced (or Saalschützian) if the product of the denominator parameters exceeds that of the numerator parameters by q. Both series will be said well–poised if $r = s$ and the parameters in the numerator can be paired off with those in the denominator so that each pair has the same product.

During the past decades, there has been growing interest in finding new formulae for basic hypergeometric series. The inversion techniques have been shown to be powerful in dealing with q-series identities (cf. [2–7]). This paper will explore further applications of the q-Legendre inversions established by Chu and Wang [8].

In the next section, we shall derive two main formulae for terminating balanced ${}_4\phi_3$-series from two particular cases of Bailey's very well–poised ${}_6\psi_6$-series identity. Then four variants of these balanced series will be evaluated by means of contiguous relations. The polynomial argument will be employed in the third section, where four further formulae for terminating balanced ${}_4\phi_3$-series will be demonstrated. Their limiting series will also be highlighted as four remarkable nonterminating ${}_2\phi_2$-series identities.

2. Closed formulae for terminating ${}_4\phi_3$-series

Chu and Wang [8] found the following bilateral form of the q-Legendre inversions, that were utilized by them to derive, from a few known ones, several terminating balanced as well as well–poised q-series identities.

Lemma 1 (Chu–Wang [8]Proposition 3)**.** *Let λ be a fixed nonnegative integer. Then for all $n \in \mathbb{N}_0$, there hold the following inverse series relations*

$$\mathcal{F}(n) = \sum_{k=-n-\lambda}^{n} (-1)^k \begin{bmatrix} \lambda+2n \\ n-k \end{bmatrix} \mathcal{G}(k), \tag{1}$$

$$\mathcal{G}(n) + (-1)^{\lambda}\mathcal{G}(-\lambda-n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} \lambda+n+k \\ n-k \end{bmatrix} \frac{1-q^{\lambda+2n}}{1-q^{\lambda+n+k}} q^{\binom{n-k}{2}}\, \mathcal{F}(k), \tag{2}$$

provided that $\lambda = 0$ and 1. Furthermore, these inversions are also valid for $\lambda > 1$ under the additional conditions $\mathcal{G}(-1) = \mathcal{G}(-2) = \cdots = \mathcal{G}(1-\lambda) = 0$.

By examining particular cases of the very well–poised ${}_6\psi_6$-series identity due to Bailey [9] (see also [1]II-33 and [6])

$$\begin{aligned} {}_6\psi_6 &\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e \end{matrix} \,\Big|\, q; \frac{qa^2}{bcde}\right] \\ &= \left[\begin{matrix} q, qa, q/a, qa/bc, qa/bd, qa/be, qa/cd, qa/ce, qa/de \\ qa/b, qa/c, qa/d, qa/e, q/b, q/c, q/d, q/e, qa^2/bcde \end{matrix} \,\Big|\, q\right]_\infty \end{aligned} \tag{3}$$

provided that $|qa^2/bcde| < 1$ for convergence, we shall derive, in this section, two dual formulae by making use of Lemma 1 and four variants by means of contiguous relations.

§2.1.

According to (3), we have the following special case

$${}_6\psi_6\left[\begin{matrix} q\sqrt{-1}, -q\sqrt{-1}, & q^{-n}, & -q^{-n}, -q/d, & d/b \\ \sqrt{-1}, & -\sqrt{-1}, -q^{1+n}, q^{1+n}, & d, & -qb/d \end{matrix} \,\Big|\, q; bq^{2n}\right] = \frac{(q^2;q^2)_n(q^2;q^2)_n(b;q)_{2n}}{(q;q)_{2n}(d^2;q^2)_n(q^2b^2/d^2;q^2)_n}.$$

Denote by $\begin{bmatrix} m \\ n \end{bmatrix}_{q^2}$ the same Gaussian binomial coefficient $\begin{bmatrix} m \\ n \end{bmatrix}$, but with the base q being replaced by q^2. Observing that

$$\left[\begin{matrix} q^{-n}, -q^{-n} \\ q^{1+n}, -q^{1+n} \end{matrix} \,\Big|\, q\right]_k = (-1)^k q^{k^2-k-2nk} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_{q^2} \frac{(q^2;q^2)_n^2}{(q^2;q^2)_{2n}},$$

we can rewrite the above ${}_6\psi_6$-series explicitly as

$$\sum_{k=-n}^{n} (-1)^k \begin{bmatrix} 2n \\ n-k \end{bmatrix}_{q^2} \frac{1+q^{2k}}{2} \left[\begin{matrix} -q/d, d/b \\ d, -qb/d \end{matrix} \,\Big|\, q\right]_k q^{k^2-k} b^k = \frac{(-q;q)_{2n}(b;q)_{2n}}{(d^2;q^2)_n(q^2b^2/d^2;q^2)_n}.$$

This matches exactly with (1) for $\lambda = 0$ and the specifications

$$\mathcal{F}(n) := \frac{(-q;q)_{2n}(b;q)_{2n}}{(d^2;q^2)_n(q^2b^2/d^2;q^2)_n},$$
$$\mathcal{G}(n) := \frac{(1+q^{2n})q^{n^2-n}b^n}{2}\left[\begin{matrix}-q/d,d/b\\ d,-qb/d\end{matrix}\Big| q\right]_n.$$

Then the dual relation corresponding to (2) reads as

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n+k\\ n-k\end{bmatrix}_{q^2}\frac{1-q^{4n}}{1-q^{2n+2k}}q^{2\binom{n-k}{2}}\frac{(-q;q)_{2k}(b;q)_{2k}}{(d^2;q^2)_k(q^2b^2/d^2;q^2)_k}$$
$$=\frac{(1+q^{2n})}{2}q^{n^2-n}b^n\left\{\left[\begin{matrix}-q/d,d/b\\ d,-qb/d\end{matrix}\Big| q\right]_n+\left[\begin{matrix}q/d,-d/b\\ -d,qb/d\end{matrix}\Big| q\right]_n\right\}.$$

Keeping in mind that

$$\begin{bmatrix}n+k\\ n-k\end{bmatrix}_{q^2}=(-1)^kq^{2nk-k^2+k}\frac{(q^{-2n};q^2)_k(q^{2+2n};q^2)_k}{(q^2;q^2)_{2k}},$$

we have derived the following balanced series identity.

Theorem 1 (Balanced series identity)**.**

$${}_4\phi_3\left[\begin{matrix}q^{-2n},q^{2n},b,qb\\ q,d^2,q^2b^2/d^2\end{matrix}\Big| q^2;q^2\right]=\frac{b^n}{2}\left\{\left[\begin{matrix}-q/d,d/b\\ d,-qb/d\end{matrix}\Big| q\right]_n+\left[\begin{matrix}q/d,-d/b\\ -d,qb/d\end{matrix}\Big| q\right]_n\right\}.$$

According to the linear relation

$$1-q^{2k}x=\frac{b^2q^2-d^2x}{b^2q^2-bd^2}(1-q^{2k}b)+\frac{bd^2-d^2x}{bd^2-b^2q^2}\Big(1-q^{2k+2}\frac{b^2}{d^2}\Big), \tag{4}$$

we can manipulate the following ${}_5\phi_4$-series

$${}_5\phi_4\left[\begin{matrix}q^{-2n},q^{2n},q^2x,b,qb\\ q,x,d^2,q^4b^2/d^2\end{matrix}\Big| q^2;q^2\right]=\sum_{k=0}^{n}\frac{1-q^{2k}x}{1-x}\left[\begin{matrix}q^{-2n},q^{2n},b,\quad qb\\ q,\quad q^2,d^2,q^4b^2/d^2\end{matrix}\Big| q^2\right]_k q^{2k}$$
$$=\frac{(1-b)(b^2q^2-d^2x)}{(1-x)(b^2q^2-bd^2)}{}_4\phi_3\left[\begin{matrix}q^{-2n},q^{2n},qb,q^2b\\ q,d^2,q^4b^2/d^2\end{matrix}\Big| q^2;q^2\right]+\frac{(1-q^2b^2/d^2)(bd^2-d^2x)}{(1-x)(bd^2-b^2q^2)}{}_4\phi_3\left[\begin{matrix}q^{-2n},q^{2n},b,qb\\ q,d^2,q^2b^2/d^2\end{matrix}\Big| q^2;q^2\right].$$

Evaluating the above two ${}_4\phi_3$-series by Theorem 1

$${}_5\phi_4\left[\begin{matrix}q^{-2n},q^{2n},q^2x,b,qb\\ q,x,d^2,q^4b^2/d^2\end{matrix}\Big| q^2;q^2\right]$$
$$=\frac{(qb)^n(1-b)(b^2q^2-d^2x)}{2(1-x)(b^2q^2-bd^2)}\left\{\left[\begin{matrix}-q/d,d/qb\\ d,-q^2b/d\end{matrix}\Big| q\right]_n+\left[\begin{matrix}q/d,-d/qb\\ -d,q^2b/d\end{matrix}\Big| q\right]_n\right\}$$
$$+\frac{b^n(1-q^2b^2/d^2)(bd^2-d^2x)}{2(1-x)(bd^2-b^2q^2)}\left\{\left[\begin{matrix}-q/d,d/b\\ d,-qb/d\end{matrix}\Big| q\right]_n+\left[\begin{matrix}q/d,-d/b\\ -d,qb/d\end{matrix}\Big| q\right]_n\right\}$$

and then regrouping the terms, we find, after some simplifications, the identity below with an extra free parameter x.

Proposition 1 (Balanced series identity)**.**

$${}_5\phi_4\left[\begin{matrix}q^{-2n},q^{2n},q^2x,b,qb\\ q,x,d^2,q^4b^2/d^2\end{matrix}\Big| q^2;q^2\right]=\frac{b^n}{2}\left\{\Lambda_n(b,d)+\Lambda_n(b,-d)\right\},$$

where

$$\Lambda_n(b,d)=\left\{q^n-\frac{q(b-x)(1-q^{2n})}{d(1-x)(1-q^2b/d^2)}\right\}\left[\begin{matrix}-q/d,d/qb\\ d,-q^2b/d\end{matrix}\Big|\,q\right]_n.$$

When $x=q^{2n}$, this proposition reduces to another balanced series identity.

Corollary 2 (Balanced series identity)**.**

$${}_4\phi_3\left[\begin{matrix}q^{-2n},q^{2+2n},b,qb\\ q,d^2,q^4b^2/d^2\end{matrix}\Big|\,q^2;q^2\right]=\frac{b^n}{2}\left\{\Lambda_n(b,d)+\Lambda_n(b,-d)\right\},$$

where

$$\Lambda_n(b,d)=\frac{(1+q/d)(1-qb/d)}{1-q^2b/d^2}\left[\begin{matrix}-q^2/d,d/b\\ d,-q^2b/d\end{matrix}\Big|\,q\right]_n.$$

§2.2.

Alternatively, consider another particular case of (3)

$${}_6\psi_6\left[\begin{matrix}q\sqrt{-q},-q\sqrt{-q}, & q^{-n}, & -q^{-n},-q^2/d, & qd/b\\ \sqrt{-q}, & -\sqrt{-q},-q^{2+n}, & q^{2+n}, & d, & -qb/d\end{matrix}\Big|\,q;bq^{2n}\right]=\frac{2(1-q/b)}{(1+q/d)(1-d/b)}\frac{(q^4;q^2)_n(q^2;q^2)_n(b;q)_{2n}}{(q^2;q)_{2n}\,[d^2,q^2b^2/d^2;q^2\,]_n}.$$

In view of

$$\left[\begin{matrix}q^{-n},-q^{-n}\\ q^{2+n},-q^{2+n}\end{matrix}\Big|\,q\right]_k=(-1)^kq^{k^2-k-2nk}\begin{bmatrix}2n+1\\ n-k\end{bmatrix}_{q^2}\frac{(q^2;q^2)_n(q^2;q^2)_{n+1}}{(q^2;q^2)_{2n+1}},$$

we can express it as the following equality

$$\sum_{k=-n-1}^{n}(-1)^k\begin{bmatrix}2n+1\\ n-k\end{bmatrix}_{q^2}\frac{1+q^{2k+1}}{2(1-q/b)}\frac{[-q/d,d/b;q]_{k+1}}{[d,-qb/d;q]_k}q^{k^2-k}b^k=\frac{(-q;q)_{2n+1}(b;q)_{2n}}{(d^2;q^2)_n(q^2b^2/d^2;q^2)_n}.$$

This matches exactly with (1) for $\lambda=1$ and the specifications

$$\mathcal{F}(n):=\frac{(-q;q)_{2n+1}(b;q)_{2n}}{(d^2;q^2)_n(q^2b^2/d^2;q^2)_n},$$
$$\mathcal{G}(n):=\frac{(1+q^{2n+1})q^{n^2-n}b^n}{2(1-q/b)}\frac{[-q/d,d/b;q\,]_{n+1}}{[d,-qb/d;q\,]_n}.$$

Then the dual relation displayed in (2) reads as

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n+k+1\\ n-k\end{bmatrix}_{q^2}\frac{1-q^{4n+2}}{1-q^{2n+2k+2}}q^{2\binom{n-k}{2}}\frac{(-q;q)_{2k+1}(b;q)_{2k}}{(d^2;q^2)_k(q^2b^2/d^2;q^2)_k}$$
$$=\frac{(1+q^{2n+1})}{2(1-q/b)}q^{n^2-n}b^n\left\{\frac{[-q/d,d/b;q\,]_{n+1}}{[d,-qb/d;q\,]_n}+\frac{[q/d,-d/b;q\,]_{n+1}}{[-d,qb/d;q\,]_n}\right\}.$$

Noting that

$$\begin{bmatrix}n+k+1\\ n-k\end{bmatrix}_{q^2}=(-1)^kq^{2nk-k^2+k}\frac{(q^{-2n};q^2)_k(q^{2+2n};q^2)_{k+1}}{(q^2;q^2)_{2k+1}},$$

we get the following balanced series identity.

Theorem 3 (Balanced series identity)**.**

$$ {}_4\phi_3\left[\begin{matrix} q^{-2n}, q^{2+2n}, b, qb \\ q^3, d^2, q^2b^2/d^2 \end{matrix} \,\middle|\, q^2; q^2\right] = \frac{b^n}{2}\left\{\Lambda_n(b,d) + \Lambda_n(b,-d)\right\}, $$

where

$$ \Lambda_n(b,d) = \frac{(1-q)(1+q/d)(1-d/b)}{(1-q/b)(1-q^{2n+1})}\left[\begin{matrix} -q^2/d, qd/b \\ d, -qb/d \end{matrix} \,\middle|\, q\right]_n. $$

Analogously, applying the linear relation (4), we can reformulate the ${}_5\phi_4$-series

$$ {}_5\phi_4\left[\begin{matrix} q^{-2n}, q^{2+2n}, q^2x, b, qb \\ q^3, x, d^2, q^4b^2/d^2 \end{matrix} \,\middle|\, q^2; q^2\right] $$
$$ = \frac{(1-b)(b^2q^2-d^2x)}{(1-x)(b^2q^2-bd^2)}{}_4\phi_3\left[\begin{matrix} q^{-2n}, q^{2+2n}, qb, q^2b \\ q^3, d^2, q^4b^2/d^2 \end{matrix} \,\middle|\, q^2; q^2\right] \frac{(1-q^2b^2/d^2)(bd^2-d^2x)}{(1-x)(bd^2-b^2q^2)}{}_4\phi_3\left[\begin{matrix} q^{-2n}, q^{2+2n}, b, qb \\ q^3, d^2, q^2b^2/d^2 \end{matrix} \,\middle|\, q^2; q^2\right]. $$

By means of Theorem 3, we establish another balanced series identity with an extra parameter x.

Proposition 2 (Balanced series identity)**.**

$$ {}_5\phi_4\left[\begin{matrix} q^{-2n}, q^{2+2n}, q^2x, b, qb \\ q^3, x, d^2, q^4b^2/d^2 \end{matrix} \,\middle|\, q^2; q^2\right] = \frac{b^n(1-q)(1-x/b)}{2(1-x)(1-q/b)}\left\{\Lambda_n(b,d) + \Lambda_n(b,-d)\right\}, $$

where

$$ \Lambda_n(b,d) = \frac{(1+q/d)(1-qb/d)}{(1-q^2b/d^2)}\left\{1 - q^n\frac{d(1-x/q)(1-q^2b/d^2)}{b(1-x/b)(1-q^{2n+1})}\right\}\left[\begin{matrix} -q^2/d, d/b \\ d, -q^2b/d \end{matrix} \,\middle|\, q\right]_n. $$

Letting $x = q^{2n+2}$ in Proposition 2, we obtain another balanced ${}_4\phi_3$-series identity.

Corollary 4 (Balanced series identity)**.**

$$ {}_4\phi_3\left[\begin{matrix} q^{-2n}, q^{4+2n}, b, qb \\ q^3, d^2, q^4b^2/d^2 \end{matrix} \,\middle|\, q^2; q^2\right] = \frac{b^{n+1}d(1-q)}{2q(1-q^{2n+2})}\left\{\Lambda_n(b,d) - \Lambda_n(b,-d)\right\}, $$

where

$$ \Lambda_n(b,d) = \frac{(1-q^2/d^2)(1-q^2b^2/d^2)}{(q-b)(1-q^2b/d^2)}\left[\begin{matrix} -q^2/d, d/b \\ d/q, -qb/d \end{matrix} \,\middle|\, q\right]_{n+1}. $$

3. Polynomial argument and limiting series

Let $P(y)$ and $Q(y)$ be the two polynomials of degrees at most m. If they agree at more than $m+1$ distinct values of y, then $P(y)$ and $Q(y)$ are identical.

For $b = q^{-m}$ with $m \in \mathbb{N}$ and $m \leq 2n$, the identity in Theorem 1 becomes

$$ {}_4\phi_3\left[\begin{matrix} q^{-2n}, q^{2n}, q^{-m}, q^{1-m} \\ q, d^2, q^{2-2m}/d^2 \end{matrix} \,\middle|\, q^2; q^2\right] = \frac{q^{-mn}}{2}\left\{\left[\begin{matrix} -q/d, q^m d \\ d, -q^{1-m}/d \end{matrix} \,\middle|\, q\right]_n + \left[\begin{matrix} q/d, -q^m d \\ -d, q^{1-m}/d \end{matrix} \,\middle|\, q\right]_n\right\}. $$

Multiplying across the last equation by q^{mn} and then making use of the equality

$$ \left[\begin{matrix} -q/d, q^m d \\ d, -q^{1-m}/d \end{matrix} \,\middle|\, q\right]_n = \left[\begin{matrix} q^n d, -q^{1+n-m}/d \\ d, -q^{1-m}/d \end{matrix} \,\middle|\, q\right]_m, $$

we can reformulate the resulting equality as

$$\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{q^{mn}\,[q^{-2n},q^{2n};q^2]_k\,(q^{-m};q)_{2k}}{(q;q)_{2k}\,[d^2,q^{2-2m}/d^2;q^2]_k} q^{2k} = \frac{1}{2}\left\{\left[\begin{matrix} q^n d, -q^{1+n-m}/d \\ d, -q^{1-m}/d \end{matrix}\,\Big|\, q\right]_m + \left[\begin{matrix} -q^n d, q^{1+n-m}/d \\ -d, q^{1-m}/d \end{matrix}\,\Big|\, q\right]_m\right\},$$

which is valid for infinitely many $n \in \mathbb{N}$ with $2n \geq m$. Moreover, both sides of the last equation are polynomials of degree at most $2m$ in q^n. Hence, this is a polynomial identity in the variable q^n. Renaming q^n by b, we highlight the resulting identity in the following proposition, where the other three balanced ${}_4\phi_3$-series identities can analogously be deduced.

Proposition 3 (Four balanced series identities)**.**

a. *The counterpart formula from Theorem 1:*

$${}_4\phi_3\left[\begin{matrix} q^{-m}, q^{1-m}, b^2, 1/b^2 \\ q, d^2, q^{2-2m}/d^2 \end{matrix}\,\Big|\, q^2; q^2\right] = \frac{1}{2}\left\{\Lambda_m(b,d) + \Lambda_m(b,-d)\right\},$$

where

$$\Lambda_m(b,d) = \left[\begin{matrix} bd, -d/b \\ d, -d \end{matrix}\,\Big|\, q\right]_m.$$

b. *The counterpart formula from Corollary 2:*

$${}_4\phi_3\left[\begin{matrix} q^{-m}, q^{1-m}, b^2, q^2/b^2 \\ q, d^2, q^{4-2m}/d^2 \end{matrix}\,\Big|\, q^2; q^2\right] = \frac{1}{2}\left\{\Lambda_m(b,d) + \Lambda_m(b,-d)\right\},$$

where

$$\Lambda_m(b,d) = \frac{(1+q/d)(1-q^{1-m}/d)}{(1-q^{2-m}/d^2)}\left[\begin{matrix} bd/q, -d/b \\ d, -d/q \end{matrix}\,\Big|\, q\right]_m.$$

c. *The counterpart formula from Theorem 3:*

$${}_4\phi_3\left[\begin{matrix} q^{-m}, q^{1-m}, b^2, q^2/b^2 \\ q^3, d^2, q^{2-2m}/d^2 \end{matrix}\,\Big|\, q^2; q^2\right] = \frac{1}{2}\left\{\Lambda_m(b,d) + \Lambda_m(b,-d)\right\},$$

where

$$\Lambda_m(b,d) = \frac{(1-q)(1+b/d)(1-bdq^{m-1})}{(1-q^{1+m})(1-b^2/q)}\left[\begin{matrix} bd/q, -qd/b \\ d, -d \end{matrix}\,\Big|\, q\right]_m.$$

d. *The counterpart formula from Corollary 4:*

$${}_4\phi_3\left[\begin{matrix} q^{-m}, q^{1-m}, q^2b^2, q^2/b^2 \\ q^3, d^2, q^{4-2m}/d^2 \end{matrix}\,\Big|\, q^2; q^2\right] = \frac{1-q}{2q}\left\{\Lambda_m(b,d) + \Lambda_m(b,-d)\right\},$$

where

$$\Lambda_m(b,d) = \frac{bd(1-q^2/d^2)(1-q^{2-2m}/d^2)}{(q-q^{-m})(1-b^2)(1-q^{2-m}/d^2)}\left[\begin{matrix} bd/q, -d/bq \\ d/q, -d/q \end{matrix}\,\Big|\, q\right]_{m+1}.$$

Letting $m \to \infty$ in the last proposition, we derive the following four interesting nonterminating ${}_2\phi_2$-series identities, that resemble somewhat the q-analogue of Bailey's ${}_2F_1(-1)$-sum (cf. [1]II-10):

$$
{}_2\phi_2\left[\begin{matrix} a, q/a \\ -q,\ c \end{matrix} \,\Big|\, q; -c\right] = \left[\begin{matrix} ac, qc/a \\ c,\ qc \end{matrix} \,\Big|\, q^2\right]_\infty .
$$

Corollary 5 (Four nonterminating ${}_2\phi_2$-series identities)**.**

(a) $$ {}_2\phi_2\left[\begin{matrix} b^2, 1/b^2 \\ q,\ d^2 \end{matrix} \,\Big|\, q^2; qd^2\right] = \frac{1}{2}\left\{ \left[\begin{matrix} bd, -d/b \\ d, -d \end{matrix} \,\Big|\, q\right]_\infty + \left[\begin{matrix} -bd, d/b \\ -d, d \end{matrix} \,\Big|\, q\right]_\infty \right\}. $$

(b) $$ {}_2\phi_2\left[\begin{matrix} b^2, q^2/b^2 \\ q,\ d^2 \end{matrix} \,\Big|\, q^2; d^2/q\right] = \frac{1}{2}\left\{ \left[\begin{matrix} bd/q, -d/b \\ d, -d \end{matrix} \,\Big|\, q\right]_\infty + \left[\begin{matrix} -bd/q, d/b \\ -d, d \end{matrix} \,\Big|\, q\right]_\infty \right\}. $$

(c) $$ {}_2\phi_2\left[\begin{matrix} b^2, q^2/b^2 \\ q^3,\ d^2 \end{matrix} \,\Big|\, q^2; qd^2\right] = \frac{b(1-q)}{2d(1-b^2/q)}\left[\begin{matrix} bd/q, -d/b \\ d, -d \end{matrix} \,\Big|\, q\right]_\infty - \frac{b(1-q)}{2d(1-b^2/q)}\left[\begin{matrix} -bd/q, d/b \\ -d, d \end{matrix} \,\Big|\, q\right]_\infty . $$

(d) $$ {}_2\phi_2\left[\begin{matrix} q^2b^2, q^2/b^2 \\ q^3,\ d^2 \end{matrix} \,\Big|\, q^2; d^2/q\right] = \frac{qb(1-q)}{2d(1-b^2)}\left[\begin{matrix} bd/q, -d/bq \\ d, -d \end{matrix} \,\Big|\, q\right]_\infty - \frac{qb(1-q)}{2d(1-b^2)}\left[\begin{matrix} -bd/q, d/bq \\ -d, d \end{matrix} \,\Big|\, q\right]_\infty . $$

These identities do not seem to have appeared previously. We offer a classical proof for the first one. Others can be done similarly. Recall the transformation due to Jackson (cf. Gasper–Rahman [**?** , III-4]):

$$
{}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix} \,\Big|\, q; z\right] = \frac{(az;q)_\infty}{(z;q)_\infty}\, {}_2\phi_2\left[\begin{matrix} a,\ c/b \\ az,\ c \end{matrix} \,\Big|\, q; bz\right].
$$

The first ${}_2\phi_2$-series in the above corollary can be reformulated as follows:

$$
{}_2\phi_2\left[\begin{matrix} b^2, 1/b^2 \\ q,\ d^2 \end{matrix} \,\Big|\, q^2; qd^2\right] = \frac{(d^2/b^2;q^2)_\infty}{(d^2;q^2)_\infty}\, {}_2\phi_1\left[\begin{matrix} b^2, qb^2 \\ q \end{matrix} \,\Big|\, q^2; d^2/b^2\right].
$$

Evaluating the last ${}_2\phi_1$-series by the q-binomial series

$$
\begin{aligned}
{}_2\phi_1\left[\begin{matrix} b^2, qb^2 \\ q \end{matrix} \,\Big|\, q^2; d^2/b^2\right] &= \frac{1}{2}\left\{ {}_1\phi_0\left[\begin{matrix} b^2 \\ - \end{matrix} \,\Big|\, q; d/b\right] + {}_1\phi_0\left[\begin{matrix} b^2 \\ - \end{matrix} \,\Big|\, q; -d/b\right] \right\} \\
&= \frac{1}{2}\left\{ \frac{(bd;q)_\infty}{(d/b;q)_\infty} + \frac{(-bd;q)_\infty}{(-d/b;q)_\infty} \right\}
\end{aligned}
$$

and making some simplifications, we confirm the identity. □

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References

[1] Gasper, G., & Rahman, R. (2004). *Basic Hypergeometric Series* (2nd ed.) Cambridge University Press, Cambridge.

[2] Chen, X., & Chu, W. (2014). Carlitz inversions and identities of Rogers–Ramanujan type. *Rocky Mountain Journal of Mathematics*, *44*(4), 1125–1143.

[3] Chu, W. (1994). Inversion techniques and combinatorial identities: Basic hypergeometric identities. *Publicationes Mathematicae Debrecen*, *44*(3/4), (1994), 301–320.

[4] Chu, W. (1994). Inversion techniques and combinatorial identities: Strange evaluations of basic hypergeometric series. *Compositio Mathematica*, *91*, 121–144.

[5] Chu, W. (1995). Basic Hypergeometric Identities: An introductory revisiting through the Carlitz inversions. *Forum Mathematicum*, *7*, 117–129.

[6] Chu, W. (2006). Bailey's very well–poised ${}_6\psi_6$-series identity. *Journal of Combinatorial Theory, Series A*, *113*(6), 966–979.

[7] Gessel, I., & Stanton, D. (1983). Applications of q-Lagrange inversion to basic hypergeometric series. *Transactions of the American Mathematical Society*, *277*, 173–201.

[8] Chu, W., & Wang, C. (2009). Bilateral inversions and terminating basic hypergeometric series identities. *Discrete Mathematics, 309*, 3888–3904.

[9] Bailey, W. N. (1936). Series of hypergeometric type which are infinite in both directions. *Quarterly Journal of Mathematics, 7*, 105–115.