Open Journal of
Mathematical Analysis

PSR Press

*Article*

# Further on quantum-plank derivatives and integrals in composite forms

**Ghulam Farid[1,*] and Zeeshan Afzal[2]**

[1] Department of Mathematics, COMSATS University of Islamabad, Attock Campus, Attock, Pakistan.
[2] Department of Mathematics and Statistics, The University of Lahore, Lahore, Pakistan.
* Correspondence: ghlmfarid@ciit-attock.edu.pk

Communicated by: Absar Ul Haq
Received: 7 August 2022; Accepted: 26 December 2022; Published: 31 December 2022.

**Abstract:** In quantum-plank calculus, $q$-derivatives and $h$-derivatives are fundamental factors. Recently, a composite form of both derivatives is introduced and called $q-h$-derivative. This paper aims to present a further generalized notion of derivatives will be called $(q,p-h)$-derivatives. This will produce $q$-derivative, $h$-derivative, $q-h$-derivative and $(p,q)$-derivative. Theory based on all aforementioned derivatives can be generalized via this new notion. It is expected, this paper will be useful and beneficial for researchers working in diverse fields of sciences and engineering.

**Keywords:** $q$-derivative; $q$-integral; $h$-derivative; $h$-integral; $q-h$-derivative; $q-h$-integral; inequalities.

**MSC:** 26A51; 26A33; 33E12.

---

## 1. Introduction

The $h$-derivative, $q$-derivative, $q-h$-derivative and $(p,q)$-derivative are given by the following notions and quotients:

- the $h$-derivative of $g$: $D_h g(t) = \frac{d_h g(t)}{d_h t} = \frac{g(t+h)-g(t)}{h}$,
- the $q$-derivative of $g$: $D_q g(t) = \frac{d_q g(t)}{d_q t} = \frac{g(qt)-g(t)}{(q-1)t}$,
- the $q-h$-derivative of $g$: $C_h D_q f(t) = \frac{{}_h d_q f(t)}{{}_h d_q t} = \frac{g(q(t+h))-g(t)}{(q-1)t+qh}$,
- the $(p,q)$-derivative of $g$: $D_q^p g(t) = \frac{d_q^p g(t)}{d_q^p t} = \frac{g(qt)-g(pt)}{(q-p)t}$,

respectively.

The equations; $d_h g(t) = g(t+h) - g(t)$, $d_q g(t) = g(qt) - g(t)$, ${}_h d_q g(t) = g(q(t+h)) - g(t)$ and $d_q^p g(t) = g(qt) - g(pt)$ provide $h$-differential, $q$-differential, $q-h$-differential and $(p,q)$-differential for the function $g$ respectively.

As an example $h$-derivative, $q$-derivative, $q-h$-derivative and $(p,q)$-derivative of $t^n$ can be computed in the forms $\frac{(t+h)^n - t^n}{h} = nt^{n-1} + \frac{n(n-1)}{2} t^{n-2} h + ... + h^{n-1}$, $\frac{q^n-1}{q-1} t^{n-1} = (q^{n-1} + ... + 1)t^{n-1}$, $\frac{q^n (t+h)^n - p^n t^n}{(q-p)t+qh}$ and $\frac{(q^n-p^n)t^n}{(q-p)t} = (q^{n-1} + ... + p^{n-1})t^{n-1}$ respectively. For the sake of simplicity the notations $[n]_q$ and $[n]_{q,p}$ are used instead of $\frac{q^n-1}{q-1}$ and $\frac{q^n-p^n}{q-p}$. Then $D_q t^n = [n]_q t^{n-1}$ and $D_q^p t^n = [n]_{q,p} t^{n-1}$. Since $\lim_{q\to 1} D_q g(t) = \lim_{h\to 0} D_h g(t) = \frac{dg(t)}{dt}$, $h$-derivative, $q$-derivative, $q-h$-derivative and $(q,p)$-derivative are generalized notions of ordinary derivative provided that $g$ is differentiable function, therefore, these notions of derivatives are used to generalize the theory based on ordinary derivatives. Especially, the $q$-derivative leads to the subject of $q$-calculus, for detailed study one can see [8]. In the following we give rules of the $q$-derivative and the $h$-derivative as follows:

The formulae of $q$-derivative of sum and product of two functions $g_1$ and $g_2$ are given by;

$$D_q\{g_1(t) + g_2(t)\} = D_q g_1(t) + D_q g_2(t)\,, \tag{1}$$

and

$$D_q\{g_1(t) g_2(t)\} = g_1(qt) D_q g_2(t) + g_2(t) D_q g_1(t)\,, \tag{2}$$

respectively.

Since $g_1(t)g_2(t) = g_2(t)g_1(t)$, the above formula is equivalent to the following one

$$D_q\{g_1(t)g_2(t)\} = g_1(t)D_q g_2(t) + g_2(qt)D_q g_1(t). \tag{3}$$

Using (2), the $q$-derivative of quotient of two functions $g_1$ and $g_2$ is given by the formula

$$D_q\left(\frac{g_1(t)}{g_2(t)}\right) = \frac{g_2(t)D_q g_1(t) - g_1(t)D_q g_2(t)}{g_2(t)g_2(qt)}. \tag{4}$$

While, by using (3), the $q$-derivative of quotient of two functions $g_1$ and $g_2$ is given by the formula

$$D_q\left(\frac{g_1(t)}{g_2(t)}\right) = \frac{g_2(qt)D_q g_1(t) - g_1(qt)D_q g_2(t)}{g_2(t)g_2(qt)}. \tag{5}$$

The formulae of $h$-derivative of sum and product of two functions $g_1$ and $g_2$ are given by;

$$D_h\{g_1(t) + g_2(t)\} = D_h g_1(t) + D_h g_2(t)\,, \tag{6}$$

and

$$D_h\{g_1(t)g_2(t)\} = g_1(t)D_h g_2(t) + g_2(t+h)D_h g_1(t), \tag{7}$$

respectively.

The $h$-derivative of quotient of two functions $g_1$ and $g_2$ is given by the formula

$$D_h\left(\frac{g_1(t)}{g_2(t)}\right) = \frac{g_2(t)D_h g_1(t) - g_1(t)D_h g_2(t)}{g_2(t)g_2(t+h)}. \tag{8}$$

The above $h$-derivative and $q$-derivative formulas are unified in the following $q-h$-derivative formulas: The $q-h$-derivative is linear, i.e., the following equation holds:

$$C_h D_q(\alpha f(t) + \beta g(t)) = \alpha\, C_h D_q f(t) + \beta\, C_h D_q g(t).$$

The $q-h$-derivative of product of two functions is given by the following equation:

$$C_h D_q(f(t)g(t)) = \frac{{}_h d_q(f(t)g(t))}{{}_h d_q x} = f(q(t+h))C_h D_q g(t) + g(t)C_h D_q f(t). \tag{9}$$

The $q-h$-derivative of quotient of two functions is given by the following equation:

$$C_h D_q\left(\frac{f(t)}{g(t)}\right) = \frac{C_h D_q(f(t))g(q(t+h)) - f(q(t+h))C_h D_q(g(t))}{g(q(t+h))g(t)}. \tag{10}$$

One can note, the $q-h$-derivative formulas generate both $q$-derivative and $h$-derivative formulas.

Next, we give the definitions of $q$-derivative, $(p,q)$-derivative and $q-h$-derivative on a finite interval.

**Definition 1.** [2] Let $0 < q < 1$. For a continuous function $f : I = [a,b] \to \mathbb{R}$ the $q$-derivative on $I$ denoted by ${}_aD_q f$ is defined by

$${}_aD_q f(x) := \frac{f(qx + (1-q)a) - f(x)}{(q-1)(x-a)},\ x \neq a,\ {}_aD_q f(a) = \lim_{x\to a} {}_aD_q f(x). \tag{11}$$

**Definition 2.** [3,4] Let $0 < q < p \leq 1$. For a continuous function $f : I = [a,b] \to \mathbb{R}$ the $(p,q)$-derivative on $I$ denoted by ${}_aD_{p,q} f$ is defined by

$$\begin{aligned}
{}_aD_{p,q} f(x) &:= \frac{f(qx + (1-q)a) - f(px + (1-p)a)}{(q-p)(x-a)},\ x \neq a,\\
{}_aD_{p,q} f(a) &= \lim_{x\to a} {}_aD_{p,q} f(x).
\end{aligned}$$

**Definition 3.** [5] Let $0 < q < 1$, $h \in \mathbb{R}$ and $x \in I$. For a continuous function $f : I \to \mathbb{R}$ the left and right $q-h-$derivatives on $I$ denoted by $C_hD_q^{a^+}f$ and $C_hD_q^{b-}f$ are defined with the following equations respectively;

$$C_hD_q^{a^+}f(x) := \frac{f((1-q)a+q(x+h))-f(x)}{(1-q)(a-x)+qh};\ x \neq \frac{qh+(1-q)a}{1-q} := u, \tag{12}$$

$$C_hD_q^{b-}f(x) := \frac{f((1-q)x+q(b+h))-f(b)}{(1-q)(x-b)+qh};\ x \neq \frac{-qh+(1-q)b}{1-q} := v, \tag{13}$$

provided that $(1-q)a+q(x+h) \in [a,x]$ and $(1-q)x+q(b+h) \in [x,b]$. Also, $C_hD_q^{a^+}f(u) = \lim_{x\to u} C_hD_q^{a^+}f(x)$ and $C_hD_q^{b-}f(v) = \lim_{x\to v} C_hD_q^{b-}f(x)$.

The definitions of $q$-integral, $(p,q)$-integral and $q-h$-integral of function $f$ on interval $[a,b]$ are given as follows:

**Definition 4.** [2] Let $0 < q < 1$ and function $f : I = [a,b] \to \mathbb{R}$. The $q$-definite integral on $I$ is defined by the following formula:

$$\int_a^x f(t)\, {}_ad_qt = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a),\ x \in [a,b]. \tag{14}$$

**Definition 5.** [3] Let $0 < q < p \leq 1$ and function $f : I = [a,b] \to \mathbb{R}$. The $(p,q)$-definite integral on $I$ is defined by the following formula:

$$\int_a^x f(t)\, {}_ad_q^pt = (p-q)(x-a)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}b + \left(1-\frac{q^n}{p^{n+1}}\right)a\right),\ x \in [a,b]. \tag{15}$$

**Definition 6.** [5] Let $0 < q < 1$ and $f : I = [a,b] \to \mathbb{R}$ be a continuous function. Then the left and right $q-h$-integrals on $I$ denoted by $I_{q-h}^{a+}f$ and $I_{q-h}^{b}f$ are defined as follows:

$$I_{q-h}^{a+}f(x) := \int_a^x f(t)\, {}_hd_qt = ((1-q)(x-a)+qh)\sum_{n=0}^{\infty} q^n f(q^n a + (1-q^n)x + nq^n h),\ x > a, \tag{16}$$

$$I_{q-h}^{b-}f(x) := \int_x^b f(t)\, {}_hd_qt = ((1-q)(b-x)+qh)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)b + nq^n h),\ x < b. \tag{17}$$

In (15), if $a = 0$, then the Jackson $q$-definite integral on $[0,x]$ is obtained as follows [8]:

$$\int_0^x f(t)\, {}_0d_qt = \int_0^x f(t)d_qt = (1-q)x\sum_{n=0}^{\infty} q^n f(q^n x),\ x \in [a,b]. \tag{18}$$

The aim in this paper is to define a generalize notion of derivative that includes $q$-derivative (quantum derivative), $h$-derivative (plank derivative), $(p,q)$-derivative and $q-h$-derivative (quantum-plank-derivative). This will be called $(q,p-h)$-derivative. We derive formulas for $q-h$-derivative of sum/difference, product and quotient of two functions. We will give the definition of $(q,p-h)$-integral, moreover the definitions of $(q,p-h)$-derivative as well as $(q,p-h)$-integral are given on a finite interval of the real line.

## 2. $(q,p-h)$-Derivatives

Let we define the $(q,p-h)$-differential of a real valued function $f$ as follows:

$${}_hd_q^pf(x) = f(q(x+h)) - f(px). \tag{19}$$

Then for "$h=0$", "$p=1$", "$h=0,\ p=1$" and "$p=1,\ q\to 1$" in (19), we get $(q,p)$-differential, $(q,p-h)$-differential, $q$-differential and $h$-differential respectively as follows:

$$\begin{aligned}
{}_0d_q^p f(x) &= f(qx)-f(px) = d_q^p f(x),\\
{}_hd_q^1 f(x) &= f(q(x+h))-f(x) = {}_hd_q f(x),\\
{}_0d_q^1 f(x) &= f(qx)-f(x) = d_q f(x)
\end{aligned}$$

and

$$ {}_hd_1^1 f(x) = f(x+h)-f(x) = {}_hd_q f(x).$$

In particular,

$$ {}_hd_q^p(x) = qx+qh-px = (q-p)x+qh. \tag{20}$$

Then for "$h=0$", "$p=1$", "$h=0,\ p=1$" and "$p=1,\ q\to 1$" in (20), we have

$$\begin{cases}
{}_0d_q^p(x) = (q-p)x = d_q^p(x),\\
{}_hd_q^1(x) = (q-1)x+qh = {}_hd_q(x),\\
{}_0d_q^1(x) = (q-1)x = d_q(x),\\
{}_hd_1^1(x) = h = d_h(x),
\end{cases} \tag{21}$$

respectively.

For $S(x)=f(x)+g(x)$ the $(q,p-h)$-differential of $S$ is given by;

$$ {}_hd_q^p(S(x)) = {}_hd_q^p(f(x)+g(x)) = (f+g)(q(x+h))-(f+g)(px) = {}_hd_q^p f(x) + {}_hd_q^p g(x). \tag{22}$$

For $\beta\in\mathbb{R}$, the $(q,p-h)$-differential of $\beta f$ is given by;

$$ {}_hd_q^p(\beta f)(x) = (\beta f)(q(x+h)) - (\beta f)(px) = \beta\ {}_hd_q^p f(x). \tag{23}$$

From (22) and (23), it can be concluded that $(q,p-h)$-differential is linear. For the product function $P$ of $f$ and $g$ i.e. $P(x)=f(x)g(x)$, the $(q,p-h)$-differential is calculated as follows:

$$\begin{aligned}
{}_hd_q^p(P(x)) = {}_hd_q^p((fg)(x)) &= (fg)(q(x+h))-(fg)(px)\\
&= f(q(x+h))g(q(x+h)) + f(q(x+h))g(px)\\
&\quad - f(q(x+h))g(px) - f(px)g(px)\\
&= f(q(x+h))[g(q(x+h))-g(px)]\\
&\quad + g(px)[f(q(x+h))-f(px)].
\end{aligned}$$

Hence we have the following formula for $(q,p-h)$-differential of product of two functions:

$$ {}_hd_q^p(P(x)) = {}_hd_q^p(f(x)g(x)) = f(q(x+h))\,{}_hd_q^p g(x) + g(px)\,{}_hd_q^p f(x). \tag{24}$$

For "$h=0$", "$p=1$", "$h=0,p=1$" and "$p=1,q\to 1$" in (24), we get $(q,p)$-differential, $(q,p-h)$-differential, $q$-differential and $h$-differential of product $P$ of functions $f$ and $g$, respectively as follows:

$$\begin{aligned}
{}_0d_q^p(P(x)) = {}_0d_q^p(f(x)g(x)) &= d_q^p(f(x)g(x))\\
&= f(qx)\,{}_0d_q^p g(x) + g(px)\,{}_0d_q^p f(x)\\
&= f(qx)d_q^p g(x) + g(px)d_q^p f(x)
\end{aligned}$$

$$\begin{aligned}
{}_hd_q^1(P(x)) = {}_hd_q^1(f(x)g(x)) &= {}_hd_q(f(x)g(x))\\
&= f(q(x+h))\,{}_hd_q^1 g(x) + g(x)\,{}_hd_q^1 f(x)\\
&= f(q(x+h))\,{}_hd_q g(x) + g(x)\,{}_hd_q f(x),
\end{aligned}$$

$$
\begin{aligned}
{}_0d_q^1(P(x)) &= {}_0d_q^1(f(x)g(x)) = d_q(f(x)g(x)) \\
&= f(qx)_0d_qg(x) + g(x)\,{}_0d_qf(x) \\
&= f(qx)d_qg(x) + g(x)d_qf(x),
\end{aligned}
$$

$$
\begin{aligned}
{}_hd_1^1(P(x)) &= {}_hd_1^1(f(x)g(x)) = d_h(f(x)g(x)) \\
&= f(x+h)_hd_1^1g(x) + g(x)_hd_1^1f(x) \\
&= f(x+h)d_hg(x) + g(x)d_hf(x),
\end{aligned}
$$

respectively.

Now, we define composite derivative as follows:

**Definition 7.** Let $0 < q < p \leq 1$, $h \in \mathbb{R}$ and $f : I \to \mathbb{R}$ be a continuous function. Then the $(q, p-h)$-derivative of $f$ is defined by

$$
\begin{cases}
C_hD_q^pf(x) = \frac{{}_hD_q^pf(x)}{{}_hd_q^px} = \frac{f(q(x+h))-f(px)}{(q-p)x+qh},\ x \neq \frac{qh}{p-q} := x_\circ \\
C_hd_q^pf(x_\circ) = \lim\limits_{x\to x_\circ} C_hD_q^pf(x).
\end{cases}
\tag{25}
$$

For $h = 0$ and $q \to 1$ in (25), we have

$$
C_0D_q^pf(x) = D_q^pf(x) = \frac{d_qf(x)}{d_qx} = \frac{f(qx)-f(px)}{(q-p)x}\,, \tag{26}
$$

and

$$
C_hD_1f(x) = D_hf(x) = \frac{d_hf(x)}{d_hx} = \frac{f(x+h)-f(x)}{h}, \tag{27}
$$

respectively.

If $f$ is differentiable and $h = 0$, $q \to 1$ in (25), we get the ordinary derivative of $f$.

**Remark 1.** It is notable that if we put $p = 1$, $h = \frac{\omega}{q}$ where $\omega > 0$, the Wolfgang Hahn difference operator given in [6] is obtained.

**Example 1.** The $(q, p-h)$-derivative of $x^n$, $n \in \mathbb{N}$ is calculated as follows:

$$
C_hD_q^p(x^n) = \frac{q^n(x+h)^n - p^nx^n}{(q-p)x+qh} = \frac{(q^n-p^n)x^n}{(q-p)x+qh} + \frac{q^n(nx^{n-1}h+...+h^n)}{(q-p)x+qh}. \tag{28}
$$

For "$p = 1$", "$h = 0$", "$h = 0$, $p = 1$" and "$p = 1$, $q \to 1$" in (28), we get quantum-plank derivative, $(p, q)$-derivative, quantum-derivative and plank-derivative of function $x^n$ respectively as follows:

$$
C_hD_q^1(x^n) = \frac{q^n(x+h)^n - x^n}{(q-1)x+qh} = \frac{(q^n-1)x^n}{(q-1)x+qh} + \frac{q^n(nx^{n-1}h+...+h^n)}{(q-1)x+qh}, \tag{29}
$$

$$
C_0D_q^p(x^n) = \frac{q^nx^n - p^nx^n}{(q-1)x} = \frac{q^n-p^n}{q-p}x^{n-1} = [n]_{p,q}x^{n-1} = D_q^p(x^n), \tag{30}
$$

$$
C_0D_q^1(x^n) = \frac{q^nx^n - x^n}{(q-1)x} = \frac{q^n-1}{q-1}x^{n-1} = [n]x^{n-1} = D_q(x^n), \tag{31}
$$

and

$$
C_hD_1^1(x^n) = \frac{(x+h)^n - x^n}{h} = nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + ..... + h^{n-1}. \tag{32}
$$

In particular, we have $\lim\limits_{h\to 0} C_hD_1^1(x^n) = nx^{n-1}$.

### 2.1. Linearity of $(q, p - h$-derivative

The $(q, p - h)$-derivative is linear, for real valued functions $f, g$ and $\alpha, \beta \in \mathbb{R}$ one can have linearity from the linearity of $(q, p - h)$-differentials as follows:

$$C_h D_q^p(\alpha f(x) + \beta g(x)) = \alpha\, C_h D_q^p f(x) + \beta\, C_h D_q^p g(x).$$

### 2.2. Product formula for $(q, p - h)$-derivatives

By using the $q - h$-differential of product of functions from (24), the product formula is stated as follows:

$$\begin{aligned} C_h D_q^p(f(x)g(x)) &= \frac{{}_h d_q^p(f(x)g(x))}{{}_h d_q^p x} \\ &= \frac{f(q(x+h))_h d_q^p g(x) +_h d_q^p f(x) g(x)}{{}_h d_q^p x} \\ &= f(q(x+h)) C_h D_q^p g(x) + g(x) C_h D_q^p f(x). \end{aligned} \tag{33}$$

It generates both $q$-derivative product formula and $h$-derivative product formula simultaneously as follows:

For $h = 0$ we have $q$-derivative formula for products of functions is obtained as follows:

$$\begin{aligned} C_0 D_q(f(x)g(x)) &= \frac{d_q(f(x)g(x))}{d_q x} \\ &= D_q(f(x)g(x)) \\ &= f(qx) C_0 D_q g(x) + g(x) C_0 D_q^p f(x) \\ &= f(qx) D_q g(x) + g(x) D_q f(x). \end{aligned}$$

For $q \to 1$ we have $h$-derivative formula for products of functions is obtained as follows:

$$\begin{aligned} C_h D_1(f(x)g(x)) &= \frac{d_h(f(x)g(x))}{d_h x} \\ &= D_h(f(x)g(x)) \\ &= f(x+h) C_h D_1 g(x) + g(x) C_h D_1 f(x) \\ &= f(x+h) D_h g(x) + g(x) D_h f(x). \end{aligned}$$

By using symmetry we can have from (33):

$$C_h d_q^p(g(x)f(x)) = g(q(x+h)) C_h D_q^p f(x) + f(x) C_h d_q^p g(x). \tag{34}$$

Both (33) and (34) are equivalent.

**Remark 2.** It is notable that if we put $p = 1$, $h = \frac{\omega}{q}$ for $\omega > 0$, equation (33) provides the product formula for $(q, \omega)$-derivatives given in [6].

### 2.3. Quotient formula for $(q, p - h)$-derivatives

The quotient formula of $(q, p - h)$-derivatives for quotient of two functions by using (33) and (34) are given as follows: We have for $g(x) \neq 0$

$$g(x)\frac{f(x)}{g(x)} = f(x). \tag{35}$$

By taking $q - h$-derivative on both sides, we have

$$C_h d_q^p\left(g(x)\frac{f(x)}{g(x)}\right) = C_h d_q^p(f(x)). \tag{36}$$

By using (33), one can get

$$g(q(x+h))C_hd_q^p\left(\frac{f(x)}{g(x)}\right)+\frac{f(x)}{g(x)}C_hd_q^pg(x)=C_hd_q^p(f(x)).$$

Now

$$\begin{aligned} C_hd_q^p\left(\frac{f(x)}{g(x)}\right) &= \frac{C_hd_q^p(f(x))-\frac{f(x)}{g(x)}C_hd_q^p(g(x))}{g(q(x+h))} \\ &= \frac{g(x)C_hd_q^p(f(x))-f(x)C_hd_q^p(g(x))}{g(q(x+h))g(x)}. \end{aligned} \tag{37}$$

By using (34), one can get

$$\frac{f(q(x+h))}{g(q(x+h))}C_hd_q^p\Big(g(x)\Big)+g(x)C_hd_q^p\left(\frac{f(x)}{g(x)}\right)=C_hd_q^p\Big(f(x)\Big),$$

that is:

$$C_hd_q^p\left(\frac{f(x)}{g(x)}\right)=\frac{C_hd_q^p(f(x))g(q(x+h))-f(q(x+h))C_hd_q^p(g(x))}{g(q(x+h))g(x)}.$$

**Remark 3.** It is notable that if we put $p=1$, $h=\frac{\omega}{q}$ for $\omega>0$, equation (37) provides the quotient formula for $(q,\omega)$-derivatives given in [6].

If $f$ is the $(q,p-h)$-derivative of $F$ that is $f(x) = C_hd_q^pF(x)$, then $F$ will be called the $(q,p-h)$-anti-derivative of $f$. The $(q,p-h)$-anti-derivative will be denoted by $\int f(x)\,{}_hd_q^px$.

## 3. $(q,p-h)$-derivative on a finite interval

In this section we consider a finite interval $I:=[a,b]$ for $a,b$ real numbers. We define $(q,p-h)$-derivative on this interval in the following definition.

**Definition 8.** Let $0<q<p\leq 1$, $h\in\mathbb{R}$ and $x\in I$. For a continuous function $f:I\to\mathbb{R}$ the left and right $q-h-$derivatives on $I$ denoted by $C_hD_q^{a^+}f$ and $C_hD_q^{b-}f$ are defined with the following equations respectively;

$$C_hD_{p,q}^{a^+}f(x):=\frac{f((1-q)a+q(x+h))-f((1-p)a+px)}{(p-q)(a-x)+qh};\ x\neq\frac{qh+(p-q)a}{p-q}:=u, \tag{38}$$

$$C_hD_{p,q}^{b_-}f(x):=\frac{f((1-q)x+q(b+h))-f((1-p)x+pb)}{(p-q)(x-b)+qh};\ x\neq\frac{-qh+(p-q)b}{p-q}:=v, \tag{39}$$

provided that $(p-q)a+q(x+h)\in[a,x]$ and $(p-q)x+q(b+h)\in[x,b]$. Also, $C_hD_{p,q}^{a^+}f(u)=\lim\limits_{x\to u}C_hD_{p,q}^{a^+}f(x)$ and $C_hD_{p,q}^{b_-}f(v)=\lim\limits_{x\to v}C_hD_{p,q}^{b_-}f(x)$.

The function $f$ will be called left $(q,p-h)$-differentiable on $(a,x+h)$, if $C_hD_{p,q}^{a^+}f(x)$ exists for each of its points, on the other hand $f$ will be called right $(q,p-h)$-differentiable on $(x+h,b)$, if $C_hD_{p,q}^{b_-}f(x)$ exists at each of its points. It is noted that $C_hD_{p,q}^{a^+}f(b)=C_hD_{p,q}^{b_-}f(a)$. In (38), the value $h=0$ gives the $(p,q)$-derivative on interval $I$ stated in Definition 2, i.e., $C_0D_{p,q}^{a^+}f(x)={}_aD_{p,q}f(x)$; the setting $h=0$, $p=1$ gives the $q$-derivative on interval $I$ stated in Definition 1, i.e., $C_0D_{1,q}^{a^+}f(x)={}_aD_qf(x)$; the value $p=1$ gives the $q-h$-derivative on interval $I$ stated in Definition 3, i.e., $C_hD_{1,q}^{a^+}f(x)=C_hD_q^{a^+}f(x)$. Also for $a=0$ one can have $C_hD_{p,q}^{0^+}f(x)=C_hD_q^pf(x)$, i.e., the $(q,p-h)$-derivative given in (25) is recovered; for $h=0=a$ one can have $C_0D_{p,q}^{0^+}f(x)=D_q^pf(x)$, i.e., the $(p,q)$-derivative is recovered; for $a=0$, $q=p=1$ one can have $C_hD_{1,1}^{0^+}f(x)=D_hf(x)$ i.e., the $h$-derivative is recovered; for $a=0$, $p=1$ one can have $C_hD_{1,q}^{0^+}f(x)=C_hD_qf(x)$ i.e., the $q-h$-derivative is recovered; for $h=0=a=p$ and taking limit $q\to 1$ one can get the usual derivative for a differentiable function $f$ i.e., $\lim\limits_{q\to 1}C_0D_{1,q}^{0^+}f(x)=\frac{d}{dx}f(x)$. The similar consequences can be found from the equation (39). We give the definition of left and right $q-$derivatives on $I$ as follows:

**Definition 9.** Let $0 < q < p \leq 1$, $h \in \mathbb{R}$ and $x \in I$. For a continuous function $f : I \to \mathbb{R}$ the left and right composite $(p, q)$-derivatives on $I$ denoted by $D_{p,q}^{a^+} f$ and $D_{p,q}^{b_-} f$ are defined with the following equations respectively;

$$D_{p,q}^{a^+} f(x) := \frac{f(qx + (1-q)a) - f(px + (1-p)a)}{(p-q)(a-x)};\ x > a, \tag{40}$$

$$D_{p,q}^{b_-} f(x) := \frac{f(qb + (1-q)x) - f(pb + (1-p)x)}{(p-q)(x-b)};\ x < b. \tag{41}$$

From (40) we have $D_{p,q}^{0^+} f(x) = D_{p,q} f(x)$. Next, we give the definition of left and right $(q, p-h)$-integrals as follows:

**Definition 10.** Let $0 < q < p \leq 1$ and $f : I = [a, b] \to \mathbb{R}$ be a continuous function. Then the left and right $q, p-h$-integrals on $I$ denoted by $I_{q,p-h}^{a+} f$ and $I_{q,p-h}^{b} f$ are defined as follows:

$$\begin{aligned} I_{q,p-h}^{a+} f(x) :&= \int_a^x f(t)\, {}_h d_q^p t \\ &= ((p-q)(x-a) + qh) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}} a + \left(1 - \frac{q^n}{p^{n+1}}\right) x + \frac{nq^n h}{p^{n+1}}\right),\ x > a, \end{aligned} \tag{42}$$

$$\begin{aligned} I_{q,p-h}^{b-} f(x) :&= \int_x^b f(t)\, {}_h d_q^p t \\ &= ((p-q)(b-x) + qh) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}} x + \left(1 - \frac{q^n}{p^{n+1}}\right) b + \frac{nq^n h}{p^{n+1}}\right),\ x < b. \end{aligned} \tag{43}$$

**Example 2.** Let $f(t) = t - a$ and $g(t) = b - t$. Then we have

$$\begin{aligned} I_{q,p-h}^{a+} f(x) = \int_a^x (t-a)\, {}_h d_q^p t = & \frac{(p-q)(x-a) + qh}{p-q} \\ & \times \left(\frac{(p+q-1)(px-a)}{p+q} + \frac{h(p-q)}{p^2} \sum_{n=0}^{\infty} n \left(\frac{q}{p}\right)^{2n}\right) \end{aligned} \tag{44}$$

and

$$\begin{aligned} I_{q,p-h}^{b-} g(x) = \int_x^b (b-t)\, {}_h d_q^p t = & \frac{(p-q)(b-x) + qh}{p-q} \\ & \times \left(\frac{b-x}{p+q} - \frac{h(p-q)}{p^2} \sum_{n=0}^{\infty} n \left(\frac{q}{p}\right)^{2n}\right). \end{aligned} \tag{45}$$

**Example 3.** Let $f(t) = x - t$ and $g(t) = t - x$. Then we have

$$\begin{aligned} I_{q,p-h}^{a+} f(x) &= \int_a^x (x-t)\, {}_h d_q^p t \\ &= \frac{(p-q)(x-a) + qh}{p-q} \left(\frac{x-a}{p+q} - \frac{h(p-q)}{p^2} \sum_{n=0}^{\infty} n \left(\frac{q}{p}\right)^{2n}\right) \end{aligned} \tag{46}$$

and

$$\begin{aligned} I_{q,p-h}^{b-} g(x) &= \int_x^b (t-x)\, {}_h d_q^p t \\ &= \frac{(p-q)(b-x) + qh}{p-q} \left(\frac{(p+q-1)(b-x)}{p+q} + \frac{h(p-q)}{p^2} \sum_{n=0}^{\infty} n \left(\frac{q}{p}\right)^{2n}\right). \end{aligned} \tag{47}$$

By considering $h = 0$ the corresponding left and right $(p, q)$-integrals are defined as follows:

**Definition 11.** Let $0 < q < p \leq 1$ and $f : I = [a, b] \to \mathbb{R}$ be a continuous function. Then the left and right $(p, q)$-integrals on $I$ denoted by $I^{a+}_{p,q}f$ and $I^{b}_{p,q}f$ are defined as follows:

$$\begin{aligned} I^{a+}_{q,p-0}f(x) :&= I^{a+}_{p,q}f(x) = \int_a^x f(t)d_q^p t \\ &= (p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}a + \left(1-\frac{q^n}{p^{n+1}}\right)x\right), \; x > a, \end{aligned} \tag{48}$$

$$\begin{aligned} I^{b-}_{q,p-0}f(x) :&= I^{b-}_{p,q}f(x) \\ &= \int_x^b f(t)d_q^p t = (p-q)(b-x)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}x + \left(1-\frac{q^n}{p^{n+1}}\right)b\right), \; x < b. \end{aligned} \tag{49}$$

The left $(p, q)$-integral is equivalent to the $(p, q)$-definite integral defined in [3]. For $p = 1$; the left $(p, q)$-integral is equivalent to the $q_a$-definite integral defined in [2], while the right $(p, q)$-integral is defined in [1] which is called $q^b$-definite integral.

**Example 4.** Let $f(t) = t - a$ and $g(t) = b - t$. Then from Example 2 for $h = 0$ we have $I^{a+}_{q,p-0}f(x) = I^{a+}_{p,q}f(x) = \int_a^x (t-a)d_q^p t = \frac{(p+q-1)(px-a)(x-a)}{p+q}$ and $I^{b-}_{q,p-0}g(x) = I^{b-}_{p,q}f(x) = \int_x^b (b-t)d_q^p t = \frac{(b-x)^2}{p+q}$.

By considering $p = 1$, $q \to 1$ the corresponding left and right $h$-integrals are defined as follows:

**Definition 12.** Let $f : I = [a, b] \to \mathbb{R}$ be a continuous function. Then the left and right $h$-integrals on $I$ denoted by $I^{a+}_h f$ and $I^{b}_h f$ are defined as follows:

$$I^{a+}_h f(x) = \lim_{q\to 1} I^{a+}_{q,1-h}f(x), \; x > a, \tag{50}$$

$$I^{b-}_h f(x) = \lim_{q\to 1} I^{b-}_{q,1-h}f(x), \; x < b. \tag{51}$$

It is noted from Definition 10 that $I^{a+}_{q,p-h}f(b) = I^{b-}_{q,p-h}f(a) = \int_a^b f(t)\,{}_h d_q^p t$.

**Conflicts of Interest:** The author declares no conflict of interest.

**Data Availability:** All data required for this research is included within this paper.

**Funding Information:** This research is funded by Higher Education Commission of Pakistan.

## References

[1] Bermudo, S., Kórus, P., & Nápoles Valdés, J. E. (2020). On $q-$Hermite-Hadamard inequalities for general convex functions. *Acta Mathematica Hungarica, 162*, 364-374.

[2] Tariboon, J., & Ntouyas, S. K. (2014). Quantum integral inequalities on finite intervals. *Journal of Inequalities and Applications, 2014*, Article No. 121. https://doi.org/10.1186/1029-242X-2014-121.

[3] Tunç, M., & Göv, E. (2016). $(p, q)$-integral inequalities. *RGMIA, 19*, 1-13.

[4] Tunç, M., & Göv, E. (2021). Some integral inequalities via (p, q)-calculus on finite intervals. *Filomat, 35*(5), 1421-1430.

[5] Farid, G., Anwar, M., & Shoaib, M., (2023). On generalizations of $q-$ and $h-$integrals and some related inequalties. *Preprint*.

[6] Hahn, W. (1983). Ein beitrag zur theorie der orthogonalpolynome. *Monatshefte Für Mathematik, 95*(1), 19-24.

[7] Farid, G. (2019). Some Riemann-Liouville fractional integral inequalities for convex functions. *The Journal of Analysis, 27*(4), 1095-1102.

[8] Kac, V., & Cheung, V. (2002). *Quantum Calculus*. Springer, New York, NY.