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Estimation to the number of limit cycles for generalized Kukles differential system

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Abstract: This article considers the limit cycles of a class of Kukles polynomial differential systems of the form Eq. (5). We obtain the maximum number of limit cycles that bifurcate from the periodic orbits of a linear center $\dot{x} = y, \dot{y} = -x$, by using the averaging theory of first and second order.

Keywords: Limit cycle; Averaging theory; Kukles systems.

MSC: 34C29; 34C25; 47H11.

1. Introduction

The study of limit cycles, which are isolated periodic orbits in the set of solutions of differential equations, is one of the main problems in the theory of differential equations. It is done by checking their existence, number, and stability. Many mathematicians, physicists, chemists, biologists, and others were interested in knowing and discovering those properties related to the limit cycles. The origin or the motivation of limit cycles emerged from the second part of the 16^{th} Hilbert problem [1], which involves finding the maximum number of limit cycles of polynomial vector fields with fixed degrees.

There are several methods exist to study the number of limit cycles that bifurcate from the periodic orbits such as the abelian integral method [2], the integrating factor [3], the Poincaré return map [4], Poincaré-Melnikov integral method [5] and averaging theory [6,7]. The study of limit cycles for differential equations or planar differential systems by applying the averaging method has been considered by several authors see, for instance, [8–11].

Here we consider a particular case of the 16^{th} Hilbert problem to study the upper bound of the generalized polynomial Kukles system,

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = Q(x,y), \end{cases} \tag{1}$$

where $Q(x,y)$ is a polynomial with real coefficients of degree n. In [12], Kukles introduced the following differential system

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3, \end{cases} \tag{2}$$

and he gave the necessary and sufficient conditions under which the system (2) has a center at the origin. In [13], it was solved the center problem for system (2) when $a_2 = 0$ and it was proved that at most six limit cycles bifurcate from the origin. In [14], Sadovskii solved the center-focus problem for system (2) with $a_2a_7 \neq 0$ and proved that systems (2) can have seven limit cycles bifurcate from the origin.

In [8], Llibre and Mereu used the averaging theory to study the maximum number of limit cycles of a class of generalized polynomial Kukles differential system of the form

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - \sum\limits_{k\geq 1} \varepsilon^k \left(f_{n_1}^k(x) + g_{n_2}^k(x)\, y + h_{n_3}^k(x)\, y^2 + d_0^k y^3 \right), \end{cases} \tag{3}$$

where for every k the polynomials $f_{n_1}^k, g_{n_2}^k$ and $h_{n_3}^k$ have degree n_1, n_2 and n_3 respectively, d_0^k real number different from zero and ε is a small parameter.

In [9], Boulfoul *et al.*, used the averaging theory to study the maximum number of limit cycles of a class of generalized polynomial Kukles differential system of the form

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x - f(x) - g(x)y - h(x)y^2 - l(x)y^3, \end{cases} \tag{4}$$

where $f(x) = \varepsilon f_1(x) + \varepsilon^2 f_2(x), g(x) = \varepsilon g_1(x) + \varepsilon^2 g_2(x), h(x) = \varepsilon h_1(x) + \varepsilon^2 h_2(x)$ and $l(x) = \varepsilon l_1(x) + \varepsilon^2 l_2(x)$ have degree n_1, n_2, n_3 and n_4 respectively, and ε is a small parameter.

In this paper, by using the averaging theory, we study the maximum number of limit cycles which can bifurcate from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$ perturbed inside the class of generalized polynomial Kukles differential system of the form

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - f(x)y^{2p} - g(x)y^{2p+1} - h(x)y^{2p+2} - l(x)y^{2p+3}, \end{cases} \tag{5}$$

where $f(x) = \varepsilon f^1(x) + \varepsilon^2 f^2(x)$, $g(x) = \varepsilon g^1(x) + \varepsilon^2 g^2(x)$, $h(x) = \varepsilon h^1(x) + \varepsilon^2 h^2(x)$, and $l(x) = \varepsilon l^1(x) + \varepsilon^2 l^2(x)$, where $f^k(x), g^k(x), h^k(x)$ and $l^k(x)$ have degree n_1, n_2, n_3, n_4 respectively for $k = 1, 2$. p is a positive integer and ε is a small parameter. The main results of this paper is the following theorem:

Theorem 1. *For $|\varepsilon|$ sufficiently small, the maximum number of limit cycles of the polynomial Kukles differential system (5) which can bifurcate from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$,*

(a) is $\max\left\{\left[\frac{n_2}{2}\right], \left[\frac{n_4}{2}\right] + 1\right\}$, by using averaging theory of first order,
(b) and $\max\left\{\left[\frac{n_2}{2}\right], \left[\frac{n_4}{2}\right] + 1, \left[\frac{n_1}{2}\right] + \left[\frac{n_2-1}{2}\right] + p, \left[\frac{n_1}{2}\right] + \left[\frac{n_4-1}{2}\right] + p + 1, \left[\frac{n_1-1}{2}\right] + \mu + p, \left[\frac{n_2-1}{2}\right] + \left[\frac{n_3}{2}\right] + p + 1, \left[\frac{n_3}{2}\right] + \left[\frac{n_4-1}{2}\right] + p + 2, \left[\frac{n_3-1}{2}\right] + \mu + p + 1\right\}$, by using averaging theory of second order, where $\mu = \min\left\{\left[\frac{n_2}{2}\right], \left[\frac{n_4}{2}\right] + 1\right\}$.

Our paper is organized as; in §2, we introduce the averaging theory of first and second order. Then in §3, we prove our main theorem using the tools mentioned in §2. And finally, we concluded our study by giving some applications.

2. The averaging theory of first and second order

The averaging theory of first and second order, for studying periodic orbits, was developed in [6,15]. The following result is Theorem 4.2 of [6].

Theorem 2. *Consider the differential system*

$$\dot{x}(t) = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 R(t, x, \varepsilon), \tag{6}$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n, R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume that

(i) $F_1(t,.) \in C^1(D)$ *for all* $t \in \mathbb{R}$, F_2, R *and* $D_x F_1$ *are locally Lipschitz with respect to* x, *and* R *is differentiable with respect to* ε. *Define* $f_1, f_2 : D \to \mathbb{R}$ *by*

$$\left.\begin{aligned} f_1(z) = &\ \frac{1}{T}\int_0^T F_1(s,z)ds, \\ f_2(z) = &\ \frac{1}{T}\int_0^T \left[D_z F_1(s,z)\int_0^s F_1(t,z)dt + F_2(s,z)\right] ds. \end{aligned}\right\} \tag{7}$$

(ii) *For* $V \subset D$ *an open and bounded set and for* $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$, *there exists* $a_\varepsilon \in V$ *such that* $f_1(a_\varepsilon) + \varepsilon f_2(a_\varepsilon) = 0$ *and* $d_B(f_1 + \varepsilon f_2, V, a) \neq 0$.

Then for $|\varepsilon| > 0$ *sufficiently small there exists a* T*-periodic solution* $\varphi(\cdot, \varepsilon)$ *of the system* (6) *such that* $\varphi(0,\varepsilon) \to a$ *when* $\varepsilon \to 0$.

If f_1 *is not identically zero, then the zeros of* $f_1 + \varepsilon f_2$ *are mainly the zeros of* f_1 *for* ε *sufficiently small. In this case the previous result provides the averaging theory of first order.*

If f_1 *is identically zero and* f_2 *is not identically zero, then the zeros of* $f_1 + \varepsilon f_2$ *are mainly the zeros of* f_2 *for* ε *sufficiently small. In this case the previous result provides the averaging theory of second order. For additional information on the averaging theory see the books [7,16].*

3. Proof of statement (a) and (b) of Theorem 1

In this proof, we use the first order averaging theory. So, we write the system (5) in polar coordinates (r,θ) where $x = rcos\theta$, $y = rsin\theta$, $r > 0$. We write the polynomials $f^1(x), g^1(x), h^1(x)$ and $l^1(x)$ which appear in (5) as,

$$f^1(x) = \sum_{i=0}^{n_1} a_i x^i, g^1(x) = \sum_{i=0}^{n_2} b_i x^i, h^1(x) = \sum_{i=0}^{n_3} c_i x^i \text{ and } l^1(x) = \sum_{i=0}^{n_4} d_i x^i. \tag{8}$$

Therefore system (5) becomes

$$\begin{aligned} \dot{r} &= -\varepsilon\left(\sum_{i=0}^{n_1} a_i r^{i+2p}\cos^i\theta\sin^{2p+1}\theta + \sum_{i=0}^{n_2} b_i r^{i+2p+1}\cos^i\theta\sin^{2p+2}\theta\right. \\ &\quad \left.+ \sum_{i=0}^{n_3} c_i r^{i+2p+2}\cos^i\theta\sin^{2p+3}\theta + \sum_{i=0}^{n_4} d_i r^{i+2p+3}\cos^i\theta\sin^{2p+4}\theta\right), \\ \dot{\theta} &= -1 - \frac{\varepsilon}{r}\left(\sum_{i=0}^{n_1} a_i r^{i+2p}\cos^{i+1}\theta\sin^{2p}\theta + \sum_{i=0}^{n_2} b_i r^{i+2p+1}\cos^{i+1}\theta\sin^{2p+1}\theta\right. \\ &\quad \left.+ \sum_{i=0}^{n_3} c_i r^{i+2p+2}\cos^{i+1}\theta\sin^{2p+2}\theta + \sum_{i=0}^{n_4} d_i r^{i+2p+3}\cos^{i+1}\theta\sin^{2p+3}\theta\right). \end{aligned}$$

Taking θ as the new independent variable, we get

$$\begin{aligned} \frac{dr}{d\theta} &= \varepsilon\left(\sum_{i=0}^{n_1} a_i r^{i+2p}\cos^i\theta\sin^{2p+1}\theta + \sum_{i=0}^{n_2} b_i r^{i+2p+1}\cos^i\theta\sin^{2p+2}\theta\right. \\ &\quad \left.+ \sum_{i=0}^{n_3} c_i r^{i+2p+2}\cos^i\theta\sin^{2p+3}\theta + \sum_{i=0}^{n_4} d_i r^{i+2p+3}\cos^i\theta\sin^{2p+4}\theta\right) + O(\varepsilon^2), \\ &= \varepsilon F_1(r,\theta) + O(\varepsilon^2). \end{aligned}$$

Let F_{10} be the averaging equation of first order associated with system (5).

Using the notation introduced in Theorem 2, we compute F_{10} by integrating F_1 with respect to θ,

$$F_{10}(r) = \frac{1}{2\pi}\int_0^{2\pi} F_1(r,\theta)d\theta.$$

Lemma 1. *Let $A_{i,j}(\theta) = \cos^i\theta \sin^j\theta$ and $\theta\xi_{i,j}(\theta) = \int_0^\theta A_{i,j}(s)ds$, where*

$$\int_0^{2\pi} A_{i,j}(\theta)d\theta = \begin{cases} 0, & \text{if } i \text{ is odd or } j \text{ is odd}, \\ 2\pi\xi_{i,j}(2\pi), & \text{if } i \text{ and } j \text{ are even}, \end{cases}$$

and

$$\xi_{2i,2j+4}(2\pi) = \frac{2j+3}{2i+2j+4}\xi_{2i,2j+2}(2\pi).$$

Using Lemma (1), we obtain the integral of the function $F_{10}(r)$

$$\begin{aligned} F_{10}(r) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{\substack{i=0\\ i\text{ even}}}^{n_2} b_i r^{i+2p+1}A_{i,2p+2}(\theta) + \sum_{\substack{i=0\\ i\text{ even}}}^{n_4} d_i r^{i+2p+3}A_{i,2p+4}(\theta)\right)d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{[\frac{n_2}{2}]} b_{2i} r^{2i+2p+1}A_{2i,2p+2}(\theta) + \sum_{i=0}^{[\frac{n_4}{2}]} d_{2i} r^{2i+2p+3}A_{2i,2p+4}(\theta)\right)d\theta \\ &= r^{2p+1}\left(\sum_{i=0}^{[\frac{n_2}{2}]} b_{2i} r^{2i}\xi_{2i,2p+2}(2\pi) + \sum_{i=0}^{[\frac{n_4}{2}]} d_{2i} r^{2i+2}\xi_{2i,2p+4}(2\pi)\right) \\ &= r^{2p+1}\left(\sum_{i=0}^{[\frac{n_2}{2}]} b_{2i} r^{2i}\xi_{2i,2p+2}(2\pi) + \sum_{i=0}^{[\frac{n_4}{2}]} \frac{2p+3}{2i+2p+4} d_{2i} r^{2i+2}\xi_{2i,2p+2}(2\pi)\right). \end{aligned} \tag{9}$$

Then the polynomial $F_{10}(r)$ has at most $\max\left\{\left[\frac{n_2}{2}\right], \left[\frac{n_4}{2}\right]+1\right\}$ positive roots. Hence statement (a) of Theorem (1) is proved.

For proving statement (b) of Theorem 1, we will use the second-order averaging theory. We take $f^1(x), g^1(x), h^1(x)$ and $l^1(x)$ as defined in (8), and let

$$f^2(x) = \sum_{i=0}^{n_1} \bar{a}_i x^i, g^2(x) = \sum_{i=0}^{n_2} \bar{b}_i x^i, h^2(x) = \sum_{i=0}^{n_3} \bar{c}_i x^i \text{ and } l^2(x) = \sum_{i=0}^{n_4} \bar{d}_i x^i. \tag{10}$$

In polar coordinates (r,θ) where $x = r\cos\theta$, $y = r\sin\theta$, $r > 0$, the differential system (5) becomes

$$\begin{cases} \dot{r} = -\varepsilon G_1(r,\theta) - \varepsilon^2 H_1(r,\theta), \\ \dot{\theta} = -1 - \frac{\varepsilon}{r}G_2(r,\theta) - \frac{\varepsilon^2}{r}H_2(r,\theta). \end{cases} \tag{11}$$

Taking θ as new independent variable, we find $\dfrac{dr}{d\theta} = \varepsilon F_1(r,\theta) + \varepsilon^2 F_2(r,\theta) + O(\varepsilon^3)$, where

$$F_1(r,\theta) = G_1 \quad \text{and} \quad F_2(r,\theta) = H_1 - \frac{1}{r}G_1G_2. \tag{12}$$

And

$$\begin{cases} G_1 = \sum\limits_{i=0}^{n_1} a_i r^{i+2p} A_{i,2p+1}(\theta) + \sum\limits_{i=0}^{n_2} b_i r^{i+2p+1} A_{i,2p+2}(\theta) + \sum\limits_{i=0}^{n_3} c_i r^{i+2p+2} A_{i,2p+3}(\theta) + \sum\limits_{i=0}^{n_4} d_i r^{i+2p+3} A_{i,2p+4}(\theta), \\ H_1 = \sum\limits_{i=0}^{n_1} \bar{a}_i r^{i+2p} A_{i,2p+1}(\theta) + \sum\limits_{i=0}^{n_2} \bar{b}_i r^{i+2p+1} A_{i,2p+2}(\theta) + \sum\limits_{i=0}^{n_3} \bar{c}_i r^{i+2p+2} A_{i,2p+3}(\theta) + \sum\limits_{i=0}^{n_4} \bar{d}_i r^{i+2p+3} A_{i,2p+4}(\theta), \\ G_2 = \sum\limits_{i=0}^{n_1} a_i r^{i+2p} A_{i+1,2p}(\theta) + \sum\limits_{i=0}^{n_2} b_i r^{i+2p+1} A_{i+1,2p+1}(\theta) + \sum\limits_{i=0}^{n_3} c_i r^{i+2p+2} A_{i+1,2p+2}(\theta) + \sum\limits_{i=0}^{n_4} d_i r^{i+2p+3} A_{i+1,2p+3}(\theta), \\ H_2 = \sum\limits_{i=0}^{n_1} \bar{a}_i r^{i+2p} A_{i+1,2p}(\theta) + \sum\limits_{i=0}^{n_2} \bar{b}_i r^{i+2p+1} A_{i+1,2p+1}(\theta) + \sum\limits_{i=0}^{n_3} \bar{c}_i r^{i+2p+2} A_{i+1,2p+2}(\theta) + \sum\limits_{i=0}^{n_4} \bar{d}_i r^{i+2p+3} A_{i+1,2p+3}(\theta). \end{cases} \tag{13}$$

In order to apply averaging theory of second order, F_{10} must be identically equal to zero. Therefore from (9), we take

$$\begin{cases} b_{2i} = -\frac{2p+3}{2i-1} d_{2i-2} & 1 \le i \le \mu, \\ b_0 = b_{2i} = d_{2i-2} = 0 & \mu+1 \le i \le \lambda, \end{cases} \tag{14}$$

where $\lambda = \max\left\{\left[\frac{n_2}{2}\right], \left[\frac{n_4}{2}\right]+1\right\}$.

Let F_{20} be the averaging equation of second order associated with system (5). Now, we determine F_{20} by integrating with respect to θ

$$F_{20}(r) = F_{20}^1(r) + F_{20}^2(r),$$

where

$$F_{20}^1(r) = \frac{1}{2\pi}\int_0^{2\pi} \frac{d}{dr}F_1(r,\theta)y(r,\theta)d\theta \quad \text{and} \quad F_{20}^2(r) = \frac{1}{2\pi}\int_0^{2\pi} F_2(r,\theta)d\theta.$$

By substituting (14) in (13) and (12), we get

$$\begin{aligned} F_1(r,\theta) = & \sum_{i=0}^{\left[\frac{n_1-1}{2}\right]} a_{2i+1}A_{2i+1,2p+1}(\theta)\, r^{2i+2p+1} + \sum_{i=0}^{\left[\frac{n_1}{2}\right]} a_{2i}A_{2i,2p+1}(\theta)\, r^{2i+2p} + \sum_{i=0}^{\left[\frac{n_2-1}{2}\right]} b_{2i+1}A_{2i+1,2p+2}(\theta)\, r^{2i+2p+2} + \\ & \sum_{i=0}^{\left[\frac{n_3-1}{2}\right]} c_{2i+1}A_{2i+1,2p+3}(\theta)\, r^{2i+2p+3} + \sum_{i=0}^{\left[\frac{n_3}{2}\right]} c_{2i}A_{2i,2p+3}(\theta)\, r^{2i+2p+2} + \sum_{i=0}^{\left[\frac{n_4-1}{2}\right]} d_{2i+1}A_{2i+1,2p+4}(\theta)\, r^{2i+2p+4} + \\ & \sum_{i=1}^{\mu}\left(A_{2i-2,2p+4}(\theta) - \frac{2p+3}{2i-1}A_{2i,2p+2}(\theta)\right) d_{2i-2}r^{2i+2p+1}. \end{aligned}$$

To compute F_{20}^1, we must derive F_1, so

$$\begin{aligned} \frac{dF_1(r,\theta)}{dr} = & \sum_{i=0}^{\left[\frac{n_1-1}{2}\right]} (2i+2p+1)a_{2i+1}A_{2i+1,2p+1}(\theta)\, r^{2i+2p} + \sum_{i=0}^{\left[\frac{n_1}{2}\right]} (2i+2p)a_{2i}A_{2i,2p+1}(\theta)\, r^{2i+2p-1} + \sum_{i=0}^{\left[\frac{n_2-1}{2}\right]} (2i+2p+ \\ & 2)b_{2i+1}A_{2i+1,2p+2}(\theta)\, r^{2i+2p+1} + \sum_{i=0}^{\left[\frac{n_3-1}{2}\right]} (2i+2p+3)c_{2i+1}A_{2i+1,2p+3}(\theta)\, r^{2i+2p+2} + \sum_{i=0}^{\left[\frac{n_3}{2}\right]} (2i+2p+ \\ & 2)c_{2i}A_{2i,2p+3}(\theta)\, r^{2i+2p+1} + \sum_{i=0}^{\left[\frac{n_4-1}{2}\right]} (2i+2p+4)d_{2i+1}A_{2i+1,2p+4}(\theta)\, r^{2i+2p+3} + \sum_{i=1}^{\mu} (2i+2p+ \\ & 1)d_{2i-2}\left(A_{2i-2,2p+4}(\theta) - \frac{2p+3}{2i-1}A_{2i,2p+2}(\theta)\right) r^{2i+2p}. \end{aligned}$$

And we have

$$\begin{aligned} y(r,\theta) = \int_0^\theta F_1(r,\theta)d\theta = & \sum_{i=0}^{\left[\frac{n_1-1}{2}\right]} a_{2i+1}r^{2i+2p+1}\left(\beta_{i,p,0} + \sum_{l=1}^{i+p+1} \beta_{i,p,l}\cos(2l)\theta\right) + \\ & \sum_{i=0}^{\left[\frac{n_1}{2}\right]} a_{2i}r^{2i+2p}\left(\tilde{\beta}_{i,p,0} + \sum_{l=1}^{i+p+1} \tilde{\beta}_{i,p,l}\cos(2l-1)\theta\right) + \sum_{i=0}^{\left[\frac{n_2-1}{2}\right]} b_{2i+1}r^{2i+2p+2} \sum_{l=0}^{i+p+1} \bar{\beta}_{i,p,l}\sin(2l+1)\theta + \\ & \sum_{i=0}^{\left[\frac{n_3-1}{2}\right]} c_{2i+1}r^{2i+2p+3}\left(\gamma_{i,p,0} + \sum_{l=1}^{i+p+2} \gamma_{i,p,l}\cos(2l)\theta\right) + \sum_{i=0}^{\left[\frac{n_3}{2}\right]} c_{2i}r^{2i+2p+2}\left(\tilde{\gamma}_{i,p,0} + \sum_{l=1}^{i+p+2} \tilde{\gamma}_{i,p,l}\cos(2l-1)\theta\right) + \\ & \sum_{i=0}^{\left[\frac{n_4-1}{2}\right]} d_{2i+1}r^{2i+2p+4} \sum_{l=0}^{i+p+2} \bar{\gamma}_{i,p,l}\sin(2l+1)\theta + \sum_{i=1}^{\mu} d_{2i-2}r^{2i+2p+1} \sum_{l=1}^{i+p+1} \delta_{i,p,l}\sin(2l)\theta, \end{aligned}$$

where $\beta_{i,p,l}, \tilde{\beta}_{i,p,l}, \bar{\beta}_{i,p,l}, \gamma_{i,p,l}, \tilde{\gamma}_{i,p,l}, \bar{\gamma}_{i,p,l}$ and $\delta_{i,p,l}$ are constants.

Now, we define as

$$F_{20}^1(r) = \mathrm{Y}_1(r) + \mathrm{Y}_2(r) + \mathrm{Y}_3(r) + \mathrm{Y}_4(r) + \mathrm{Y}_5(r) + \mathrm{Y}_6(r) + \mathrm{Y}_7(r),$$

such that

$$\begin{aligned}
\Upsilon_1(r) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_1-1}{2}\right]}(2i+2p+1)a_{2i+1}A_{2i+1,2p+1}(\theta)\,r^{2i+2p}\right)y(r,\theta)d\theta,\\
\Upsilon_2(r) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_1}{2}\right]}(2i+2p)a_{2i}A_{2i,2p+1}(\theta)\,r^{2i+2p-1}\right)y(r,\theta)d\theta,\\
\Upsilon_3(r) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_2-1}{2}\right]}(2i+2p+2)b_{2i+1}A_{2i+1,2p+2}(\theta)\,r^{2i+2p+1}\right)y(r,\theta)d\theta,\\
\Upsilon_4(r) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_3-1}{2}\right]}(2i+2p+3)c_{2i+1}A_{2i+1,2p+3}(\theta)\,r^{2i+2p+2}\right)y(r,\theta)d\theta,\\
\Upsilon_5(r) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_3}{2}\right]}(2i+2p+2)c_{2i}A_{2i,2p+3}(\theta)\,r^{2i+2p+1}\right)y(r,\theta)d\theta,\\
\Upsilon_6(r) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_4-1}{2}\right]}(2i+2p+4)d_{2i+1}A_{2i+1,2p+4}(\theta)\,r^{2i+2p+3}\right)y(r,\theta)d\theta,\\
\Upsilon_7(r) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=1}^{\mu}(2i+2p+1)d_{2i-2}\left(A_{2i-2,2p+4}(\theta)-\frac{2p+3}{2i-1}A_{2i,2p+2}(\theta)\right)r^{2i+2p}\right)y(r,\theta)d\theta.
\end{aligned}$$

In the following Lemmas, we will compute the integrals $\Upsilon_1(r)-\Upsilon_7(r)$.

Lemma 2. *The integral* $\Upsilon_1(r)$ *is given by the following*

$$\Upsilon_1(r) = \sum_{i=0}^{\left[\frac{n_1-1}{2}\right]}\sum_{s=1}^{\mu}\frac{(2i+2p+1)}{2}a_{2i+1}d_{2s-2}\sum_{l=1}^{s+p+1}\delta_{s,p,l}D_{i,p,l}r^{2i+2s+4p+1}. \tag{15}$$

Proof. By using the integrals in Appendix, we get

$$(a_1)\quad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_1-1}{2}\right]}(2i+2p+1)a_{2i+1}A_{2i+1,2p+1}(\theta)\,r^{2i+2p}\right)\times$$
$$\left(\sum_{s=0}^{\left[\frac{n_1-1}{2}\right]}a_{2s+1}r^{2s+2p+1}\left(\beta_{s,p,0}+\sum_{l=1}^{s+p+1}\beta_{s,p,l}\cos(2l)\theta\right)\right)d\theta=0,$$

.

$$(b_1)\quad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_1-1}{2}\right]}(2i+2p+1)a_{2i+1}A_{2i+1,2p+1}(\theta)\,r^{2i+2p}\right)\times$$
$$\left(\sum_{s=0}^{\left[\frac{n_1}{2}\right]}a_{2s}r^{2s+2p}\left(\tilde{\beta}_{s,p,0}+\sum_{l=1}^{s+p+1}\tilde{\beta}_{s,p,l}\cos(2l-1)\theta\right)\right)d\theta=0,$$

$$(c_1)\quad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_1-1}{2}\right]}(2i+2p+1)a_{2i+1}A_{2i+1,2p+1}(\theta)\,r^{2i+2p}\right)\times$$

$$\left(\sum_{s=0}^{\left[\frac{n_2-1}{2}\right]} b_{2s+1} r^{2s+2p+2} \sum_{l=0}^{s+p+1} \bar{\beta}_{s,p,l} \sin\left(2l+1\right)\theta\right) d\theta = 0,$$

$$(d_1) \qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_1-1}{2}\right]}(2i+2p+1)a_{2i+1}A_{2i+1,2p+1}\left(\theta\right)r^{2i+2p}\right)\times$$
$$\left(\sum_{s=0}^{\left[\frac{n_3-1}{2}\right]} c_{2s+1} r^{2s+2p+3}\left(\gamma_{s,p,0}+\sum_{l=1}^{s+p+2}\gamma_{s,p,l}\cos\left(2l\right)\theta\right)\right) d\theta = 0,$$

$$(e_1) \qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_1-1}{2}\right]}(2i+2p+1)a_{2i+1}A_{2i+1,2p+1}\left(\theta\right)r^{2i+2p}\right)\times$$
$$\left(\sum_{s=0}^{\left[\frac{n_3}{2}\right]} c_{2s} r^{2s+2p+2}\left(\tilde{\gamma}_{s,p,0}+\sum_{l=1}^{s+p+2}\tilde{\gamma}_{s,p,l}\cos\left(2l-1\right)\theta\right)\right) d\theta = 0,$$

$$(f_1) \qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_1-1}{2}\right]}(2i+2p+1)a_{2i+1}A_{2i+1,2p+1}\left(\theta\right)r^{2i+2p}\right)\times$$
$$\left(\sum_{s=0}^{\left[\frac{n_4-1}{2}\right]} d_{2s+1} r^{2s+2p+4} \sum_{l=0}^{s+p+2} \bar{\gamma}_{s,p,l}\sin(2l+1)\theta\right) d\theta = 0,$$

$$(g_1) \qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_1-1}{2}\right]}(2i+2p+1)a_{2i+1}A_{2i+1,2p+1}\left(\theta\right)r^{2i+2p}\right)\times$$
$$\left(\sum_{s=1}^{\mu} d_{2s-2} r^{2s+2p+1} \sum_{l=1}^{s+p+1} \delta_{s,p,l}\sin(2l)\theta\right) d\theta =$$
$$\sum_{i=0}^{\left[\frac{n_1-1}{2}\right]}\sum_{s=1}^{\mu}\frac{(2i+2p+1)}{2}a_{2i+1}d_{2s-2}\sum_{l=1}^{s+p+1}\delta_{s,p,l}D_{i,p,l}r^{2i+2s+4p+1}.$$

We observe that the sum of the integrals $(a_1)-(g_1)$ is the polynomial (15). This ends the proof of Lemma (2). □

Lemma 3. *The integral* $\Upsilon_2(r)$ *is given by the following,*

$$\begin{aligned}\Upsilon_2(r) &= \sum_{i=0}^{\left[\frac{n_1}{2}\right]}\sum_{s=0}^{\left[\frac{n_2-1}{2}\right]}(i+p)a_{2i}b_{2s+1}\sum_{l=0}^{s+p+1}\bar{\beta}_{s,p,l}C_{i,p,l}r^{2i+2s+4p+1}\\ &\quad+\sum_{i=0}^{\left[\frac{n_1}{2}\right]}\sum_{s=0}^{\left[\frac{n_4-1}{2}\right]}(i+p)a_{2i}d_{2s+1}\sum_{l=0}^{s+p+2}\bar{\gamma}_{s,p,l}C_{i,p,l}r^{2i+2s+4p+3}.\end{aligned} \tag{16}$$

Proof. By using the integrals in Appendix, we get

$$(a_2) \qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_1}{2}\right]}(2i+2p)a_{2i}A_{2i,2p+1}\left(\theta\right)r^{2i+2p-1}\right)\times$$

$$\left(\sum_{s=0}^{\left[\frac{n_1-1}{2}\right]} a_{2s+1} r^{2s+2p+1}\left(\beta_{s,p,0}+\sum_{l=1}^{s+p+1}\beta_{s,p,l}\cos\left(2l\right)\theta\right)\right)d\theta=0,$$

$$(b_2)\qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_1}{2}\right]}(2i+2p)a_{2i}A_{2i,2p+1}\left(\theta\right)r^{2i+2p-1}\right)\times$$

$$\left(\sum_{s=0}^{\left[\frac{n_1}{2}\right]} a_{2s} r^{2s+2p}\left(\tilde{\beta}_{s,p,0}+\sum_{l=1}^{s+p+1}\tilde{\beta}_{s,p,l}\cos\left(2l-1\right)\theta\right)\right)d\theta=0,$$

$$(c_2)\qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_1}{2}\right]}(2i+2p)a_{2i}A_{2i,2p+1}\left(\theta\right)r^{2i+2p-1}\right)\times$$

$$\left(\sum_{s=0}^{\left[\frac{n_2-1}{2}\right]} b_{2s+1} r^{2s+2p+2}\sum_{l=0}^{s+p+1}\bar{\beta}_{s,p,l}\sin\left(2l+1\right)\theta\right)d\theta=$$

$$\sum_{i=0}^{\left[\frac{n_1}{2}\right]}\sum_{s=0}^{\left[\frac{n_2-1}{2}\right]}(i+p)a_{2i}b_{2s+1}\sum_{l=0}^{s+p+1}\bar{\beta}_{s,p,l}C_{i,p,l}r^{2i+2s+4p+1},$$

$$(d_2)\qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_1}{2}\right]}(2i+2p)a_{2i}A_{2i,2p+1}\left(\theta\right)r^{2i+2p-1}\right)\times$$

$$\left(\sum_{s=0}^{\left[\frac{n_3-1}{2}\right]} c_{2s+1} r^{2s+2p+3}\left(\gamma_{s,p,0}+\sum_{l=1}^{s+p+2}\gamma_{s,p,l}\cos\left(2l\right)\theta\right)\right)d\theta=0,$$

$$(e_2)\qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_1}{2}\right]}(2i+2p)a_{2i}A_{2i,2p+1}\left(\theta\right)r^{2i+2p-1}\right)\times$$

$$\left(\sum_{s=0}^{\left[\frac{n_3}{2}\right]} c_{2s} r^{2s+2p+2}\left(\tilde{\gamma}_{s,p,0}+\sum_{l=1}^{s+p+2}\tilde{\gamma}_{s,p,l}\cos\left(2l-1\right)\theta\right)\right)d\theta=0,$$

$$(f_2)\qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_1}{2}\right]}(2i+2p)a_{2i}A_{2i,2p+1}\left(\theta\right)r^{2i+2p-1}\right)\times$$

$$\left(\sum_{s=0}^{\left[\frac{n_4-1}{2}\right]} d_{2s+1} r^{2s+2p+4}\sum_{l=0}^{s+p+2}\bar{\gamma}_{s,p,l}\sin(2l+1)\theta\right)d\theta=$$

$$\sum_{i=0}^{\left[\frac{n_1}{2}\right]}\sum_{s=0}^{\left[\frac{n_4-1}{2}\right]}(i+p)a_{2i}d_{2s+1}\sum_{l=0}^{s+p+2}\bar{\gamma}_{s,p,l}C_{i,p,l}r^{2i+2s+4p+3},$$

$$(g_2)\qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_1}{2}\right]}(2i+2p)a_{2i}A_{2i,2p+1}\left(\theta\right)r^{2i+2p-1}\right)\times$$

$$\left(\sum_{s=1}^{\mu} d_{2s-2} r^{2s+2p+1}\sum_{l=1}^{s+p+1}\delta_{s,p,l}\sin(2l)\theta\right)d\theta=0.$$

We observe that the sum of the integrals $(a_2)-(g_2)$ is the polynomial (16). This ends the proof of Lemma 3. □

Lemma 4. *The integral* $\Upsilon_3(r)$ *is given by the following,*

$$\begin{aligned}\Upsilon_3(r) &= \sum_{i=0}^{\left[\frac{n_2-1}{2}\right]}\sum_{s=0}^{\left[\frac{n_1}{2}\right]}(i+p+1)b_{2i+1}a_{2s}\sum_{l=1}^{s+p+1}\tilde{\beta}_{s,p,l}E_{i,p,l}r^{2i+2s+4p+1}\\ &+\sum_{i=0}^{\left[\frac{n_2-1}{2}\right]}\sum_{s=0}^{\left[\frac{n_3}{2}\right]}(i+p+1)b_{2i+1}c_{2s}\sum_{l=1}^{s+p+2}\tilde{\gamma}_{s,p,l}E_{i,p,l}r^{2i+2s+4p+3}. \end{aligned}\tag{17}$$

Proof. By using the integrals in Appendix, we get

$$(a_3)\qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_2-1}{2}\right]}(2i+2p+2)b_{2i+1}A_{2i+1,2p+2}(\theta)\,r^{2i+2p+1}\right)\times$$

$$\left(\sum_{s=0}^{\left[\frac{n_1-1}{2}\right]}a_{2s+1}r^{2s+2p+1}\left(\beta_{s,p,0}+\sum_{l=1}^{s+p+1}\beta_{s,p,l}\cos(2l)\,\theta\right)\right)d\theta=0,$$

$$(b_3)\qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_2-1}{2}\right]}(2i+2p+2)b_{2i+1}A_{2i+1,2p+2}(\theta)\,r^{2i+2p+1}\right)\times$$

$$\left(\sum_{s=0}^{\left[\frac{n_1}{2}\right]}a_{2s}r^{2s+2p}\left(\tilde{\beta}_{s,p,0}+\sum_{l=1}^{s+p+1}\tilde{\beta}_{s,p,l}\cos(2l-1)\,\theta\right)\right)d\theta=$$

$$\sum_{i=0}^{\left[\frac{n_2-1}{2}\right]}\sum_{s=0}^{\left[\frac{n_1}{2}\right]}(i+p+1)b_{2i+1}a_{2s}\sum_{l=1}^{s+p+1}\tilde{\beta}_{s,p,l}E_{i,p,l}r^{2i+2s+4p+1},$$

$$(c_3)\qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_2-1}{2}\right]}(2i+2p+2)b_{2i+1}A_{2i+1,2p+2}(\theta)\,r^{2i+2p+1}\right)\times$$

$$\left(\sum_{s=0}^{\left[\frac{n_2-1}{2}\right]}b_{2s+1}r^{2s+2p+2}\sum_{l=0}^{s+p+1}\bar{\beta}_{s,p,l}\sin(2l+1)\,\theta\right)d\theta=0,$$

$$(d_3)\qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_2-1}{2}\right]}(2i+2p+2)b_{2i+1}A_{2i+1,2p+2}(\theta)\,r^{2i+2p+1}\right)\times$$

$$\left(\sum_{s=0}^{\left[\frac{n_3-1}{2}\right]}c_{2s+1}r^{2s+2p+3}\left(\gamma_{s,p,0}+\sum_{l=1}^{s+p+2}\gamma_{s,p,l}\cos(2l)\,\theta\right)\right)d\theta=0,$$

$$(e_3)\qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_2-1}{2}\right]}(2i+2p+2)b_{2i+1}A_{2i+1,2p+2}(\theta)\,r^{2i+2p+1}\right)\times$$

$$\left(\sum_{s=0}^{\left[\frac{n_3}{2}\right]}c_{2s}r^{2s+2p+2}\left(\tilde{\gamma}_{s,p,0}+\sum_{l=1}^{s+p+2}\tilde{\gamma}_{s,p,l}\cos(2l-1)\,\theta\right)\right)d\theta=$$

$$\sum_{i=0}^{\left[\frac{n_2-1}{2}\right]}\sum_{s=0}^{\left[\frac{n_3}{2}\right]}(i+p+1)b_{2i+1}c_{2s}\sum_{l=1}^{s+p+2}\tilde{\gamma}_{s,p,l}E_{i,p,l}r^{2i+2s+4p+3},$$

$$(f_3)\qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_2-1}{2}\right]}(2i+2p+2)b_{2i+1}A_{2i+1,2p+2}(\theta)\,r^{2i+2p+1}\right)\times$$
$$\left(\sum_{s=0}^{\left[\frac{n_4-1}{2}\right]}d_{2s+1}r^{2s+2p+4}\sum_{l=0}^{s+p+2}\bar{\gamma}_{s,p,l}\sin(2l+1)\theta\right)d\theta=0,$$

$$(g_3)\qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_2-1}{2}\right]}(2i+2p+2)b_{2i+1}A_{2i+1,2p+2}(\theta)\,r^{2i+2p+1}\right)\times$$
$$\left(\sum_{s=1}^{\mu}d_{2s-2}r^{2s+2p+1}\sum_{l=1}^{s+p+1}\delta_{s,p,l}\sin(2l)\theta\right)d\theta=0.$$

We observe that the sum of the integrals $(a_3)-(g_3)$ is the polynomial (17). This ends the proof of Lemma 4. □

Lemma 5. *The integral $\Upsilon_4(r)$ is given by the following*

$$\Upsilon_4(r) = \sum_{i=0}^{\left[\frac{n_3-1}{2}\right]}\sum_{s=1}^{\mu}\frac{(2i+2p+3)}{2}c_{2i+1}d_{2s-2}\sum_{l=1}^{s+p+1}\delta_{s,p,l}\tilde{D}_{i,p,l}r^{2i+2s+4p+3}. \tag{18}$$

Proof. By using the integrals in Appendix, we get

$$(a_4)\qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_3-1}{2}\right]}(2i+2p+3)c_{2i+1}A_{2i+1,2p+3}(\theta)\,r^{2i+2p+2}\right)\times$$
$$\left(\sum_{s=0}^{\left[\frac{n_1-1}{2}\right]}a_{2s+1}r^{2s+2p+1}\left(\beta_{s,p,0}+\sum_{l=1}^{s+p+1}\beta_{s,p,l}\cos(2l)\,\theta\right)\right)d\theta=0,$$

$$(b_4)\qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_3-1}{2}\right]}(2i+2p+3)c_{2i+1}A_{2i+1,2p+3}(\theta)\,r^{2i+2p+2}\right)\times$$
$$\left(\sum_{s=0}^{\left[\frac{n_1}{2}\right]}a_{2s}r^{2s+2p}\left(\tilde{\beta}_{s,p,0}+\sum_{l=1}^{s+p+1}\tilde{\beta}_{s,p,l}\cos(2l-1)\,\theta\right)\right)d\theta=0,$$

$$(c_4)\qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_3-1}{2}\right]}(2i+2p+3)c_{2i+1}A_{2i+1,2p+3}(\theta)\,r^{2i+2p+2}\right)\times$$
$$\left(\sum_{s=0}^{\left[\frac{n_2-1}{2}\right]}b_{2s+1}r^{2s+2p+2}\sum_{l=0}^{s+p+1}\bar{\beta}_{s,p,l}\sin(2l+1)\,\theta\right)d\theta=0,$$

$$(d_4)\qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{\left[\frac{n_3-1}{2}\right]}(2i+2p+3)c_{2i+1}A_{2i+1,2p+3}(\theta)\,r^{2i+2p+2}\right)\times$$

$$\left(\sum_{s=0}^{\left[\frac{n_3-1}{2}\right]} c_{2s+1} r^{2s+2p+3} \left(\gamma_{s,p,0} + \sum_{l=1}^{s+p+2} \gamma_{s,p,l} \cos(2l)\theta\right)\right) d\theta = 0,$$

$$(e_4) \qquad \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{i=0}^{\left[\frac{n_3-1}{2}\right]} (2i+2p+3) c_{2i+1} A_{2i+1,2p+3}(\theta)\, r^{2i+2p+2}\right) \times$$

$$\left(\sum_{s=0}^{\left[\frac{n_3}{2}\right]} c_{2s} r^{2s+2p+2} \left(\tilde{\gamma}_{s,p,0} + \sum_{l=1}^{s+p+2} \tilde{\gamma}_{s,p,l} \cos(2l-1)\theta\right)\right) d\theta = 0,$$

$$(f_4) \qquad \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{i=0}^{\left[\frac{n_3-1}{2}\right]} (2i+2p+3) c_{2i+1} A_{2i+1,2p+3}(\theta)\, r^{2i+2p+2}\right) \times$$

$$\left(\sum_{s=0}^{\left[\frac{n_4-1}{2}\right]} d_{2s+1} r^{2s+2p+4} \sum_{l=0}^{s+p+2} \bar{\gamma}_{s,p,l} \sin(2l+1)\theta\right) d\theta = 0,$$

$$(g_4) \qquad \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{i=0}^{\left[\frac{n_3-1}{2}\right]} (2i+2p+3) c_{2i+1} A_{2i+1,2p+3}(\theta)\, r^{2i+2p+2}\right) \times$$

$$\left(\sum_{s=1}^{\mu} d_{2s-2} r^{2s+2p+1} \sum_{l=1}^{s+p+1} \delta_{s,p,l} \sin(2l)\theta\right) d\theta =$$

$$\sum_{i=0}^{\left[\frac{n_3-1}{2}\right]} \sum_{s=1}^{\mu} \frac{(2i+2p+3)}{2} c_{2i+1} d_{2s-2} \sum_{l=1}^{s+p+1} \delta_{s,p,l} \tilde{D}_{i,p,l} r^{2i+2s+4p+3}.$$

We observe that the sum of the integrals $(a_4)-(g_4)$ is the polynomial (18). This ends the proof of Lemma 5. □

Lemma 6. *The integral $\Upsilon_5(r)$ is given by the following*

$$\begin{aligned}\Upsilon_5(r) \;=\; & \sum_{i=0}^{\left[\frac{n_3}{2}\right]} \sum_{s=0}^{\left[\frac{n_2-1}{2}\right]} (i+p+1) c_{2i} b_{2s+1} \sum_{l=0}^{s+p+1} \bar{\beta}_{s,p,l} \tilde{C}_{i,p,l} r^{2i+2s+4p+3} \\ & + \sum_{i=0}^{\left[\frac{n_3}{2}\right]} \sum_{s=0}^{\left[\frac{n_4-1}{2}\right]} (i+p+1) c_{2i} d_{2s+1} \sum_{l=0}^{s+p+2} \bar{\gamma}_{s,p,l} \tilde{C}_{i,p,l} r^{2i+2s+4p+5}. \end{aligned} \tag{19}$$

Proof. By using the integrals in Appendix, we get

$$(a_5) \qquad \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{i=0}^{\left[\frac{n_3}{2}\right]} (2i+2p+2) c_{2i} A_{2i,2p+3}(\theta)\, r^{2i+2p+1}\right) \times$$

$$\left(\sum_{s=0}^{\left[\frac{n_1-1}{2}\right]} a_{2s+1} r^{2s+2p+1} \left(\beta_{s,p,0} + \sum_{l=1}^{s+p+1} \beta_{s,p,l} \cos(2l)\theta\right)\right) d\theta = 0,$$

$$(b_5) \qquad \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{i=0}^{\left[\frac{n_3}{2}\right]} (2i+2p+2) c_{2i} A_{2i,2p+3}(\theta)\, r^{2i+2p+1}\right) \times$$

$$\left(\sum_{s=0}^{\left[\frac{n_1}{2}\right]} a_{2s} r^{2s+2p} \left(\tilde{\beta}_{s,p,0} + \sum_{l=1}^{s+p+1} \tilde{\beta}_{s,p,l} \cos\left(2l-1\right)\theta\right)\right) d\theta = 0,$$

$$(c_5) \qquad \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{i=0}^{\left[\frac{n_3}{2}\right]} (2i+2p+2) c_{2i} A_{2i,2p+3}\left(\theta\right) r^{2i+2p+1}\right) \times$$

$$\left(\sum_{s=0}^{\left[\frac{n_2-1}{2}\right]} b_{2s+1} r^{2s+2p+2} \sum_{l=0}^{s+p+1} \bar{\beta}_{s,p,l} \sin\left(2l+1\right)\theta\right) d\theta =$$

$$\sum_{i=0}^{\left[\frac{n_3}{2}\right]} \sum_{s=0}^{\left[\frac{n_2-1}{2}\right]} (i+p+1) c_{2i} b_{2s+1} \sum_{l=0}^{s+p+1} \bar{\beta}_{s,p,l} \tilde{C}_{i,p,l} r^{2i+2s+4p+3},$$

$$(d_5) \qquad \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{i=0}^{\left[\frac{n_3}{2}\right]} (2i+2p+2) c_{2i} A_{2i,2p+3}\left(\theta\right) r^{2i+2p+1}\right) \times$$

$$\left(\sum_{s=0}^{\left[\frac{n_3-1}{2}\right]} c_{2s+1} r^{2s+2p+3} \left(\gamma_{s,p,0} + \sum_{l=1}^{s+p+2} \gamma_{s,p,l} \cos\left(2l\right)\theta\right)\right) d\theta = 0,$$

$$(e_5) \qquad \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{i=0}^{\left[\frac{n_3}{2}\right]} (2i+2p+2) c_{2i} A_{2i,2p+3}\left(\theta\right) r^{2i+2p+1}\right) \times$$

$$\left(\sum_{s=0}^{\left[\frac{n_3}{2}\right]} c_{2s} r^{2s+2p+2} \left(\tilde{\gamma}_{s,p,0} + \sum_{l=1}^{s+p+2} \tilde{\gamma}_{s,p,l} \cos\left(2l-1\right)\theta\right)\right) d\theta = 0,$$

$$(f_5) \qquad \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{i=0}^{\left[\frac{n_3}{2}\right]} (2i+2p+2) c_{2i} A_{2i,2p+3}\left(\theta\right) r^{2i+2p+1}\right) \times$$

$$\left(\sum_{s=0}^{\left[\frac{n_4-1}{2}\right]} d_{2s+1} r^{2s+2p+4} \sum_{l=0}^{s+p+2} \bar{\gamma}_{s,p,l} \sin(2l+1)\theta\right) d\theta =$$

$$\sum_{i=0}^{\left[\frac{n_3}{2}\right]} \sum_{s=0}^{\left[\frac{n_4-1}{2}\right]} (i+p+1) c_{2i} d_{2s+1} \sum_{l=0}^{s+p+2} \bar{\gamma}_{s,p,l} \tilde{C}_{i,p,l} r^{2i+2s+4p+5},$$

$$(g_5) \qquad \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{i=0}^{\left[\frac{n_3}{2}\right]} (2i+2p+2) c_{2i} A_{2i,2p+3}\left(\theta\right) r^{2i+2p+1}\right) \times$$

$$\left(\sum_{s=1}^{\mu} d_{2s-2} r^{2s+2p+1} \sum_{l=1}^{s+p+1} \delta_{s,p,l} \sin(2l)\theta\right) d\theta = 0.$$

We observe that the sum of the integrals $(a_5) - (g_5)$ is the polynomial (19). This ends the proof of Lemma 6. □

Lemma 7. *The integral* $\Upsilon_6(r)$ *is given by the following,*

$$\begin{aligned} \Upsilon_6(r) &= \sum_{i=0}^{\left[\frac{n_4-1}{2}\right]} \sum_{s=0}^{\left[\frac{n_1}{2}\right]} (i+p+2) d_{2i+1} a_{2s} \sum_{l=1}^{s+p+1} \tilde{\beta}_{s,p,l} \tilde{E}_{i,p,l} r^{2i+2s+4p+3} \\ &+ \sum_{i=0}^{\left[\frac{n_4-1}{2}\right]} \sum_{s=0}^{\left[\frac{n_3}{2}\right]} (i+p+2) d_{2i+1} c_{2s} \sum_{l=1}^{s+p+2} \tilde{\gamma}_{s,p,l} \tilde{E}_{i,p,l} r^{2i+2s+4p+5}. \end{aligned} \tag{20}$$

Proof. By using the integrals in Appendix, we get

$$(a_6) \qquad \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{i=0}^{\left[\frac{n_4-1}{2}\right]} (2i+2p+4) d_{2i+1} A_{2i+1,2p+4}(\theta) \, r^{2i+2p+3} \right) \times$$

$$\left(\sum_{s=0}^{\left[\frac{n_1-1}{2}\right]} a_{2s+1} r^{2s+2p+1} \left(\beta_{s,p,0} + \sum_{l=1}^{s+p+1} \beta_{s,p,l} \cos(2l)\,\theta \right) \right) d\theta = 0,$$

$$(b_6) \qquad \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{i=0}^{\left[\frac{n_4-1}{2}\right]} (2i+2p+4) d_{2i+1} A_{2i+1,2p+4}(\theta) \, r^{2i+2p+3} \right) \times$$

$$\left(\sum_{s=0}^{\left[\frac{n_1}{2}\right]} a_{2s} r^{2s+2p} \left(\tilde{\beta}_{s,p,0} + \sum_{l=1}^{s+p+1} \tilde{\beta}_{s,p,l} \cos(2l-1)\,\theta \right) \right) d\theta =$$

$$\sum_{i=0}^{\left[\frac{n_4-1}{2}\right]} \sum_{s=0}^{\left[\frac{n_1}{2}\right]} (i+p+2) d_{2i+1} a_{2s} \sum_{l=1}^{s+p+1} \tilde{\beta}_{s,p,l} \tilde{E}_{i,p,l} r^{2i+2s+4p+3},$$

$$(c_6) \qquad \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{i=0}^{\left[\frac{n_4-1}{2}\right]} (2i+2p+4) d_{2i+1} A_{2i+1,2p+4}(\theta) \, r^{2i+2p+3} \right) \times$$

$$\left(\sum_{s=0}^{\left[\frac{n_2-1}{2}\right]} b_{2s+1} r^{2s+2p+2} \sum_{l=0}^{s+p+1} \bar{\beta}_{s,p,l} \sin(2l+1)\,\theta \right) d\theta = 0,$$

$$(d_6) \qquad \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{i=0}^{\left[\frac{n_4-1}{2}\right]} (2i+2p+4) d_{2i+1} A_{2i+1,2p+4}(\theta) \, r^{2i+2p+3} \right) \times$$

$$\left(\sum_{s=0}^{\left[\frac{n_3-1}{2}\right]} c_{2s+1} r^{2s+2p+3} \left(\gamma_{s,p,0} + \sum_{l=1}^{s+p+2} \gamma_{s,p,l} \cos(2l)\,\theta \right) \right) d\theta = 0,$$

$$(e_6) \qquad \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{i=0}^{\left[\frac{n_4-1}{2}\right]} (2i+2p+4) d_{2i+1} A_{2i+1,2p+4}(\theta) \, r^{2i+2p+3} \right) \times$$

$$\left(\sum_{s=0}^{\left[\frac{n_3}{2}\right]} c_{2s} r^{2s+2p+2} \left(\tilde{\gamma}_{s,p,0} + \sum_{l=1}^{s+p+2} \tilde{\gamma}_{s,p,l} \cos(2l-1)\,\theta \right) \right) d\theta =$$

$$\sum_{i=0}^{\left[\frac{n_4-1}{2}\right]} \sum_{s=0}^{\left[\frac{n_3}{2}\right]} (i+p+2) d_{2i+1} c_{2s} \sum_{l=1}^{s+p+2} \tilde{\gamma}_{s,p,l} \tilde{E}_{i,p,l} r^{2i+2s+4p+5},$$

$$(f_6) \qquad \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{i=0}^{\left[\frac{n_4-1}{2}\right]} (2i+2p+4) d_{2i+1} A_{2i+1,2p+4}(\theta) r^{2i+2p+3} \right) \times$$
$$\left(\sum_{s=0}^{\left[\frac{n_4-1}{2}\right]} d_{2s+1} r^{2s+2p+4} \sum_{l=0}^{s+p+2} \bar{\gamma}_{s,p,l} \sin(2l+1)\theta \right) d\theta = 0,$$

$$(g_6) \qquad \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{i=0}^{\left[\frac{n_4-1}{2}\right]} (2i+2p+4) d_{2i+1} A_{2i+1,2p+4}(\theta) r^{2i+2p+3} \right) \times$$
$$\left(\sum_{s=1}^{\mu} d_{2s-2} r^{2s+2p+1} \sum_{l=1}^{s+p+1} \delta_{s,p,l} \sin(2l)\theta \right) d\theta = 0.$$

We observe that the sum of the integrals $(a_6) - (g_6)$ is the polynomial (20). This ends the proof of Lemma 7. □

Lemma 8. *The integral $\Upsilon_7(r)$ is given by the following,*

$$\begin{aligned} \Upsilon_7(r) &= \sum_{i=1}^{\mu} \sum_{s=0}^{\left[\frac{n_1-1}{2}\right]} \frac{(2i+2p+1)}{2} d_{2i-2} a_{2s+1} \sum_{l=0}^{s+p+1} \beta_{s,p,l} \left(\tilde{F}_{i,p,l} - \frac{2p+3}{2i-1} F_{i,p,l} \right) r^{2i+2s+4p+1} \\ &\quad + \sum_{i=1}^{\mu} \sum_{s=0}^{\left[\frac{n_3-1}{2}\right]} \frac{(2i+2p+1)}{2} d_{2i-2} c_{2s+1} \sum_{l=0}^{s+p+2} \gamma_{s,p,l} \left(\tilde{F}_{i,p,l} - \frac{2p+3}{2i-1} F_{i,p,l} \right) r^{2i+2s+4p+3}. \end{aligned} \tag{21}$$

Proof. By using the integrals in Appendix, we get

$$(a_7) \qquad \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{i=1}^{\mu} (2i+2p+1) d_{2i-2} \left(A_{2i-2,2p+4}(\theta) - \frac{2p+3}{2i-1} A_{2i,2p+2}(\theta) \right) r^{2i+2p} \right) \times$$
$$\left(\sum_{s=0}^{\left[\frac{n_1-1}{2}\right]} a_{2s+1} r^{2s+2p+1} \left(\beta_{s,p,0} + \sum_{l=1}^{s+p+1} \beta_{s,p,l} \cos(2l)\theta \right) \right) d\theta =$$
$$\sum_{i=1}^{\mu} \sum_{s=0}^{\left[\frac{n_1-1}{2}\right]} \frac{(2i+2p+1)}{2} d_{2i-2} a_{2s+1} \sum_{l=0}^{s+p+1} \beta_{s,p,l} \left(\tilde{F}_{i,p,l} - \frac{2p+3}{2i-1} F_{i,p,l} \right) r^{2i+2s+4p+1},$$

$$(b_7) \qquad \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{i=1}^{\mu} (2i+2p+1) d_{2i-2} \left(A_{2i-2,2p+4}(\theta) - \frac{2p+3}{2i-1} A_{2i,2p+2}(\theta) \right) r^{2i+2p} \right) \times$$
$$\left(\sum_{s=0}^{\left[\frac{n_1}{2}\right]} a_{2s} r^{2s+2p} \left(\tilde{\beta}_{s,p,0} + \sum_{l=1}^{s+p+1} \tilde{\beta}_{s,p,l} \cos(2l-1)\theta \right) \right) d\theta = 0,$$

$$(c_7) \qquad \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{i=1}^{\mu} (2i+2p+1) d_{2i-2} \left(A_{2i-2,2p+4}(\theta) - \frac{2p+3}{2i-1} A_{2i,2p+2}(\theta) \right) r^{2i+2p} \right) \times$$
$$\left(\sum_{s=0}^{\left[\frac{n_2-1}{2}\right]} b_{2s+1} r^{2s+2p+2} \sum_{l=0}^{s+p+1} \bar{\beta}_{s,p,l} \sin(2l+1)\theta \right) d\theta = 0,$$

$$(d_7) \qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=1}^{\mu}(2i+2p+1)d_{2i-2}\left(A_{2i-2,2p+4}(\theta)-\frac{2p+3}{2i-1}A_{2i,2p+2}(\theta)\right)r^{2i+2p}\right)\times$$
$$\left(\sum_{s=0}^{\left[\frac{n_3-1}{2}\right]}c_{2s+1}r^{2s+2p+3}\left(\gamma_{s,p,0}+\sum_{l=1}^{s+p+2}\gamma_{s,p,l}\cos(2l)\theta\right)\right)d\theta=$$
$$\sum_{i=1}^{\mu}\sum_{s=0}^{\left[\frac{n_3-1}{2}\right]}\frac{(2i+2p+1)}{2}d_{2i-2}c_{2s+1}\sum_{l=0}^{s+p+2}\gamma_{s,p,l}\left(\tilde{F}_{i,p,l}-\frac{2p+3}{2i-1}F_{i,p,l}\right)r^{2i+2s+4p+3},$$
$$(e_7) \qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=1}^{\mu}(2i+2p+1)d_{2i-2}\left(A_{2i-2,2p+4}(\theta)-\frac{2p+3}{2i-1}A_{2i,2p+2}(\theta)\right)r^{2i+2p}\right)\times$$
$$\left(\sum_{s=0}^{\left[\frac{n_3}{2}\right]}c_{2s}r^{2s+2p+2}\left(\tilde{\gamma}_{s,p,0}+\sum_{l=1}^{s+p+2}\tilde{\gamma}_{s,p,l}\cos(2l-1)\theta\right)\right)d\theta=0,$$
$$(f_7) \qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=1}^{\mu}(2i+2p+1)d_{2i-2}\left(A_{2i-2,2p+4}(\theta)-\frac{2p+3}{2i-1}A_{2i,2p+2}(\theta)\right)r^{2i+2p}\right)\times$$
$$\left(\sum_{s=0}^{\left[\frac{n_4-1}{2}\right]}d_{2s+1}r^{2s+2p+4}\sum_{l=0}^{s+p+2}\bar{\gamma}_{s,p,l}\sin(2l+1)\theta\right)d\theta=0,$$
$$(g_7) \qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=1}^{\mu}(2i+2p+1)d_{2i-2}\left(A_{2i-2,2p+4}(\theta)-\frac{2p+3}{2i-1}A_{2i,2p+2}(\theta)\right)r^{2i+2p}\right)\times$$
$$\left(\sum_{s=1}^{\mu}d_{2s-2}r^{2s+2p+1}\sum_{l=1}^{s+p+1}\delta_{s,p,l}\sin(2l)\theta\right)d\theta=0.$$

We observe that the sum of the integrals $(a_7)-(g_7)$ is the polynomial (21). This ends the proof of Lemma 8. □

By Lemmas 2-8, we obtain $F_{20}^1(r)=r^{1+4p}P_1(r^2)$, where $P_1(r^2)$ is a polynomial of degree

$$\max\left\{\left[\frac{n_1}{2}\right]+\left[\frac{n_2-1}{2}\right],\left[\frac{n_1}{2}\right]+\left[\frac{n_4-1}{2}\right]+1,\left[\frac{n_1-1}{2}\right]+\mu,\right.$$
$$\left.\left[\frac{n_2-1}{2}\right]+\left[\frac{n_3}{2}\right]+1,\left[\frac{n_3}{2}\right]+\left[\frac{n_4-1}{2}\right]+2,\left[\frac{n_3-1}{2}\right]+\mu+1\right\}.$$

Again by substituting (14) in (13) and (12), we obtain

$$F_2(r,\theta)=\sum_{i=0}^{n_1}\bar{a}_i r^{i+2p}A_{i,2p+1}(\theta)+\sum_{i=0}^{n_2}\bar{b}_i r^{i+2p+1}A_{i,2p+2}(\theta)+\sum_{i=0}^{n_3}\bar{c}_i r^{i+2p+2}A_{i,2p+3}(\theta)+\sum_{i=0}^{n_4}\bar{d}_i r^{i+2p+3}A_{i,2p+4}(\theta)-$$
$$\frac{1}{r}\left(\sum_{i=0}^{\left[\frac{n_1-1}{2}\right]}a_{2i+1}A_{2i+1,2p+1}(\theta)r^{2i+2p+1}+\sum_{i=0}^{\left[\frac{n_1}{2}\right]}a_{2i}A_{2i,2p+1}(\theta)r^{2i+2p}+\sum_{i=0}^{\left[\frac{n_2-1}{2}\right]}b_{2i+1}A_{2i+1,2p+2}(\theta)r^{2i+2p+2}+\right.$$
$$\sum_{i=0}^{\left[\frac{n_3-1}{2}\right]}c_{2i+1}A_{2i+1,2p+3}(\theta)r^{2i+2p+3}+\sum_{i=0}^{\left[\frac{n_3}{2}\right]}c_{2i}A_{2i,2p+3}(\theta)r^{2i+2p+2}+\sum_{i=0}^{\left[\frac{n_4-1}{2}\right]}d_{2i+1}A_{2i+1,2p+4}(\theta)r^{2i+2p+4}+$$
$$\left.\sum_{i=1}^{\mu}\left(A_{2i-2,2p+4}(\theta)-\frac{2p+3}{2i-1}A_{2i,2p+2}(\theta)\right)d_{2i-2}r^{2i+2p+1}\right)\times$$
$$\left(\sum_{k=0}^{\left[\frac{n_1-1}{2}\right]}a_{2k+1}A_{2k+2,2p}(\theta)r^{2k+2p+1}+\sum_{k=0}^{\left[\frac{n_1}{2}\right]}a_{2k}A_{2k+1,2p}(\theta)r^{2k+2p}+\sum_{k=0}^{\left[\frac{n_2-1}{2}\right]}b_{2k+1}A_{2k+2,2p+1}(\theta)r^{2k+2p+2}+\right.$$

$$\sum_{k=0}^{\left[\frac{n_3-1}{2}\right]} c_{2k+1}A_{2k+2,2p+2}(\theta)\, r^{2k+2p+3} + \sum_{k=0}^{\left[\frac{n_3}{2}\right]} c_{2k}A_{2k+1,2p+2}(\theta)\, r^{2k+2p+2} + \sum_{k=0}^{\left[\frac{n_4-1}{2}\right]} d_{2k+1}A_{2k+2,2p+3}(\theta)\, r^{2k+2p+4} + \sum_{k=1}^{\mu}\left(A_{2k-1,2p+3}(\theta) - \frac{2p+3}{2k-1}A_{2k+1,2p+1}(\theta)\right) d_{2k-2}r^{2k+2p+1}\Bigg).$$

To find the explicit expression of $F_{20}^2(r) = \dfrac{1}{2\pi}\int_0^{2\pi} F_2(r,\theta)d\theta$, we use the Lemma 1. So, we get

$$\begin{aligned}
F_{20}^2(r) = &\left(\sum_{i=0}^{\left[\frac{n_2}{2}\right]} \bar{b}_{2i}r^{2i}\xi_{2i,2p+2}(2\pi) + \sum_{i=0}^{\left[\frac{n_4}{2}\right]} \bar{d}_{2i}r^{2i+2}\xi_{2i,2p+4}(2\pi)\right) r^{2p+1} - \\
&\sum_{i=0}^{\left[\frac{n_1-1}{2}\right]}\sum_{k=1}^{\mu} a_{2i+1}d_{2k-2}\left(\xi_{2i+2k,4p+4}(2\pi) - \frac{2p+3}{2k-1}\xi_{2i+2k+2,4p+2}(2\pi)\right) r^{2i+2k+4p+1} - \\
&\sum_{i=0}^{\left[\frac{n_1}{2}\right]}\sum_{k=0}^{\left[\frac{n_2-1}{2}\right]} a_{2i}b_{2k+1}\xi_{2i+2k+2,4p+2}(2\pi)r^{2i+2k+4p+1} - \sum_{i=0}^{\left[\frac{n_1}{2}\right]}\sum_{k=0}^{\left[\frac{n_4-1}{2}\right]} a_{2i}d_{2k+1}\xi_{2i+2k+2,4p+4}(2\pi)r^{2i+2k+4p+3} - \\
&\sum_{i=0}^{\left[\frac{n_2-1}{2}\right]}\sum_{k=0}^{\left[\frac{n_1}{2}\right]} b_{2i+1}a_{2k}\xi_{2i+2k+2,4p+4}(2\pi)\, r^{2i+2k+4p+1} - \sum_{i=0}^{\left[\frac{n_2-1}{2}\right]}\sum_{k=0}^{\left[\frac{n_3}{2}\right]} b_{2i+1}c_{2k}\xi_{2i+2k+2,4p+4}(2\pi)\, r^{2i+2k+4p+3} - \\
&\sum_{i=0}^{\left[\frac{n_3-1}{2}\right]}\sum_{k=1}^{\mu} c_{2i+1}d_{2k-2}\left(\xi_{2i+2k,4p+6}(2\pi) - \frac{2p+3}{2k-1}\xi_{2i+2k+2,4p+4}(2\pi)\right) r^{2i+2k+4p+3} - \\
&\sum_{i=0}^{\left[\frac{n_3}{2}\right]}\sum_{k=0}^{\left[\frac{n_2-1}{2}\right]} c_{2i}b_{2k+1}\xi_{2i+2k+2,4p+4}(2\pi)\, r^{2i+2k+4p+3} - \sum_{i=0}^{\left[\frac{n_3}{2}\right]}\sum_{k=0}^{\left[\frac{n_4-1}{2}\right]} c_{2i}d_{2k+1}\xi_{2i+2k+2,4p+6}(2\pi)\, r^{2i+2k+4p+5} - \\
&\sum_{i=0}^{\left[\frac{n_4-1}{2}\right]}\sum_{k=0}^{\left[\frac{n_1}{2}\right]} d_{2i+1}a_{2k}\xi_{2i+2k+2,4p+4}(2\pi)\, r^{2i+2k+4p+3} - \sum_{i=0}^{\left[\frac{n_4-1}{2}\right]}\sum_{k=0}^{\left[\frac{n_3}{2}\right]} d_{2i+1}c_{2k}\xi_{2i+2k+2,4p+6}(2\pi)\, r^{2i+2k+4p+5} - \\
&\sum_{i=1}^{\mu}\sum_{k=0}^{\left[\frac{n_1-1}{2}\right]} d_{2i-2}a_{2k+1}\left(\xi_{2i+2k,4p+4}(2\pi) - \frac{2p+3}{2i-1}\xi_{2i+2k+2,4p+2}(2\pi)\right) r^{2i+2k+4p+1} - \\
&\sum_{i=1}^{\mu}\sum_{k=0}^{\left[\frac{n_3-1}{2}\right]} d_{2i-2}c_{2k+1}\left(\xi_{2i+2k,4p+6}(2\pi) - \frac{2p+3}{2i-1}\xi_{2i+2k+2,4p+4}(2\pi)\right) r^{2i+2k+4p+3} = r^{1+2p}\left(P_2(r^2) + r^{2p}P_3(r^2)\right).
\end{aligned}$$

Where $P_2(r^2)$ is a polynomial of degree

$$\max\left\{\left[\frac{n_2}{2}\right], \left[\frac{n_4}{2}\right]+1\right\},$$

and $P_3(r^2)$ is a polynomial of degree

$$\max\left\{\left[\frac{n_1}{2}\right]+\left[\frac{n_2-1}{2}\right], \left[\frac{n_1}{2}\right]+\left[\frac{n_4-1}{2}\right]+1, \left[\frac{n_1-1}{2}\right]+\mu,\right.$$
$$\left.\left[\frac{n_2-1}{2}\right]+\left[\frac{n_3}{2}\right]+1, \left[\frac{n_3}{2}\right]+\left[\frac{n_4-1}{2}\right]+2, \left[\frac{n_3-1}{2}\right]+\mu+1\right\}.$$

Therefore $F_{20}(r)$ is a polynomial in the variable r^2 of the form

$$F_{20}(r) = F_{20}^1(r) + F_{20}^2(r) = r^{1+2p}\left(r^{2p}P_1\left(r^2\right) + P_2\left(r^2\right) + r^{2p}P_3\left(r^2\right)\right).$$

Thus, $F_{20}(r)$ has at most

$$\max\left\{\left[\frac{n_2}{2}\right], \left[\frac{n_4}{2}\right]+1, \left[\frac{n_1}{2}\right]+\left[\frac{n_2-1}{2}\right]+p, \left[\frac{n_1}{2}\right]+\left[\frac{n_4-1}{2}\right]+p+1, \left[\frac{n_1-1}{2}\right]+\right.$$
$$\left.\mu+p, \left[\frac{n_2-1}{2}\right]+\left[\frac{n_3}{2}\right]+p+1, \left[\frac{n_3}{2}\right]+\left[\frac{n_4-1}{2}\right]+p+2, \left[\frac{n_3-1}{2}\right]+\mu+p+1\right\},$$

positive roots. Hence statement (b) of Theorem 1 is proved.

4. Applications

In this section we shall give examples to illustrate statements (a) and (b) of Theorem 1. We consider the first example corresponds to statement (a)

Example 1.

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - \varepsilon(2xy^{10} + (\frac{256}{231} - \frac{2560}{33}x^2)y^{11} + x^3y^{12} + (\frac{32768}{429}x^2)y^{13}), \end{cases} \tag{22}$$

where $f^1(x), g^1(x), h^1(x), l^1(x)$ have degree $n_1 = 1, n_2 = 2, n_3 = 3$ and $n_4 = 2$ respectively.

The function of averaging theory of first order is

$$F_{10}(r) = r^{15} - \frac{5}{4}r^{13} + \frac{1}{4}r^{11},$$

which has exactly $\left[\frac{n_4}{2}\right] + 1 =$ two positive zeros $r_1 = \frac{1}{2}, r_2 = 1$. Which satisfy

$$\frac{dF_{10}(r)}{dr}|_{r=r_1} = -\frac{3}{8192} \neq 0, \quad \frac{dF_{10}(r)}{dr}|_{r=r_2} = \frac{3}{2} \neq 0,$$

so we conclude that the system (22) has an unstable limit cycle for $r_2 = 1$, and a stable limit cycle for $r_1 = \frac{1}{2}$.

Example 2. We consider an example that corresponds to statement (b) of Theorem 1

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - f(x)y^4 - g(x)y^5 - l(x)y^6 - h(x)y^7, \end{cases} \tag{23}$$

where

$$\begin{aligned} f(x) &= \varepsilon(\frac{341}{321888} + 6x) + \varepsilon^2(-1 - 3x), \\ g(x) &= \varepsilon(\frac{429184}{675}x + \frac{182773472}{1395}x^3) + \varepsilon^2(-\frac{1}{4500} + 2x^3), \\ h(x) &= \varepsilon(x^2) + \varepsilon^2(5x - 3x^2), \\ l(x) &= \varepsilon(-\frac{155357728}{3069}x) + \varepsilon^2(-x + \frac{22}{1575}), \end{aligned}$$

and $f(x), g(x), h(x), l(x)$ have degree $n_1 = 1, n_2 = 3, n_3 = 2$ and $n_4 = 1$ respectively. Since we have F_{10} identical to zero, then we must solve the equation $F_{20} = 0$ that is

$$r^{15} - \frac{5269}{3600}r^{13} + \frac{1529}{2880}r^{11} - \frac{341}{4800}r^9 + \frac{11}{2880}r^7 - \frac{1}{14400}r^5 = 0.$$

Which has exactly $\left[\frac{n_2-1}{2}\right] + \left[\frac{n_3}{2}\right] + p + 1 =$ five positive zeros $r_1 = \frac{1}{5}, r_2 = \frac{1}{4}, r_3 = \frac{1}{3}, r_4 = \frac{1}{2}$ and $r_5 = 1$. These last roots satisfy

$$\begin{aligned} \frac{dF_{20}}{dr}|_{r=r_1} &= \frac{252}{6103515625} \neq 0, \quad \frac{dF_{20}}{dr}|_{r=r_2} = -\frac{63}{671088640} \neq 0 \\ \frac{dF_{20}}{dr}|_{r=r_3} &= \frac{28}{23914845} \neq 0, \quad \frac{dF_{20}}{dr}|_{r=r_4} = -\frac{21}{163840} \neq 0 \\ \frac{dF_{20}}{dr}|_{r=r_5} &= \frac{6}{5} \neq 0. \end{aligned}$$

Therefore we conclude that system (23) has three unstable limit cycles for $r_1 = \frac{1}{5}$, $r_3 = \frac{1}{3}$ and $r_5 = 1$. In addition to two others stable limit cycles for $r_2 = \frac{1}{4}$ and $r_4 = \frac{1}{2}$.

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Appendix

We recall some formulas used during this article.

$$
\begin{aligned}
\int_0^{2\pi} A_{i,j}\sin(2l+1)\theta d\theta &\neq 0,\ if\ \ i\ even\ and\ j\ odd,\\
\int_0^{2\pi} A_{i,j}\sin(2l+1)\theta d\theta &= \begin{cases} 0\ if\ \ i\ odd\ or\ j\ even,\\ \pi C_{i,p,l}\ ,\ i\ =2k\ ,\ j\ =2p+1\ and\ l\geq 0,\\ \pi\tilde{C}_{i,p,l}\ ,\ i\ =2k\ ,\ j\ =2p+3\ and\ l\geq 0,\end{cases}\\
\int_0^{2\pi} A_{i,j}\sin(2l)\theta d\theta &\neq 0,\ if\ \ i\ odd\ and\ j\ odd,\\
\int_0^{2\pi} A_{i,j}\sin(2l)\theta d\theta &= \begin{cases} 0\ if\ \ i\ even\ or\ j\ even,\\ \pi D_{i,p,l}\ ,\ i\ =2k+1\ ,\ j\ =2p+1\ and\ l\geq 0,\\ \pi\tilde{D}_{i,p,l}\ ,\ i\ =2k+1\ ,\ j\ =2p+3\ and\ l\geq 0,\end{cases}\\
\int_0^{2\pi} A_{i,j}\cos(2l-1)\theta d\theta &\neq 0,\ if\ \ i\ odd\ and\ j\ even,\\
\int_0^{2\pi} A_{i,j}\cos(2l-1)\theta d\theta &= \begin{cases} 0\ if\ \ i\ even\ or\ j\ odd,\\ \pi E_{i,p,l}\ ,\ i\ =2k+1\ ,\ j\ =2p+2\ and\ l\geq 0,\\ \pi\tilde{E}_{i,p,l}\ ,\ i\ =2k+1\ ,\ j\ =2p+4\ and\ l\geq 0,\end{cases}\\
\int_0^{2\pi} A_{i,j}\cos(2l)\theta d\theta &\neq 0,\ if\ \ i\ even\ and\ j\ even,\\
\int_0^{2\pi} A_{i,j}\cos(2l)\theta d\theta &= \begin{cases} 0\ if\ \ i\ odd\ or\ j\ odd,\\ \pi F_{i,p,l}\ ,\ i\ =2k\ ,\ j\ =2p+2\ and\ l\geq 0,\\ \pi\tilde{F}_{i,p,l}\ ,\ i\ =2k\ ,\ j\ =2p+4\ and\ l\geq 0,\end{cases}
\end{aligned}
$$

where $C_{i,p,l}, \tilde{C}_{i,p,l}, D_{i,p,l}, \tilde{D}_{i,p,l}, E_{i,p,l}, \tilde{E}_{i,p,l}, F_{i,p,l}$ and $\tilde{F}_{i,p,l}$ are non-zero constants.