

Article

# Explicit formula for the $n$ -th derivative of a quotient

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**Abstract:** Leibniz's rule for the  $n$ -th derivative of a product is a very well-known and extremely useful formula. This article introduces an analogous explicit formula for the  $n$ -th derivative of a quotient of two functions. Later, we use this formula to derive new partition identities and to develop expressions for some particular  $n$ -th derivatives.

**Keywords:**  $n$ -th derivative of a quotient; Generalized quotient rule; Partitions.

**MSC:** 26A24; 28A15; 11P99.

## 1. Introduction

If we chose two functions  $u$  and  $v$ , and went around asking mathematicians to compute the  $n$ -th derivative of their product, the first idea that would come to their mind is to use Leibniz's formula. However, what if we asked them to compute the  $n$ -th derivative of the quotient instead? What formula would come to their mind? For many, the answer is none. Therefore, a large portion of the mathematical community needs to be made aware of the existence of such a formula.

This is because, while Leibniz's formula is a subject studied in practically all calculus courses, the topic is rarely discussed when it comes to talking about an analogous formula for the quotient of two functions. Although many wonders if such a formula exists, little work has been done on the subject. In 1967 [1], the first step was taken as a more straightforward question was answered; that is, a recursive formula for the  $n$ -th derivative of  $1/f(x)$  was presented. Later, in 1980, F. Gerrish [2] noticed an interesting pattern linking the  $n$ -th derivative of a quotient to a notable determinant. In 2008, this special connection was used to establish a recursive formula for such a derivative. So if such a formula exists, why do most of us not know about it? There are two primary reasons: The first is that the existing formulas could be more practical as they are recursive rather than explicit. The second is that such a formula was thought to be useless.

F. Gerrish [2] even went as far as calling it the "Useless Formula". However, since then, this formula has found various applications and has been used to deal with a multitude of topics [3–9]. Hence, in this article, we propose revisiting the subject and developing an explicit formula for the  $n$ -th derivative of a quotient analogous to the generalized product rule. We hope this formula will become a standard like Leibniz's rule.

More precisely, it will be analogous to the generalized product rule for the product of several functions. Note that we mean by analogous that the formula will be explicit and have the same form (that is, it will be in terms of a sum over partitions). Let us begin by noting that, in the same way, Leibniz's formula is often referred to as the product rule; in this article, for simplicity, we will refer to the formula for the  $n$ -th derivative of a quotient as the quotient rule.

We will begin by deriving a new formula for the  $n$ -th derivative of  $1/f(x)$  (§2). Although such a formula already exists, the formula presented in [1] is rather complicated. Therefore, we propose a simpler formula involving partitions. The formula we will present also has the advantage of being explicit rather than recursive. We will refer to this particular case as the common rule.

Similarly, although a recursive formula already exists for the quotient rule, no explicit formula exists. Therefore, in §3, by combining the reciprocal formula with Leibniz's formula, we develop an explicit formula for the  $n$ -th derivative of the quotient of two functions. Finally, in §4, we apply the reciprocal and quotient rules developed to derive new partition identities and expressions for some special  $n$ -th derivatives.

## 2. n-th derivative of $1/v(x)$ (Reciprocal rule)

We begin by introducing the concept of partitions as partitions are an essential part of the quotient rule we will develop. As defined by the author in [10,11], a partition can be defined as follows:

**Definition 1.** A partition of a non-negative integer  $m$  is a set of positive integers whose sum equals  $m$ . We can represent a partition of  $m$  as an ordered set  $(y_{k,1}, \dots, y_{k,m})$  that verifies

$$y_{k,1} + 2y_{k,2} + \dots + my_{k,m} = \sum_{i=1}^m i y_{k,i} = m. \tag{1}$$

The coefficient  $y_{k,i}$  is the multiplicity of the integer  $i$  in the  $k$ -th partition of  $m$ . Note that  $0 \leq y_{k,i} \leq m$  while  $1 \leq i \leq m$ . Also note that the number of partitions of an integer  $m$  is given by the partition function denoted  $p(m)$  and hence,  $1 \leq k \leq p(m)$ . In the remainder of this text, the subscript  $k$  will be added to indicate that a given parameter is associated with a given partition. Similarly, for simplicity, we will omit the bounds of  $i$  and write  $\sum i y_{k,i} = m$  and  $\sum y_{k,i}$ . Furthermore, we define the following partition notation:

$$\pi_k = \sum i y_{k,i}, \tag{2}$$

$$r_k = \sum y_{k,i}. \tag{3}$$

As partitions are not the main focus of this article, we will not go into more details. For readers interested in a more in-depth explanation about partitions, see [12].

Before we begin proving the main results of this section, let us introduce the following notation: In the remainder of this article, the letters  $u$  and  $v$  will be used to indicate a function of  $x$ . In other words,  $u$  represents  $u(x)$  and  $v$  represents  $v(x)$ .

**Definition 2.** Let us define the following shorthand notation:

$$(v)^{(n)} = v^{(n)} = \frac{d^n}{dx^n} (v(x)). \tag{4}$$

In order to prove the reciprocal rule, we need to first prove the following lemma.

**Lemma 1.** We have that

$$\sum_{j=0}^{n-1} \binom{\sum Y_{k,i} - 1}{Y_{k,1}, \dots, Y_{k,n-j} - 1, \dots, Y_{k,n}} = \sum_{j=1}^n \binom{\sum Y_{k,i} - 1}{Y_{k,1}, \dots, Y_{k,j} - 1, \dots, Y_{k,n}} = \binom{\sum Y_{k,i}}{Y_{k,1}, \dots, Y_{k,n}}.$$

**Proof.**

$$\begin{aligned} \sum_{j=0}^{n-1} \binom{\sum Y_{k,i} - 1}{Y_{k,1}, \dots, Y_{k,n-j} - 1, \dots, Y_{k,n}} &= \sum_{j=0}^{n-1} \frac{(\sum Y_{k,i} - 1)!}{Y_{k,1}! \dots Y_{k,n-j}! \dots Y_{k,n}!} (Y_{k,n-j}) \\ &= \frac{(\sum Y_{k,i} - 1)!}{Y_{k,1}! \dots Y_{k,n}!} \sum_{j=1}^n (Y_{k,j}) \\ &= \frac{(\sum Y_{k,i})!}{Y_{k,1}! \dots Y_{k,n}!} = \binom{\sum Y_{k,i}}{Y_{k,1}, \dots, Y_{k,n}}. \end{aligned}$$

□

Using the recursive formula for the quotient rule [13], we derive the reciprocal rule.

**Theorem 1** (Reciprocal rule). Let  $v$  be an  $n$  times differentiable function of  $x$ , for any  $n \in \mathbb{N}$  and at every point where  $v \neq 0$ , we have that

$$\left(\frac{1}{v}\right)^{(n)} = \frac{d^n}{dx^n} \left(\frac{1}{v}\right) = n! \sum_{\sum iy_{k,i}=n} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} \frac{(-1)^{\sum y_{k,i}}}{v^{\sum y_{k,i}+1}} \prod_{i=1}^n \left[\frac{v^{(i)}}{i!}\right]^{y_{k,i}}.$$

**Remark 1.** A very interesting and compact way of rewriting this theorem is as follows:

$$\left(\frac{1}{v}\right)^{(n)} = n! \sum_{\sum iy_{k,i}=n} C_k \prod_{i=1}^n \frac{1}{y_{k,i}!} \left[\frac{v^{(i)}}{i!}\right]^{y_{k,i}}$$

where

$$C_k = \frac{d^{\sum y_{k,i}}}{dv^{\sum y_{k,i}}} \left(\frac{1}{v}\right) = \frac{(\sum y_{k,i})! (-1)^{\sum y_{k,i}}}{v^{\sum y_{k,i}+1}} = \frac{(-1)^{r_k} r_k!}{v^{r_k+1}}. \tag{5}$$

As we can see, the general reciprocal rule using the  $C_k$  notation is simple and easy to memorize.

**Remark 2.** Let us define the notation  $\{a\}_b$  corresponds to writing  $b$  times the value  $a$ . Let  $I_k = (\{1\}_{y_{k,1}}, \dots, \{n\}_{y_{k,n}})$ . Similarly, let  $P_k = (y_{k,1}, \dots, y_{k,n})$ . Other interesting ways of writing the theorem are:

$$\left(\frac{1}{v}\right)^{(n)} = \sum_{\sum iy_{k,i}=n} \binom{n}{I_k} C_k \prod_{i=1}^n \frac{[v^{(i)}]^{y_{k,i}}}{y_{k,i}!} = \frac{1}{v} \sum_{\sum iy_{k,i}=n} \binom{n}{I_k} \binom{\sum y_{k,i}}{P_k} \prod_{i=1}^n \left[-\frac{v^{(i)}}{v}\right]^{y_{k,i}}.$$

**Proof.** 1. Base case: verify true for  $n = 1$ .

$$1! \sum_{\sum iy_{k,i}=1} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} \frac{(-1)^{\sum y_{k,i}}}{v^{\sum y_{k,i}+1}} \prod_{i=1}^n \left[\frac{v^{(i)}}{i!}\right]^{y_{k,i}} = \binom{1}{1} \frac{(-1)^1}{v^2} \left[\frac{v'}{1!}\right]^1 = -\frac{v'}{v^2} = \frac{d}{dx} \left(\frac{1}{v}\right).$$

**Remark 3.** We can also verify true for  $n = 0$ . It is important to note that the partition assumed to correspond to zero is  $(0, 0, \dots)$ . Hence,

$$0! \sum_{\sum iy_{k,i}=0} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} \frac{(-1)^{\sum y_{k,i}}}{v^{\sum y_{k,i}+1}} \prod_{i=1}^n \left[\frac{v^{(i)}}{i!}\right]^{y_{k,i}} = \binom{0}{0, 0, \dots} \frac{(-1)^0}{v^1} (1) = \frac{1}{v} = \left(\frac{1}{v}\right)^{(0)}.$$

2. Induction hypothesis: assume the statement is true until  $(n - 1) \in \mathbb{N}$ .

$$\left(\frac{1}{v}\right)^{(n-1)} = (n - 1)! \sum_{\sum iy_{k,i}=n-1} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n-1}} \frac{(-1)^{\sum y_{k,i}}}{v^{\sum y_{k,i}+1}} \prod_{i=1}^{n-1} \left[\frac{v^{(i)}}{i!}\right]^{y_{k,i}}.$$

3. Induction step: we will show that this statement is true for  $n$ .

We have to show the following statement to be true:

$$\left(\frac{1}{v}\right)^{(n)} = n! \sum_{\sum iy_{k,i}=n} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} \frac{(-1)^{\sum y_{k,i}}}{v^{\sum y_{k,i}+1}} \prod_{i=1}^n \left[\frac{v^{(i)}}{i!}\right]^{y_{k,i}}.$$

Using the recursive formula developed in [13] with  $u = 1$ , we have

$$\left(\frac{1}{v}\right)^{(n)} = \frac{(-1)n!}{v} \sum_{j=1}^n \frac{v^{(n+1-j)}}{(n+1-j)!} \frac{\left(\frac{1}{v}\right)^{(j-1)}}{(j-1)!} = \frac{(-1)n!}{v} \sum_{j=0}^{n-1} \frac{v^{(n-j)}}{(n-j)!} \frac{\left(\frac{1}{v}\right)^{(j)}}{j!}.$$

Applying the induction hypothesis, we get

$$\begin{aligned} \left(\frac{1}{v}\right)^{(n)} &= \frac{(-1)n!}{v} \sum_{j=0}^{n-1} \frac{v^{(n-j)}}{(n-j)!} \sum_{\sum iy_{k,i}=j} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,j}} \frac{(-1)^{\sum y_{k,i}}}{v^{\sum y_{k,i}+1}} \prod_{i=1}^j \left[\frac{v^{(i)}}{i!}\right]^{y_{k,i}} \\ &= n! \sum_{j=0}^{n-1} \frac{v^{(n-j)}}{(n-j)!} \sum_{\sum iy_{k,i}=j} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,j}} \frac{(-1)^{\sum y_{k,i}+1}}{v^{\sum y_{k,i}+2}} \prod_{i=1}^j \left[\frac{v^{(i)}}{i!}\right]^{y_{k,i}}. \end{aligned}$$

Let us define an extension  $(y_{k,1}, \dots, y_{k,n})$  of  $(y_{k,1}, \dots, y_{k,j})$  where  $y_{k,j+1} = \dots = y_{k,n} = 0$ . Hence, we can write that

$$\begin{aligned} \left(\frac{1}{v}\right)^{(n)} &= n! \sum_{j=0}^{n-1} \frac{v^{(n-j)}}{(n-j)!} \sum_{\sum iy_{k,i}=j} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} \frac{(-1)^{\sum y_{k,i}+1}}{v^{\sum y_{k,i}+2}} \prod_{i=1}^n \left[\frac{v^{(i)}}{i!}\right]^{y_{k,i}} \\ &= n! \sum_{j=0}^{n-1} \sum_{\sum iy_{k,i}+(n-j)\cdot 1=n} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} \frac{(-1)^{\sum y_{k,i}+1}}{v^{\sum y_{k,i}+2}} \left[\frac{v^{(n-j)}}{(n-j)!}\right] \prod_{i=1}^n \left[\frac{v^{(i)}}{i!}\right]^{y_{k,i}}. \end{aligned}$$

Notice that  $1 \leq n - j \leq n$  as  $0 \leq j \leq n - 1$ . Now, for all  $(n - j) \in [1, n]$ , let us associate with each partition  $(y_{k,1}, \dots, y_{k,n})$ , the partition  $(Y_{k,1}, \dots, Y_{k,n})$  such that

$$\begin{cases} Y_{k,i} = y_{k,i} + 1, & \text{for } i = n - j, \\ Y_{k,i} = y_{k,i}, & \text{otherwise.} \end{cases}$$

Notice that  $\sum Y_{k,i} = \sum y_{k,i} + 1$  and that  $\sum iY_{k,i} = n$ . Hence, we can write

$$\begin{aligned} \left(\frac{1}{v}\right)^{(n)} &= n! \sum_{j=0}^{n-1} \sum_{\sum iY_{k,i}=n} \binom{\sum Y_{k,i} - 1}{Y_{k,1}, \dots, Y_{k,n-j} - 1, \dots, Y_{k,n}} \frac{(-1)^{\sum Y_{k,i}}}{v^{\sum Y_{k,i}+1}} \prod_{i=1}^n \left[\frac{v^{(i)}}{i!}\right]^{Y_{k,i}} \\ &= n! \sum_{\sum iY_{k,i}=n} \frac{(-1)^{\sum Y_{k,i}}}{v^{\sum Y_{k,i}+1}} \left(\prod_{i=1}^n \left[\frac{v^{(i)}}{i!}\right]^{Y_{k,i}}\right) \sum_{j=0}^{n-1} \binom{\sum Y_{k,i} - 1}{Y_{k,1}, \dots, Y_{k,n-j} - 1, \dots, Y_{k,n}}. \end{aligned}$$

Applying Lemma 1 to the inner sum, we obtain

$$\left(\frac{1}{v}\right)^{(n)} = n! \sum_{\sum iY_{k,i}=n} \binom{\sum Y_{k,i}}{Y_{k,1}, \dots, Y_{k,n}} \frac{(-1)^{\sum Y_{k,i}}}{v^{\sum Y_{k,i}+1}} \prod_{i=1}^n \left[\frac{v^{(i)}}{i!}\right]^{Y_{k,i}}.$$

This concludes our proof by induction.  $\square$

**Remark 4.** As we can see, the reciprocal rule derived (Theorem 1) is very similar to the product rule for the product of several functions:

$$(u_1 \cdots u_m)^{(n)} = \sum_{\ell_1 + \dots + \ell_m = n} \binom{n}{\ell_1, \dots, \ell_m} \prod_{i=1}^m u_i^{(\ell_i)} \tag{6}$$

There exists other alternatives for proving Theorem 1. In what follows, we will present a few propositions that will be useful for doing so.

First, let us prove the following useful proposition for the derivative of a product.

**Proposition 1.** Let  $u_1, \dots, u_n$  be differentiable functions of  $x$ , we have that

$$\frac{d}{dx} \left(\prod_{i=1}^n u_i\right) = \left(\prod_{i=1}^n u_i\right) \sum_{j=1}^n \frac{u_j'}{u_j}.$$

**Proof.** Let  $f(x) = u_1 \cdots u_n$ . Taking the logarithm of both sides, we get

$$\ln f(x) = \ln \left( \prod_{i=1}^n u_i \right) = \sum_{i=1}^n \ln u_i.$$

Differentiating both sides, we get

$$-\frac{f'(x)}{f(x)} = -\sum_{i=1}^n \frac{u_i'}{u_i}.$$

Canceling the minus sign, we obtain the desired formula.  $\square$

Now we prove the following partition identity involving a special sum of multinomial coefficients. This expression is equivalent to Lemma 1 that we used to prove Theorem 1.

**Proposition 2.** We have that

$$\sum_{\substack{\sum \varphi_i = \sum Y_i - 1 \\ \varphi_i \leq Y_i}} \binom{\sum Y_i - 1}{\varphi_1, \dots, \varphi_n} = \binom{\sum Y_i}{Y_1, \dots, Y_n}.$$

**Proof.** Let  $Z_i = Y_i - \varphi_i$  for  $1 \leq i \leq n$ .

$$\begin{aligned} \sum_{\substack{\sum \varphi_i = \sum Y_i - 1 \\ \varphi_i \leq Y_i}} \binom{\sum Y_i - 1}{\varphi_1, \dots, \varphi_n} &= \frac{(\sum Y_i - 1)!}{Y_1! \cdots Y_n!} \sum_{\substack{\sum \varphi_i = \sum Y_i - 1 \\ \varphi_i \leq Y_i}} \frac{Y_1! \cdots Y_n!}{\varphi_1! \cdots \varphi_n!} \\ &= \frac{(\sum Y_i - 1)!}{Y_1! \cdots Y_n!} \sum_{\substack{\sum Z_i = 1 \\ Z_i \geq 0}} \frac{Y_1! \cdots Y_n!}{(Y_1 - Z_1)! \cdots (Y_n - Z_n)!}. \end{aligned}$$

Knowing that the  $Z_i$ 's are non-negative integers, the only way for their sum to be equal to 1 is if one of them is equal to 1 and the others are equal to 0. Hence, we have that

$$\sum_{\substack{\sum Z_i = 1 \\ Z_i \geq 0}} \frac{Y_1! \cdots Y_n!}{(Y_1 - Z_1)! \cdots (Y_n - Z_n)!} = \sum_{i=1}^n \frac{Y_i!}{(Y_i - 1)!} = \sum_{i=1}^n Y_i.$$

Substituting back, we obtain the proposition.  $\square$

### 3. n-th derivative of $u(x)/v(x)$ (Quotient rule)

In this section, we combine Theorem 1 with Leibniz's rule to obtain the general quotient rule.

**Theorem 2** (Quotient rule). Let  $u$  and  $v$  be  $n$  times differentiable functions of  $x$ , for any  $n \in \mathbb{N}$  and at every point where  $v \neq 0$ , we have that

$$\begin{aligned} \left(\frac{u}{v}\right)^{(n)} &= \frac{d^n}{dx^n} \left(\frac{u}{v}\right) = n! \sum_{\ell=0}^n \frac{u^{(n-\ell)}}{(n-\ell)!} \sum_{\sum y_{k,i} = \ell} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,\ell}} \frac{(-1)^{\sum y_{k,i}}}{v^{\sum y_{k,i}+1}} \prod_{i=1}^{\ell} \left[\frac{v^{(i)}}{i!}\right]^{y_{k,i}} \\ &= n! \sum_{\pi_k=0}^n \frac{u^{(n-\pi_k)}}{(n-\pi_k)!} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,\pi_k}} \frac{(-1)^{\sum y_{k,i}}}{v^{\sum y_{k,i}+1}} \prod_{i=1}^{\pi_k} \left[\frac{v^{(i)}}{i!}\right]^{y_{k,i}}. \end{aligned}$$

**Proof.** Applying Leibniz's rule to Theorem 1, we obtain this theorem.  $\square$

**Remark 5.** A much more compact way of expressing this theorem is as follows:

$$\left(\frac{u}{v}\right)^{(n)} = n! \sum_{\ell=0}^n \sum_{\sum y_{k,i} = n-\ell} C_k \left[\frac{u^{(\ell)}}{\ell!}\right] \prod_{i=1}^{n-\ell} \frac{1}{y_{k,i}!} \left[\frac{v^{(i)}}{i!}\right]^{y_{k,i}} = n! \sum_{\pi_k=0}^n C_k \frac{u^{(n-\pi_k)}}{(n-\pi_k)!} \prod_{i=1}^{\pi_k} \frac{1}{y_{k,i}!} \left[\frac{v^{(i)}}{i!}\right]^{y_{k,i}}$$

where  $C_k$  is as defined in Eq. (5) and  $\pi_k$  is as defined in Eq. (2).

Using the  $C_k$  and  $\pi_k$  notation, we obtain a simple and easy to memorize expression for the general quotient rule that could potentially be taught to university students at the same time as the general product rule.

**Remark 6.** Let  $I_k = (\{1\}_{y_{k,1}}, \dots, \{\pi_k\}_{y_{k,\pi_k}})$  and  $P_k = (y_{k,1}, \dots, y_{k,\pi_k})$ . We can write the following interesting but not very practical expressions:

$$\begin{aligned} \left(\frac{u}{v}\right)^{(n)} &= \sum_{\pi_k=0}^n C_k \binom{n}{I_k, n - \pi_k} u^{(n-\pi_k)} \prod_{i=1}^{\pi_k} \frac{[v^{(i)}]^{y_{k,i}}}{y_{k,i}!} \\ &= \sum_{\pi_k=0}^n \binom{n}{I_k, n - \pi_k} \binom{\sum y_{k,i}}{P_k} \frac{u^{(n-\pi_k)}}{v} \prod_{i=1}^{\pi_k} \left[-\frac{v^{(i)}}{v}\right]^{y_{k,i}}. \end{aligned}$$

### 4. Applications

Let us first define some notation to simplify the expressions we will derive. For a given partition  $(y_{k,1}, \dots, y_{k,n})$  of  $n$ , we define the following notation:

$$c_k = \prod_{i=1}^n \frac{1}{i^{y_{k,i}} y_{k,i}!}, \quad \bar{c}_k = \prod_{i=1}^n \frac{(-1)^{y_{k,i}}}{i^{y_{k,i}} y_{k,i}!}. \tag{7}$$

$$p_k = \prod_{i=1}^n \frac{1}{i!^{y_{k,i}} y_{k,i}!}, \quad \bar{p}_k = \prod_{i=1}^n \frac{(-1)^{y_{k,i}}}{i!^{y_{k,i}} y_{k,i}!}. \tag{8}$$

$$q_k = \prod_{i=1}^n \frac{1}{i!^{y_{k,i}}}, \quad \bar{q}_k = \prod_{i=1}^n \frac{(-1)^{y_{k,i}}}{i!^{y_{k,i}}}. \tag{9}$$

#### 4.1. Partition identities

In this section, we will show how the quotient rule developed can be used to derive partition identities. In particular, we will derive a few special partition identities.

**Proposition 3.** For any  $n \in \mathbb{N}$ , we have that

$$\sum_{\sum y_{k,i}=n} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} (-1)^{\sum y_{k,i}} \prod_{i=1}^n \frac{1}{i!^{y_{k,i}}} = \frac{(-1)^n}{n!}.$$

Using the notation, this proposition can be expressed as

$$\sum_{\sum y_{k,i}=n} \binom{r_k}{y_{k,1}, \dots, y_{k,n}} (-1)^{r_k} q_k = \sum_{\sum y_{k,i}=n} \binom{r_k}{y_{k,1}, \dots, y_{k,n}} \bar{q}_k = \frac{(-1)^n}{n!}.$$

**Remark 7.** We can also rewrite it as follows:

$$\sum_{\sum y_{k,i}=n} r_k! (-1)^{r_k} \prod_{i=1}^n \frac{1}{i!^{y_{k,i}} y_{k,i}!} = \frac{(-1)^n}{n!}.$$

Using the notation, this proposition can be expressed as

$$\sum_{\sum y_{k,i}=n} r_k! (-1)^{r_k} p_k = \sum_{\sum y_{k,i}=n} r_k! \bar{p}_k = \frac{(-1)^n}{n!}.$$

**Proof.** From Theorem 1 with  $v(x) = e^x$  and knowing that  $v^{(i)}(x) = e^x$  for all  $i$ , we get

$$\begin{aligned} \frac{d^n}{dx^n} \left( \frac{1}{e^x} \right) &= n! \sum_{\sum y_{k,i}=n} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} \frac{(-1)^{\sum y_{k,i}}}{(e^x)^{\sum y_{k,i}+1}} \prod_{i=1}^n \left[ \frac{e^x}{i!} \right]^{y_{k,i}} \\ &= n! e^{-x} \sum_{\sum y_{k,i}=n} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} (-1)^{\sum y_{k,i}} \prod_{i=1}^n \frac{1}{i!^{y_{k,i}}}. \end{aligned}$$

Noticing that

$$\frac{d^n}{dx^n} \left( \frac{1}{e^x} \right) = \frac{d^n}{dx^n} (e^{-x}) = (-1)^n e^{-x},$$

we obtain the proposition.  $\square$

**Proposition 4.** For any  $n \in \mathbb{N}$  and any  $m \in \mathbb{N}^*$ , we have that

$$\sum_{\sum y_{k,i}=n} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} (-1)^{\sum y_{k,i}} \prod_{i=1}^n \left[ \binom{m}{i} \right]^{y_{k,i}} = (-1)^n \binom{n+m-1}{m-1}.$$

**Proof.** Let  $v(x) = x^m$ , then  $v^{(i)} = i! \binom{m}{i} x^{m-i}$ . Hence, from Theorem 1, we have

$$\begin{aligned} \frac{d^n}{dx^n} \left( \frac{1}{x^m} \right) &= n! \sum_{\sum y_{k,i}=n} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} \frac{(-1)^{\sum y_{k,i}}}{x^{m \sum y_{k,i}+m}} \prod_{i=1}^n \left[ x^{m-i} \binom{m}{i} \right]^{y_{k,i}} \\ &= n! \sum_{\sum y_{k,i}=n} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} \frac{(-1)^{\sum y_{k,i}}}{x^{m \sum y_{k,i}+m}} (x^{m \sum y_{k,i}-n}) \prod_{i=1}^n \left[ \binom{m}{i} \right]^{y_{k,i}} \\ &= n! x^{-m-n} \sum_{\sum y_{k,i}=n} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} (-1)^{\sum y_{k,i}} \prod_{i=1}^n \left[ \binom{m}{i} \right]^{y_{k,i}}. \end{aligned}$$

Noticing that

$$\frac{d^n}{dx^n} \left( \frac{1}{x^m} \right) = \frac{d^n}{dx^n} (x^{-m}) = (-1)^n n! \binom{n+m-1}{m-1} x^{-m-n},$$

we obtain the proposition.  $\square$

**Corollary 1.** Setting  $m = n$ , we get

$$\sum_{\sum y_{k,i}=n} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} (-1)^{\sum y_{k,i}} \prod_{i=1}^n \left[ \binom{n}{i} \right]^{y_{k,i}} = (-1)^n \binom{2n-1}{n-1}.$$

**Proposition 5.** For any  $n \in \mathbb{N}$  and any  $m \in \mathbb{N}^*$ , we have that

$$\sum_{\sum y_{k,i}=n} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} (-1)^{\sum y_{k,i}} \prod_{i=1}^n \left[ \binom{i+m-1}{m-1} \right]^{y_{k,i}} = (-1)^n \binom{m}{n}.$$

**Proof.** Let  $v(x) = x^{-m}$ , then  $v^{(i)} = (-1)^i i! \binom{i+m-1}{m-1} x^{-(m+i)}$ . Hence, from Theorem 1, we have

$$\begin{aligned} \frac{d^n}{dx^n} \left( \frac{1}{x^m} \right) &= n! \sum_{\sum y_{k,i}=n} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} \frac{(-1)^{\sum y_{k,i}}}{x^{-m \sum y_{k,i}-m}} \prod_{i=1}^n \left[ (-1)^i \binom{i+m-1}{m-1} x^{-(m+i)} \right]^{y_{k,i}} \\ &= n! (-1)^n \sum_{\sum y_{k,i}=n} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} \frac{(-1)^{\sum y_{k,i}} (x^{-m \sum y_{k,i}-n})}{x^{-m \sum y_{k,i}-m}} \prod_{i=1}^n \left[ \binom{i+m-1}{m-1} \right]^{y_{k,i}} \\ &= (-1)^n n! x^{m-n} \sum_{\sum y_{k,i}=n} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} (-1)^{\sum y_{k,i}} \prod_{i=1}^n \left[ \binom{i+m-1}{m-1} \right]^{y_{k,i}}. \end{aligned}$$

Noticing that

$$\frac{d^n}{dx^n} \left( \frac{1}{x^m} \right) = \frac{d^n}{dx^n} (x^{-m}) = n! \binom{m}{n} x^{m-n},$$

we obtain the proposition.  $\square$

An extremely interesting result that can be derived from Proposition 5 is that for the alternating sum over partitions of multinomial coefficients.

**Corollary 2.** *Setting  $m = 1$ , we get*

$$\sum_{\sum y_{k,i}=n} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} (-1)^{\sum y_{k,i}} = (-1)^n \binom{1}{n} = \begin{cases} (-1)^n, & n = 0, 1, \\ 0, & n \geq 2. \end{cases}$$

### 4.2. Special n-th derivatives

Because of the absence of a general quotient rule, there were many  $n$ -th derivatives for which we could not obtain an explicit expression. In this section, we will use the quotient rule derived to develop an expression for some of these derivatives.

The first special  $n$ -th derivative is that of  $\log_x a$  as well as that of the reciprocal of  $\ln x$ . In 2014, Feng Qi [14] introduced the following expression for the reciprocal of  $\ln x$ :

$$\left( \frac{1}{\ln x} \right)^{(n)} = \frac{(-1)^n}{x^n} \sum_{i=2}^{n+1} \frac{a_{n,i}}{(\ln x)^i}, \tag{10}$$

where

$$a_{n,2} = (n - 1)! \tag{11}$$

and, for  $n + 1 \geq i \geq 3$ ,

$$a_{n,i} = (i - 1)!(n - 1)! \sum_{\ell_1=1}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=1}^{\ell_1-1} \frac{1}{\ell_2} \dots \sum_{\ell_{i-3}=1}^{\ell_{i-4}-1} \frac{1}{\ell_{i-3}} \sum_{\ell_{i-2}=1}^{\ell_{i-3}-1} \frac{1}{\ell_{i-2}}. \tag{12}$$

The expression seems simple, however, the  $a_{n,i}$  terms correspond to a kind of multiple harmonic sum. Such sums are very tedious to compute, thus, making Eq. (10) a bit tedious to use. In what follows, using the general reciprocal rule, we will derive a simpler expression.

**Proposition 6.** *For any  $a \in \mathbb{N}^*$ , the  $n$ -th derivative of  $\log_x a$  is given by*

$$(\log_x a)^{(n)} = \left( \frac{\ln a}{\ln x} \right)^{(n)} = (\log_x a) \frac{(-1)^n n!}{x^n} \sum_{\sum y_{k,i}=n} \frac{(\sum y_{k,i})!}{(\ln x)^{\sum y_{k,i}}} \prod_{i=1}^n \frac{1}{i^{y_{k,i}} y_{k,i}!}.$$

Using the notation, we can rewrite it as follows:

$$(\log_x a)^{(n)} = \left( \frac{\ln a}{\ln x} \right)^{(n)} = (\log_x a) \frac{(-1)^n n!}{x^n} \sum_{\sum y_{k,i}=n} c_k \frac{r_k!}{(\ln x)^{r_k}}.$$

**Proof.** From Theorem 1 with  $v = \ln x$ , we have

$$(\log_x a)^{(n)} = \left( \frac{\ln a}{\ln x} \right)^{(n)} = n! \frac{\ln a}{\ln x} \sum_{\sum y_{k,i}=n} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} \frac{(-1)^{\sum y_{k,i}}}{(\ln x)^{\sum y_{k,i}}} \prod_{i=1}^n \left[ \frac{(\ln x)^{(i)}}{i!} \right]^{y_{k,i}}.$$

Knowing that, for  $i \geq 1$ ,

$$(\ln x)^{(i)} = \frac{(-1)^{i-1} (i - 1)!}{x^i},$$



hence, by substituting back and simplifying, we get

$$\begin{aligned} (\log_x a)^{(n)} &= n!(\log_x a) \sum_{\sum iy_{k,i}=n} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} \frac{(-1)^{\sum y_{k,i}}}{(\ln x)^{\sum y_{k,i}}} \prod_{i=1}^n \left[ \frac{(-1)^{i-1}}{ix^i} \right]^{y_{k,i}} \\ &= \frac{(-1)^n n!}{x^n} (\log_x a) \sum_{\sum iy_{k,i}=n} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} \frac{1}{(\ln x)^{\sum y_{k,i}}} \prod_{i=1}^n \left[ \frac{1}{i} \right]^{y_{k,i}}. \end{aligned}$$

Replacing the multinomial coefficient by its factorial definition, we obtain this proposition.  $\square$

Another special  $n$ -th derivative is that of  $\ln v(x)$ .

**Proposition 7.** *The  $n$ -th derivative of  $\ln v(x)$  is given by*

$$\begin{aligned} (\ln v)^{(n)} &= n! \sum_{\sum iy_{k,i}=n} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n}} \frac{(-1)^{\sum y_{k,i}-1}}{(\sum y_{k,i})! v^{\sum y_{k,i}}} \prod_{i=1}^n \left[ \frac{v^{(i)}}{i!} \right]^{y_{k,i}} \\ &= n! \sum_{\sum iy_{k,i}=n} \frac{(\sum y_{k,i} - 1)! (-1)^{\sum y_{k,i}-1}}{v^{\sum y_{k,i}}} \prod_{i=1}^n \frac{1}{y_{k,i}!} \left[ \frac{v^{(i)}}{i!} \right]^{y_{k,i}}. \end{aligned}$$

**Proof.** From Theorem 2, we have

$$\begin{aligned} (\ln v)^{(n)} &= \left( \frac{v'}{v} \right)^{(n-1)} \\ &= (n-1)! \sum_{\ell=0}^{n-1} \frac{(v')^{(\ell)}}{\ell!} \sum_{\sum iy_{k,i}=n-\ell-1} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n-\ell-1}} \frac{(-1)^{\sum y_{k,i}}}{v^{\sum y_{k,i}+1}} \prod_{i=1}^{n-\ell-1} \left[ \frac{v^{(i)}}{i!} \right]^{y_{k,i}} \\ &= (n-1)! \sum_{\ell=0}^{n-1} \frac{v^{(\ell+1)}(\ell+1)}{(\ell+1)!} \sum_{\sum iy_{k,i}=n-\ell-1} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n-\ell-1}} \frac{(-1)^{\sum y_{k,i}}}{v^{\sum y_{k,i}+1}} \prod_{i=1}^{n-\ell-1} \left[ \frac{v^{(i)}}{i!} \right]^{y_{k,i}} \\ &= (n-1)! \sum_{\ell=1}^n \frac{v^{(\ell)}}{\ell!} \ell \sum_{\sum iy_{k,i}=n-\ell} \binom{\sum y_{k,i}}{y_{k,1}, \dots, y_{k,n-\ell}} \frac{(-1)^{\sum y_{k,i}}}{v^{\sum y_{k,i}+1}} \prod_{i=1}^{n-\ell} \left[ \frac{v^{(i)}}{i!} \right]^{y_{k,i}}. \end{aligned}$$

Similar to what was done in the proof of Theorem 1, we defined an extension  $(y_{k,1}, \dots, y_{k,n})$  of each partition  $(y_{k,1}, \dots, y_{k,n-\ell})$  such that  $y_{k,n-\ell+1} = \dots = y_{k,n} = 0$ . Now, for every  $\ell \in [1, n]$ , let us associate with each partition  $(y_{k,1}, \dots, y_{k,n})$ , the partition  $(Y_{k,1}, \dots, Y_{k,n})$  such that

$$\begin{cases} Y_{k,i} = y_{k,i} + 1, & \text{for } i = \ell, \\ Y_{k,i} = y_{k,i}, & \text{otherwise.} \end{cases}$$

Notice that  $\sum Y_{k,i} = \sum y_{k,i} + 1$  and that  $\sum iY_{k,i} = n$ . Hence, we can write

$$\begin{aligned} (\ln v)^{(n)} &= (n-1)! \sum_{\ell=1}^n \ell \sum_{\sum iY_{k,i}=n} \binom{\sum Y_{k,i} - 1}{Y_{k,1}, \dots, Y_{k,\ell} - 1, \dots, Y_{k,n}} \frac{(-1)^{\sum Y_{k,i}-1}}{v^{\sum Y_{k,i}}} \prod_{i=1}^n \left[ \frac{v^{(i)}}{i!} \right]^{Y_{k,i}} \\ &= (n-1)! \sum_{\sum iY_{k,i}=n} \frac{(-1)^{\sum Y_{k,i}-1}}{v^{\sum Y_{k,i}}} \prod_{i=1}^n \left[ \frac{v^{(i)}}{i!} \right]^{Y_{k,i}} \sum_{\ell=1}^n \ell \binom{\sum Y_{k,i} - 1}{Y_{k,1}, \dots, Y_{k,\ell} - 1, \dots, Y_{k,n}} \\ &= (n-1)! \sum_{\sum iY_{k,i}=n} \binom{\sum Y_{k,i}}{Y_{k,1}, \dots, Y_{k,n}} \frac{(-1)^{\sum Y_{k,i}-1}}{(\sum Y_{k,i})! v^{\sum Y_{k,i}}} \prod_{i=1}^n \left[ \frac{v^{(i)}}{i!} \right]^{Y_{k,i}} \sum_{\ell=1}^n \ell Y_{k,\ell} \\ &= n! \sum_{\sum iY_{k,i}=n} \binom{\sum Y_{k,i}}{Y_{k,1}, \dots, Y_{k,n}} \frac{(-1)^{\sum Y_{k,i}-1}}{(\sum Y_{k,i})! v^{\sum Y_{k,i}}} \prod_{i=1}^n \left[ \frac{v^{(i)}}{i!} \right]^{Y_{k,i}}. \end{aligned}$$

$\square$

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