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Global asymptotic stability of constant equilibrium point in attraction-repulsion chemotaxis model with logistic source term

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Abstract: This paper deals with nonnegative solutions of the Neumann initial-boundary value problem for an attraction-repulsion chemotaxis model with logistic source term of Eq. (1) in bounded convex domains $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with smooth boundary. It is shown that if the ratio $\frac{\mu}{\chi\alpha-\xi\gamma}$ is sufficiently large, then the unique nontrivial spatially homogeneous equilibrium given by $(u_1, u_2, u_3) = (1, \frac{\alpha}{\beta}, \frac{\gamma}{\eta})$ is globally asymptotically stable in the sense that for any choice of suitably regular nonnegative initial data (u_{10}, u_{20}, u_{30}) such that $u_{10} \not\equiv 0$, the above problem possesses uniquely determined global classical solution (u_1, u_2, u_3) with $(u_1, u_2, u_3)|_{t=0} = (u_{10}, u_{20}, u_{30})$ which satisfies $\|u_1(\cdot, t) - 1\|_{L^\infty(\Omega)} \to 0$, $\left\|u_2(\cdot, t) - \frac{\alpha}{\beta}\right\|_{L^\infty(\Omega)} \to 0$, $\left\|u_3(\cdot, t) - \frac{\gamma}{\eta}\right\|_{L^\infty(\Omega)} \to 0$, as $t \to \infty$.

Keywords: Keller-Segel model; Logistic source; Chemotaxis; Attraction-Repulsion; Asymptotic Stability.

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1. Introduction

In this study, we consider the following attraction-repulsion chemotaxis system

$$\begin{cases} u_{1t} = d_1\Delta u_1 - \chi\nabla(u_1\nabla u_2) + \xi\nabla(u_1\nabla u_3) + g(u_1), & \mathbf{x} \in \Omega,\ t > 0, \\ u_{2t} = d_2\Delta u_2 + \alpha u_1 - \beta u_2, & \mathbf{x} \in \Omega,\ t > 0, \\ u_{3t} = d_3\Delta u_3 + \gamma u_1 - \eta u_3, & \mathbf{x} \in \Omega,\ t > 0, \end{cases} \tag{1}$$

where $u_1 = u_1(x, t)$ denotes the cell density, $u_2 = u_2(x, t)$ represents the concentration of an attractive signal, and $u_3 = u_3(x, t)$ is the concentration of a repulsive signal. The parameters $\chi, \xi, \mu, \alpha, \beta, \gamma$, and η are assumed to be positive.

This model with $g(u_1) \equiv 0$ was proposed in [1] for describing the quorum effect in a chemotaxis process and in [2] for describing the aggregation of microglia in Alzheimer's disease. This system describes the competition between attractive and repulsive signals, which are both produced by the cells themselves. These signals diffuse much faster than cells, so (1) reads

$$\begin{cases} u_{1t} = d_1\Delta u_1 - \chi\nabla(u_1\nabla u_2) + \xi\nabla(u_1\nabla u_3) + g(u_1), & \mathbf{x} \in \Omega,\ t > 0, \\ 0 = d_2\Delta u_2 + \alpha u_1 - \beta u_2, & \mathbf{x} \in \Omega,\ t > 0, \\ 0 = d_3\Delta u_3 + \gamma u_1 - \eta u_3, & \mathbf{x} \in \Omega,\ t > 0. \end{cases} \tag{2}$$

Suppose we simplify the chemical processes above by assuming that there is no chemorepulsive signal (i.e., $\xi = 0$). In that case, we obtain the well-known (attractive) chemotaxis model proposed by Keller and Segel [3]. Based on an apparent Lyapunov functional for this system of two coupled equations, some basic properties have been discussed in previous studies, such as the global solvability, blow up, and large-time asymptotics (see [4–10] and the references therein).

Most of the results based on (1) or (2) appear to focus on the question of global existence versus blow-up. To the best of our knowledge, few results have been reported that involve the qualitative behavior of (1) or (2) in higher dimensions. In [11], for any $\beta > 0$ and $\eta > 0$, the large-time behavior of (1) was explored in the one-dimensional case. For a higher-dimensional case, [12] showed that each solution of (1) or (2) converges to a unique trivial stationary solution.

Proposed first by Keller and Segel [3], the classical (attractive) chemotaxis model was a system of two partial differential equations (i.e., the first two equations of (1) with $\xi = 0$), which possess an apparent Lyapunov functional. This particular structure motivated a vast amount of mathematical studies in [10,13–15,17], where most of the works were focused on whether the solution blows up or not (see some early works in [8,18,19] in this area). On the other hand, for the repulsive Keller-Segel model (i.e., the coupling of first and third equations of (1) with $\chi = 0$), a Lyapunov function (which was different from that of the attractive Keller-Segel model) was found in [20] which leads to the global existence of classical solutions in two dimensions and weak solutions in three and four dimensions. Compared to the classical Keller-Segel model, the three-component system of attraction-repulsion Keller-Segel (ARKS) model (1) is much more complex to analyze due to the lack of an apparent Lyapunov functional.

The present study aims to investigate in more detail how the destabilizing and aggregation-supporting properties of chemotactic-repulsive cross-diffusion interact with growth limitations of logistic type. we shall see that largeness of the coefficient μ, as related to $\chi\alpha > \xi\gamma$, fully stabilizes the unique spatially homogeneous steady state $(u_1, u_2, u_3) = (u_{1c}, u_{2c}, u_{3c}) = (1, \frac{\alpha}{\beta}, \frac{\gamma}{\eta})$ in the sense that whenever $\frac{\mu}{\chi\alpha - \xi\gamma}$ is suitably large, the equilibrium (u_{1c}, u_{2c}, u_{3c}) becomes globally asymptotically stable. Thus, we impose boundary and initial conditions to close system (1). Therefore, we may suppose that

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial u_3}{\partial \nu} = 0, \qquad x \in \partial\Omega, \ t > 0 \tag{3}$$

and

$$u_1(x,0) = u_{10}(x), \ u_2(x,0) = u_{20}(x), \ u_3(x,0) = u_{30}(x), \quad x \in \Omega. \tag{4}$$

satisfy

$$u_{10} \geq (\not\equiv) 0, \ u_{20}, \ u_{30} \geq 0, \text{ and } u_{10} \in C^0(\bar{\Omega}), \ u_{20}, u_{30} \in W^{1,\infty}(\Omega). \tag{5}$$

Our main result is as follows.

Theorem 1. Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a smooth bounded convex domain. Then there exist $M > 0$ such that

$$\frac{\mu}{\chi\alpha - \xi\gamma} > M \tag{6}$$

for $\chi\alpha > \xi\gamma$ and (u_{10}, u_{20}, u_{30}) fulfilling *(5)*, the problem *(1)* possesses a uniquely determined global-in-time classical solution (u_1, u_2, u_3) such that

$$u_1 \in C^0(\bar{\Omega} \times [0, T_{\max})) \bigcap C^{2,1}(\bar{\Omega} \times (0, T_{\max})),$$
$$u_2, u_3 \in C^0(\bar{\Omega} \times [0, T_{\max})) \bigcap C^{2,1}(\bar{\Omega} \times (0, T_{\max})) \bigcap L^\infty((0, T_{\max}); W^{1,q}(\Omega)), \quad \forall\, q > n$$

and which satisfies

$$\|u_1(\cdot,t) - 1\|_{L^\infty(\Omega)} \to 0, \ \|u_2(\cdot,t) - \frac{\alpha}{\beta}\|_{L^\infty(\Omega)} \to 0, \ \|u_3(\cdot,t) - \frac{\gamma}{\eta}\|_{L^\infty(\Omega)} \to 0, \ \text{ as } t \to \infty. \tag{7}$$

2. Preliminaries

It is easy to see that the model (1) have the trivial equilibrium point $\mathbf{E}_0 = (0,0,0)$ and the unique positive equilibrium point

$$\mathbf{w}_c := (u_{1c}, u_{2c}, u_{3c}) = \left(1, \frac{\alpha}{\beta}, \frac{\gamma}{\eta}\right).$$

For given (u_{10}, u_{20}, u_{30}) satisfying (5), let $T_{\max}$ and (u_1, u_2, u_3) be as given by Theorem 1, and the disturbance around $\mathbf{w}_c$ will be denoted as

$$U_1(x,t) = u_1(x,t) - 1, \quad U_2(x,t) = u_2(x,t) - \tfrac{\alpha}{\beta}, \quad U_3(x,t) = u_3(x,t) - \tfrac{\gamma}{\eta} \tag{8}$$

for $x \in \overline{\Omega}$ and $t \geq 0$. Then by straightforward computation it follows that (U_1, U_2, U_3) solves

$$\begin{cases} U_{1t} = \Delta U_1 - \chi\nabla(u_1\nabla U_2) + \xi\nabla(u_1\nabla U_3) - \mu U_1(1+U_1), & x \in \Omega,\ t > 0, \\ U_{2t} = \Delta U_2 + \alpha U_1 - \beta U_2, & x \in \Omega,\ t > 0, \\ U_{3t} = \Delta U_3 + \gamma U_1 - \eta U_3, & x \in \Omega,\ t > 0, \\ \frac{\partial U_1}{\partial \nu} = \frac{\partial U_2}{\partial \nu} = \frac{\partial U_3}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ U_1(x,0) = u_{10}(x),\ U_2(x,0) = u_{20}(x)\ U_3(x,0) = u_{30}(x), & x \in \Omega. \end{cases} \tag{9}$$

To estimate the cross-diffusive terms in (9), we use the transformation $S = \chi U_2 - \xi U_3$ and let $\beta = \eta$ such that (9) can be transformed into the following system;

$$\begin{cases} U_{1t} = \Delta U_1 - \nabla(u_1\nabla S) - \mu U_1(1+U_1), & x \in \Omega,\ t > 0, \\ S_t = \Delta S + (\chi\alpha - \xi\gamma)U_1 - \beta S, & x \in \Omega,\ t > 0, \\ \frac{\partial U_1}{\partial \nu} = \frac{\partial S}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ U_1(x,0) = u_{10}(x),\ S(x,0) = \chi u_{20}(x) - \xi u_3(x) = S_0(x), & x \in \Omega. \end{cases} \tag{10}$$

3. An explicit bound for u_1 via comparison: Global existence

The cornerstone of our analysis will be provided by the following lemma, which, under a largeness assumption on $\frac{\mu}{\chi\alpha - \xi\gamma}$, establishes a detailed pointwise upper estimate that is universal in the sense that for each solution it asserts the eventual validity of an appropriate bound for u_1. The proof is based on a comparison argument, inspired by [21], which makes use of a favorable parabolic differential inequality satisfied by the quantity $Z := U + \frac{1}{2(\chi\alpha-\xi\gamma)}|\nabla S|^2$ coupling both components in (8)(cf. (13) below). The following reasoning is the only place in this chapter where the convexity of Ω is explicitly used.

Lemma 1. *Suppose that $\mu > \frac{n(\chi\alpha-\xi\gamma)}{4}$. Then for any choice of u_{10}, u_{20} and u_{30} fulfilling (2), the solution of (1) is global in time and satisfies*

$$\limsup_{t\to\infty} \|u_1(.,t)\|_{L^\infty} \leq J, \tag{11}$$

where $J := 1 + \frac{\mu\theta}{1-\theta}$ and $\theta = \frac{n(\chi\alpha-\xi\gamma)}{4\mu}$.

Proof. With U_1 , U_2 and U_3 as defined in (8), we denote

$$Z := U + \frac{1}{2(\chi\alpha - \xi\gamma)}|\nabla S|^2,$$

for $x \in \bar{\Omega}$ and $t \in (0, T_{max})$.

By using (10), we have

$$\begin{aligned} Z_t &= U_t + \frac{1}{\chi\alpha - \xi\gamma}\nabla S\nabla S_t \\ &= \Delta U_1 - \nabla(u_1\nabla S) - \mu U_1(1+U_1) + \frac{1}{(\chi\alpha - \xi\gamma)}\nabla S[\nabla\Delta S + (\chi\alpha - \xi\gamma)\nabla U_1 - \beta\nabla S] \\ &= \Delta U_1 - \nabla u_1\nabla S - u_1\Delta S - \mu U_1(1+U_1) + \frac{1}{(\chi\alpha - \xi\gamma)}\nabla S\nabla\Delta S + \nabla S\nabla U_1 - \frac{\beta}{(\chi\alpha - \xi\gamma)}|\nabla S|^2, \end{aligned}$$

for all $x \in \Omega$ and $t \in (0, T_{max})$. Here the equality $U_1(x,t) = u_1(x,t) - 1$ implies the cancelation

$$-\nabla u_1\nabla S + \nabla S\nabla U_1 \equiv 0.$$

Thus

$$Z_t = \Delta U_1 - u_1 \Delta S - \mu U_1(1+U_1) + \frac{1}{(\chi\alpha - \xi\gamma)}\nabla S \nabla \triangle S - \frac{\beta}{(\chi\alpha - \xi\gamma)}|\nabla S|^2.$$

So that in light of the point wise identity

$$\nabla S \nabla \triangle S = \frac{1}{2}\triangle|\nabla S|^2 - |D^2 S|^2,$$

we have

$$Z_t = \Delta U_1 - u_1\Delta S - \mu U_1 - \mu U_1^2 - \tfrac{1}{(\chi\alpha-\xi\gamma)}|D^2S|^2 + \Delta(\tfrac{1}{2(\chi\alpha-\xi\gamma)}|\nabla S|^2) - \tfrac{\beta}{(\chi\alpha-\xi\gamma)}|\nabla S|^2. \tag{12}$$

By the Cauchy-Schwarz inequality, we have $|\triangle S|^2 \leq n|D^2 S|^2$. By Young inequality we obtain

$$\begin{aligned} u_1\Delta S &\leq \tfrac{1}{n(\chi\alpha-\xi\gamma)}|\Delta S|^2 + \tfrac{n(\chi\alpha-\xi\gamma)}{4}{u_1}^2 \\ &\leq \tfrac{|D^2S|^2}{(\chi\alpha-\xi\gamma)} + \tfrac{n(\chi\alpha-\xi\gamma)}{4}(U_1+1)^2 \\ &= \tfrac{|D^2S|^2}{(\chi\alpha-\xi\gamma)} + \tfrac{n(\chi\alpha-\xi\gamma)}{4}(U_1^2+2U_1+1) \\ &= \tfrac{|D|^2S|^2}{(\chi\alpha-\xi\gamma)} + \tfrac{n(\chi\alpha-\xi\gamma)}{4}U^2 + \tfrac{n(\chi\alpha-\xi\gamma)}{2}U + \tfrac{n(\chi\alpha-\xi\gamma)}{4}, \quad \forall\,(x,t)\in\Omega\times(0,T_{max}). \end{aligned}$$

From (12), on dropping a nonpositive term and $\beta \geq \frac{1}{2}$. We obtain that

$$Z_t - \Delta Z = -\mu U_1 - \mu U_1^2 - \frac{\beta}{(\chi\alpha-\xi\gamma)}|\nabla S|^2 + \frac{n(\chi\alpha-\xi\gamma)}{4}U^2 + \frac{n(\chi\alpha-\xi\gamma)}{2}U + \frac{n(\chi\alpha-\xi\gamma)}{4}.$$

Thus,

$$\begin{aligned} Z_t - \Delta Z + \max\{\mu,\beta\}Z &= -\mu U_1^2 - \frac{\beta}{2(\chi\alpha-\xi\gamma)}|\nabla S|^2 + \frac{n(\chi\alpha-\xi\gamma)}{4}U^2 + \frac{n(\chi\alpha-\xi\gamma)}{2}U + \frac{n(\chi\alpha-\xi\gamma)}{4} \\ &\leq -\mu U_1^2 + \frac{n(\chi\alpha-\xi\gamma)}{4}U_1^2 + \frac{n(\chi\alpha-\xi\gamma)}{2}U_1 + \frac{n(\chi\alpha-\xi\gamma)}{4} \\ &\leq -\left(\mu - \frac{n(\chi\alpha-\xi\gamma)}{4}\right)\left\{U_1^2 - 2(\frac{n(\chi\alpha-\xi\gamma)}{4\mu-n(\chi\alpha-\xi\gamma)}) - \frac{n(\chi\alpha-\xi\gamma)}{4\mu-n(\chi\alpha-\xi\gamma)}\right\} \\ &\leq -\left(\mu - \frac{n(\chi\alpha-\xi\gamma)}{4}\right)\left\{(U_1-1)^2 - \left(\frac{n(\chi\alpha-\xi\gamma)}{4\mu-n(\chi\alpha-\xi\gamma)}\right)^2 - \frac{n(\chi\alpha-\xi\gamma)}{4\mu-n(\chi\alpha-\xi\gamma)}\right\} \\ &\leq \left(\mu - \frac{n(\chi\alpha-\xi\gamma)}{4}\right)\left\{\left(\frac{n(\chi\alpha-\xi\gamma)}{4\mu-n(\chi\alpha-\xi\gamma)}\right)^2 + \frac{n(\chi\alpha-\xi\gamma)}{4\mu-n(\chi\alpha-\xi\gamma)}\right\} \\ &\leq \left(\frac{n(\chi\alpha-\xi\gamma)}{4}\right)\left\{\frac{n(\chi\alpha-\xi\gamma)}{4\mu-n(\chi\alpha-\xi\gamma)} + 1\right\}, \quad \forall\,(x,t)\in\Omega\times(0,T_{max}). \end{aligned} \tag{13}$$

In order to derive an estimate for Z itself from this, we note that since Ω is convex and $\frac{\partial S}{\partial \nu}|_{\partial\Omega} = 0$, according to a well-known result [22] we know that $\frac{\partial|\nabla S|^2}{\partial \nu}|_{\partial\Omega} \leq 0$ and hence also $\frac{\partial Z}{\partial \nu}|_{\partial\Omega} \leq 0$ on $\partial\Omega$.

We therefore may compare Z to spatially homogeneous functions having a super solution property with regard to the parabolic operator in (13). Indeed, if we abbreviate $t_0 := \min(\frac{1}{2}T_{max}, 1)$ and $c_1 := \|U(\cdot,t_0)\|_{L^\infty(\Omega)} + \frac{1}{2(\chi\alpha-\xi\gamma)}\|\nabla S(\cdot,t)\|_{L^\infty(\Omega)}$ and let $y \in C^1([t_0,\infty))$ denote the solution of then initial-value problem

$$\begin{cases} y'(t) + y(t) = \left(\frac{n(\chi\alpha-\xi\gamma)}{4}\right)\left(\frac{n(\chi\alpha-\xi\gamma)}{4\mu-n(\chi\alpha-\xi\gamma)} + 1\right), & t > t_0, \\ y(t_0) = c_1. \end{cases} \tag{14}$$

Then from the comparison principle and the initial ordering $Z(x,t_0) \leq y(t_0)$, valid thanks to our choice of c_1, we infer that

$$Z(x,t) \leq y(t), \quad \forall\, x \in \Omega, \quad t \in (t_0, T_{max}). \tag{15}$$

Upon explicitly solving (14) for instance, we see that y is bounded and moreover satisfies

$$y(t) \to \left(\frac{n(\chi\alpha - \xi\gamma)}{4}\right) \left\{ \frac{n(\chi\alpha - \xi\gamma)}{4\mu - n(\chi\alpha - \xi\gamma)} + 1 \right\} \text{ as } t \to \infty. \tag{16}$$

Along with (15), this first implies that

$$u(t,x) \le 1 + y(t) \le 1 + \|y\|_{L^\infty(t_0,\infty)}, \quad \forall\, x \in \Omega, \quad t \in (0, T_{max}),$$

which in view of (11) warrants that actually $T_{max} = \infty$. Thereupon, (15) and (16) show that

$$\lim_{t\to\infty} \sup \|u(.,t)\|_{L^\infty}(\Omega) \le 1 + \lim_{t\to\infty} \sup y(t) \le \left\{1 + \left(\frac{n(\chi\alpha-\xi\gamma)}{4}\right) \left\{\frac{n(\chi\alpha-\xi\gamma)}{4\mu-n(\chi\alpha-\xi\gamma)} + 1\right\}\right\} = 1 + \frac{\mu\theta}{1-\theta},$$

where $\theta = \frac{n(\chi\alpha-\xi\gamma)}{4\mu}$, whereby the proof is completed. □

4. Bounds in $L^p(\Omega)$ for ∇u_2, ∇u_3 and $A^\delta U_1$ for $\delta < \frac{1}{2}$

We carry on to derive from the pointwise inequality implied by Lemma 1 an estimate for ∇u_2 and ∇u_3 with respect to the norm in $L^p(\Omega)$ for arbitrary $p > 1$.

Lemma 2. *Suppose that $p > 1$ and $\mu > \frac{n(\chi\alpha-\xi\gamma)}{4}$. Then there exist $c(p) > 0$ such that if (u_1, u_2, u_3) is the solution of (1) with (u_{10}, u_{20}, u_{30}) satisfying (5), then*

$$\lim_{t\to\infty} \sup \|\nabla u_2(.,t)\|_{L^p(\Omega)} \le C(p)J, \tag{17}$$

and

$$\lim_{t\to\infty} \sup \|\nabla u_3(.,t)\|_{L^p(\Omega)} \le C(p)J, \tag{18}$$

where J is mentioned in Lemma 1.

Proof. From the Neumann heat semigroup estimate in Ω [16,17], we can choose $c_1(p) > 0$ such that

$$\|\nabla e^{\tau\Delta}\varphi\|_{L^p(\Omega)} \le c_1(p)\tau^{-\frac{1}{2}}\|\varphi\|_{L^\infty(\Omega)}, \quad \forall\, \tau > 0 \text{ and for any } \varphi \in L^\infty(\Omega). \tag{19}$$

Furthermore, an application of Lemma 1 shows that the considered solutions satisfy

$$\lim_{t\to\infty} \sup \|u_2(.,t)\|_{L^\infty(\Omega)} \le J, \quad \forall\, t > t_0.$$

So that for any (u_1, u_2, u_3) we can fix some suitably large $t_0 = t_0(\Omega) = t_0(u_{10}, u_{20}, u_{30}) > 0$ such that

$$\|u_2(.,t)\|_{L^\infty(\Omega)} \le 2J, \quad \forall\, t \ge t_0. \tag{20}$$

Then by means of the variation-of-constants representation for u_2, the second equation in (1) we can estimate

$$\|\nabla u_2(\cdot,t)\|_{L^p(\Omega)} \le \|\nabla e^{(t-t_0)(\Delta-1)}u_2(\cdot,t_0)\|_{L^p(\Omega)} + \int_{t_0}^t \|\nabla e^{(t-s)(\Delta-1)}(u_1(\cdot,s))\|_{L^p(\Omega)}ds := I_1 + I_2, \quad \forall\, t > t_0, \tag{21}$$

where (19) implies that

$$I_1 = e^{-(t-t_0)}\|\nabla e^{(t-t_0)\Delta}u_2(\cdot,t_0)\|_{L^p(\Omega)} \le c_1(p)(t-t_0)^{-\frac{1}{2}}e^{-(t-t_0)}\|u_2(\cdot,t_0)\|_{L^\infty(\Omega)}, \quad \forall\, t > t_0. \tag{22}$$

Moreover, combining (19) with (4) yields

$$\int_{t_0}^t \|\nabla e^{(t-s)(\triangle-1)}u_2(.,s)\|_{L^p(\Omega)}ds \le c_1(p)\int_{t_0}^t (t-t_0)^{-\frac{1}{2}}e^{-(t-s)}\|u_2(.,t)\|_{L^\infty(\Omega)}ds$$

$$\leq 2c_1(p)J\int_{t_0}^{t}(t-t_0)^{-\frac{1}{2}}e^{-(t-s)}ds$$
$$= 2c_1(p)J\int_{0}^{t-t_0}\sigma^{-\frac{1}{2}}e^{-\sigma}ds$$
$$= C(p)J, \quad \forall\, t > t_0, \tag{23}$$

where $C(p) = 2c_1(p)\int_0^\infty \sigma^{-\frac{1}{2}}e^{\sigma}d\sigma$. From (21)-(23) we therefore obtain that

$$\limsup_{t\to\infty}\|\nabla u_2(.,t)\|_{L^p(\Omega)} \leq C(p)J \tag{24}$$

as desired. Applying the same procedure to u_3, we finish the proof. □

For the next Lemma and also for Lemma 4 below, we fix any number $\lambda \in (0,1)$ and, given $p > 1$, let $A = A_p$ denote the realization of the operator $-\Delta + \lambda$ under homogeneous Neumann boundary conditions in $L^P(\Omega)$. Then it is known [23,24] that A is sectorial and thus possesses closed fractional powers A^k for arbitrary $k > 0$, and the corresponding domains $D(A^k)$ are known to have the embedding property

$$D(A^k) \to W^{2,\infty}(\Omega) \text{ if } 2k - \frac{n}{p} > 2. \tag{25}$$

Moreover, if $(e^{-tA}\varphi)_{t\geq 0}$ denotes the corresponding analytic semigroup, then for each $k > 0$, there exists $K(p,k) > 0$ such that

$$\|A^k e^{-tA}\varphi\|_{L^p(\Omega)} \leq K(p,k)t^{-k}\|\varphi\|_{L^p(\Omega)} \text{ for all } t > 0 \text{ and each } \varphi \in L^p(\Omega). \tag{26}$$

These properties allow us to turn the result from Lemma 2 into an estimate for U_1 which entails some uniform regularity property beyond mere integrability.

Lemma 3. *Let $\mu > \frac{n(\chi\alpha-\xi\gamma)}{4}$. Then for all $p > 1$ and any $\delta \in (0,\frac{1}{2})$, there exists $C(p,\delta) > 0$ such that if $(u_{10},\ u_{20},\ u_{30})$ satisfy (5) and (u_1,u_2,u_3) is the solution of (1), then for $U_1 = u_1 - 1$ we have*

$$\limsup_{t\to\infty}\|A^\delta U_1(.,t)\|_{L^\infty(\Omega)} \leq C(p,\delta)J(J+1), \tag{27}$$

where J is mentioned in Lemma 1.

Proof. From Lemma 1, we have

$$\limsup_{t\to\infty}\|u_1(.,t)\|_{L^\infty(\Omega)} \leq J,$$

whereas Lemma 2 yields $C(p)J > 0$ such that

$$\limsup_{t\to\infty}\|\nabla u_2(.,t)\|_{L^p(\Omega)} \leq C(p)J,$$

$$\limsup_{t\to\infty}\|\nabla u_3(.,t)\|_{L^p(\Omega)} \leq C(p)J.$$

We can thus fix $t_0 = t_0(u,v) > 0$ such that

$$\|u_1(.,t)\|_{L^\infty(\Omega)} \leq 2J, \quad \forall\, t \geq t_0, \tag{28}$$

$$\|\nabla u_2(.,t)\|_{L^p(\Omega)} \leq 2C(p)J, \quad \forall\, t \geq t_0, \tag{29}$$

and

$$|\nabla u_3(.,t)\|_{L^p(\Omega)} \leq 2C(p)J, \quad \forall\, t \geq t_0. \tag{30}$$

Next, according to standard estimates for the Neumann heat semigroup, we can fine $C(p) > 0$ such that

$$\|e^{\tau\triangle}\nabla\cdot\varphi\| \leq c_2(p)(1+\tau^{\frac{1}{2}})\|\varphi\|_{L^p(\Omega)},$$

$$\text{for all } \tau > 0 \text{ and any } \varphi \in C^1(\bar{\Omega};\, \mathbb{R}^n) \text{ with } \varphi\nu = 0 \text{ on } \partial\Omega. \tag{31}$$

Now using (9) and recalling that $\nabla U_2 \equiv \nabla u_2$, $\nabla U_3 \equiv \nabla u_3$ and that $A = \triangle + \lambda$, we represent U_1 according to

$$\begin{aligned}
U(.,t) =& e^{(t-t_0)\triangle}U_1(.,t) - \chi \int_{t_0}^{t} e^{(t-s)\triangle}\nabla(u_1(.,s)\nabla u_2(.,s))ds + \xi \int_{t_0}^{t} e^{(t-s)\triangle}\nabla(u_1(.,s)\nabla u_2(.,s))ds \\
& - \mu \int_{t_0}^{t} e^{(t-s)\triangle}U_1(.,t)ds - \mu \int_{t_0}^{t} e^{(t-s)(\triangle-1)}U_1^2(.,s)ds \\
=& e^{-(-\lambda)(t-t_0)}e^{-(t-t_0)A}U_1(.,t_0) - \chi \int_{t_0}^{t} e^{-(-\lambda)(t-s)}e^{-(t-s)A}\nabla(u_1(.,s)\nabla u_2(.,s))ds \\
& + \xi \int_{t_0}^{t} e^{-(-\lambda)(t-s)}e^{-(t-s)A}\nabla(u_1(.,s)\nabla u_3(.,s))ds - \mu \int_{t_0}^{t} e^{-(-\lambda)(t-s)}e^{-(t-s)A}U_1(.,s)ds \\
& - \mu \int_{t_0}^{t} e^{-(-\lambda)(t-s)}e^{-(t-s)A}U_1^2(.,s)ds, \quad \forall\, t \geq t_0,
\end{aligned}$$

and thus we can estimate

$$\begin{aligned}
\|A^\delta U(\cdot,t)\|_{L^p(\Omega)} =& e^{-(-\lambda)(t-t_0)}\|A^\delta e^{-(t-t_0)A}U(\cdot,t_0)\|_{L^p(\Omega)} \\
& + \chi \int_{t_0}^{t} e^{-(-\lambda)(t-t_0)}\|A^\delta e^{-(t-s)A}\nabla(u_1(.,s)\nabla u_2(\cdot,s))\|_{L^p(\Omega)}ds \\
& + \xi \int_{t_0}^{t} e^{-(-\lambda)(t-t_0)}\|A^\delta e^{-(t-s)A}\nabla(u_1(.,s)\nabla u_3(\cdot,s))\|_{L^p(\Omega)}ds \\
& + \mu \int_{t_0}^{t} e^{-(-\lambda)(t-s)}\|A^\delta e^{-(t-s)A}U_1(\cdot,s)\|_{L^p(\Omega)}ds \\
& + \mu \int_{t_0}^{t} e^{-(-\lambda)(t-s)}\|A^\delta e^{-(t-s)A}U_1^2(\cdot,s)\|_{L^p(\Omega)}ds \\
=& I_3 + I_4 + I_5 + I_6 + I_7, \quad \forall\, t \geq t_0.
\end{aligned} \tag{32}$$

By (26),

$$\begin{aligned}
I_3 :=& e^{-(-\lambda)(t-t_0)}\|A^\delta e^{-(t-t_0)A}U_1(.,t_0)\|_{L^p(\Omega)} \\
\leq& K(p,\gamma)(t-t_0)^{-\delta}e^{-(-\lambda)(t-t_0)}\|U_1(.,t)\|_{L^p(\Omega)} \to 0 \ \text{ as } \ t \to \infty,
\end{aligned} \tag{33}$$

while (26), (28) and the Hölder inequality, we get

$$\begin{aligned}
I_7 :=& \mu \int_0^t (e)^{-(-\lambda)(t-s)}\|A^\delta e^{-(t-s)A}U_1^2(.,s)\|_{L^p(\Omega)}ds \\
\leq& \mu K(p,\delta) \int_{t_0}^{t} (t-s)^{-\delta}e^{-(-\lambda)(t-s)}\|U_1^2(.,s)\|_{L^p(\Omega)}ds \\
\leq& 4\mu J^2 K(p,\delta)|\Omega|^{\frac{1}{2}} \int_{t_0}^{t} (t-s)^{-\gamma}e^{-(1-\lambda)(t-s)}ds \\
=& 4\mu J^2 K(p,\delta)|\Omega|^{\frac{1}{p}} \int_0^\infty \sigma^{-\delta}e^{-(1-\lambda)\sigma}d\sigma = c_3(p,\delta)J^2, \quad \forall\, t \geq t_0\,,
\end{aligned} \tag{34}$$

with $c_3(p,\gamma) := 4\mu K(p,\delta)|\Omega|^{\frac{1}{p}} \int_0^\infty \sigma^{-\delta}e^{-(1-\lambda)\sigma}d\sigma$.

$$\begin{aligned}
I_6 :=& \mu K(p,\delta) \int_{t_0}^{t} e^{-(-\lambda)(t-s)}\|U_1(.,s)\|_{L^p(\Omega)}ds \\
\leq& 2\mu K(p,\delta) \int_{t_0}^{t} e^{-(\lambda)(t-s)}(t-s)^{-\delta}\|U_1(.,s)\|_{L^p(\Omega)}ds \\
=& 2JK(p,\delta)|\Omega|^{\frac{1}{p}} \int_0^t \sigma^{-\delta}e^{-(-\lambda)\sigma}d\sigma \\
\leq& c_4(p,\delta)J, \quad \forall\, t \geq t_0.
\end{aligned} \tag{35}$$

Employing (26), (28)and (30), we furthermore obtain that

$$\begin{aligned}
I_4 \leq & \chi K(p,\beta)\int_{t_0}^{t} e^{-(1-\frac{\lambda}{2})(t-s)}\left(\frac{t-s}{2}\right)^{-\delta}\|e^{\frac{t-s}{2}\Delta}\nabla\cdot(u_1(\cdot,s)\nabla u_2(\cdot,s))\|_{L^p(\Omega)}ds\\
\leq & \chi K(p,\beta)\int_{t_0}^{t} e^{-(1-\frac{\lambda}{2})(t-s)}\left(\frac{t-s}{2}\right)^{-\delta}c_2(p)\left[1+\left(\frac{t-s}{2}\right)^{-\frac{1}{2}}\right]\|u_1(\cdot,s)\nabla u_2(\cdot,s)\|_{L^p(\Omega)}ds\\
\leq & \chi K(p,\beta)c_2(p)\int_{t_0}^{t}\left(\frac{t-s}{2}\right)^{-\delta}\left[1+\left(\frac{t-s}{2}\right)^{-\frac{1}{2}}\right]e^{-(1-\frac{\lambda}{2})(t-s)}\|u_1(\cdot,s)\|_{L^p(\Omega)}\|\nabla u_2(\cdot,s)\|_{L^p(\Omega)}ds\\
\leq & \chi K(p,\beta)c_2(p)\int_{t_0}^{t}\left(\frac{t-s}{2}\right)^{-\delta}\left[1+\left(\frac{t-s}{2}\right)^{-\frac{1}{2}}\right]e^{-(1-\frac{\lambda}{2})(t-s)}\|u_1(\cdot,s)\|_{L^\infty(\Omega)}\|\nabla u_2(\cdot,s)\|_{L^p(\Omega)}ds\\
\leq & \chi K(p,\beta)c_2(p)\int_{t_0}^{t}\left(\frac{t-s}{2}\right)^{-\delta}\left[1+\left(\frac{t-s}{2}\right)^{-\frac{1}{2}}\right]e^{-(1-\frac{\lambda}{2})(t-s)}2J\cdot 2C_1(p)Jds\\
= & 4\chi K(p,\beta)c_2(p)C_1(p)J^2\int_0^{t-t_0}\left(\frac{\sigma}{2}\right)^{-\delta}\left[1+\left(\frac{\sigma}{2}\right)^{-\frac{1}{2}}\right]e^{-(1-\frac{\lambda}{2})\sigma}d\sigma\\
= & 2^{2+\delta}\chi K(p,\beta)c_2(p)C_1(p)J^2\int_0^{t-t_0}\sigma^{-\delta}\left[1+\left(\frac{\sigma}{2}\right)^{-\frac{1}{2}}\right]e^{-(1-\frac{\lambda}{2})\sigma}d\sigma\\
= & 2^{2+\delta}\chi K(p,\beta)c_2(p)C_1(p)J^2\int_0^{t-t_0}\left(\sigma^{-\delta}+\sqrt{2}\sigma^{-(\delta+\frac{1}{2})}\right)e^{-(1-\frac{\lambda}{2})\sigma}d\sigma\\
\leq & C_3(p,\beta)J^2, \quad \forall\, t>t_0.
\end{aligned} \tag{36}$$

Here we used $\delta \in (0,\frac{1}{2})$ and

$$C_3(p,\beta)=2^{2+\delta}\chi K(p,\beta)c_2(p)C_1(p)\int_0^{\infty}\left(\sigma^{-\delta}+\sqrt{2}\sigma^{-(\delta+\frac{1}{2})}\right)e^{-(1-\frac{\lambda}{2})\sigma}d\sigma.$$

By applying (26), (28) and (30), we have

$$\begin{aligned}
I_5 \leq & \xi K(p,\beta)\int_{t_0}^{t} e^{-(1-\frac{\lambda}{2})(t-s)}\left(\frac{t-s}{2}\right)^{-\delta}\|e^{\frac{t-s}{2}\Delta}\nabla\cdot(u_1(\cdot,s)\nabla u_3(\cdot,s))\|_{L^p(\Omega)}ds\\
\leq & \xi K(p,\beta)\int_{t_0}^{t} e^{-(1-\frac{\lambda}{2})(t-s)}\left(\frac{t-s}{2}\right)^{-\delta}c_2(p)\left[1+\left(\frac{t-s}{2}\right)^{-\frac{1}{2}}\right]\|u_1(\cdot,s)\nabla u_3(\cdot,s)\|_{L^p(\Omega)}ds\\
\leq & \xi K(p,\beta)c_2(p)\int_{t_0}^{t}\left(\frac{t-s}{2}\right)^{-\delta}\left[1+\left(\frac{t-s}{2}\right)^{-\frac{1}{2}}\right]e^{-(1-\frac{\lambda}{2})(t-s)}\|u_1(\cdot,s)\|_{L^p(\Omega)}\|\nabla u_2(\cdot,s)\|_{L^p(\Omega)}ds\\
\leq & \xi K(p,\beta)c_2(p)\int_{t_0}^{t}\left(\frac{t-s}{2}\right)^{-\delta}\left[1+\left(\frac{t-s}{2}\right)^{-\frac{1}{2}}\right]e^{-(1-\frac{\lambda}{2})(t-s)}\|u_1(\cdot,s)\|_{L^\infty(\Omega)}\|\nabla u_3(\cdot,s)\|_{L^p(\Omega)}ds\\
\leq & \xi K(p,\beta)c_2(p)\int_{t_0}^{t}\left(\frac{t-s}{2}\right)^{-\delta}\left[1+\left(\frac{t-s}{2}\right)^{-\frac{1}{2}}\right]e^{-(1-\frac{\lambda}{2})(t-s)}2J\cdot 2C_1(p)Jds\\
= & 4\xi K(p,\beta)c_2(p)C_1(p)J^2\int_0^{t-t_0}\left(\frac{\sigma}{2}\right)^{-\delta}\left[1+\left(\frac{\sigma}{2}\right)^{-\frac{1}{2}}\right]e^{-(1-\frac{\lambda}{2})\sigma}d\sigma\\
= & 2^{2+\delta}\xi K(p,\beta)c_2(p)C_1(p)J^2\int_0^{t-t_0}\sigma^{-\delta}\left[1+\left(\frac{\sigma}{2}\right)^{-\frac{1}{2}}\right]e^{-(1-\frac{\lambda}{2})\sigma}d\sigma\\
= & 2^{2+\delta}\xi K(p,\beta)c_2(p)C_1(p)J^2\int_0^{t-t_0}\left(\sigma^{-\delta}+\sqrt{2}\sigma^{-(\delta+\frac{1}{2})}\right)e^{-(1-\frac{\lambda}{2})\sigma}d\sigma\\
\leq & C_3(p,\beta)J^2, \quad \forall\, t>t_0.
\end{aligned} \tag{37}$$

Here we used $\delta \in (0, \frac{1}{2})$ and

$$C_3(p,\beta) = 2^{2+\delta}\xi K(p,\beta)c_2(p)C_1(p)\int_0^\infty \left(\sigma^{-\delta} + \sqrt{2}\sigma^{-(\delta+\frac{1}{2})}\right)e^{-(1-\frac{\lambda}{2})\sigma}d\sigma.$$

By substituting (33)-(37) in (32) we get

$$\limsup_{t\to\infty} \|A^\beta U_1(\cdot,t)\|_{L^p(\Omega)} \leq 3C_3(p,\beta)J^2 + C_4(p,\beta)J \leq C_5(p,\beta)J(J+1),$$

as $C(p,\beta) = C_5(p,\beta)$ and the proof is completes. □

5. A pointwise estimate for Δu_2 and Δu_3

We now choose p suitably large and δ sufficiently close to $\frac{1}{2}$ to establish, the following pointwise estimate for Δu_2 and Δu_3, using another parabolic regularization argument.

Lemma 4. *Let $\mu > \frac{n(\chi\alpha-\xi\gamma)}{4}$. Then there exists $\bar{C} > 0$, such that for any choice of u_{10},u_{20} and u_{30} fulfilling (5), the corresponding solution of (1) satisfies*

$$\limsup_{t\to\infty} \|\Delta u_2(\cdot,t)\|_{L^\infty(\Omega)} \leq \bar{C}J(J+1) \tag{38}$$

and

$$\limsup_{t\to\infty} \|\Delta u_3(\cdot,t)\|_{L^\infty(\Omega)} \leq \bar{C}J(J+1), \tag{39}$$

where J is mentioned in Lemma 1.

Proof. We can choose positive number ρ and p, and then fix an arbitrary $\rho \in (1, \frac{3}{2})$ such that

$$\rho - 1 < \delta < \frac{1}{2}, \tag{40}$$

and

$$p > \frac{n}{2(\rho-1)}. \tag{41}$$

In particular, $2\rho - \frac{n}{p} > 2\rho - 2(\rho-1) = 2$. By (25), there exists $\tilde{c}_1 > 0$ such that

$$\|\varphi\|_{W^{2,\infty}} \leq \tilde{c}_1\|A^\rho\varphi\|_{L^p(\Omega)}, \quad \forall\, \varphi \in D(A^\rho), \tag{42}$$

where $A = A_p$. Moreover, the right inequality in (40) allows us to infer from Lemma 3 that for any such solution, with (U_1, U_2, U_3) as in (8), we get

$$\limsup_{t\to\infty} \|A^\delta U_1(\cdot,t)\|_{L^\infty(\Omega)} \leq \tilde{c}_2 J(J+1),$$

with some $\tilde{c}_2 > 0$ depending on p and δ only, so that we can fix $t_0 = t_0(u_1,u_2,u_3)$ such that

$$\|A^\delta U_1(\cdot,t)\|_{L^\infty(\Omega)} \leq 2\tilde{c}_2 J(J+1), \quad \forall\, t \geq t_0. \tag{43}$$

Now according to a variation-of-constants formula associated with the second equation in (9), we can write

$$\begin{aligned} U_2(\cdot,t) &= \beta e^{(t-t_0)(\triangle-1)}U_2(\cdot,t_0) + \alpha\int_{t_0}^t e^{(t-s)(\triangle-1)}U_1(\cdot,s)ds \\ &= e^{-(1-\lambda)(t-t_0)}e^{-(t-t_0)A}U_2(\cdot,t_0) + \alpha\int_{t_0}^t e^{-(1-\lambda)(t-s)}e^{-(t-s)A}U_1(\cdot,s)ds \end{aligned}$$

for all $t > t_0$, and hence use (42) to estimate

$$\begin{aligned} \|U_2(\cdot,t)\|_{W^{2,\infty}(\Omega)} &\leq \tilde{c}_1 \|A^\kappa U_2(\cdot,t)\|_{L^p(\Omega)} \\ &\leq \beta \tilde{c}_1 e^{-(1-\lambda)(t-t_0)} \|A^\kappa e^{-(t-t_0)A} U_2(\cdot,t_0)\|_{L^p(\Omega)} \\ &\quad + \alpha \tilde{c}_1 \int_{t_0}^t e^{-(1-\lambda)(t-s)} \|A^\kappa e^{-(t-s)A} U_1(\cdot,s)\|_{L^p(\Omega)} ds, \ \ \forall\, t > t_0. \end{aligned} \tag{44}$$

By (26), we have

$$\begin{aligned} &\beta \tilde{c_1} e^{-(1-\lambda)(t-t_0)} \|A^\kappa e^{-(t-t_0)A} U_2(.,t_0)\|_{L^p(\Omega)} ds \\ &\leq \beta \tilde{c_1} e^{-(1-\lambda)(t-t_0)} \|A^\kappa e^{-(t-s)A} U_2(.,t_0)\|_{L^p(\Omega)} ds \to 0 \ \text{ as } t \to \infty, \end{aligned} \tag{45}$$

whereas (26) combined with (43) shows that

$$\begin{aligned} &\alpha \tilde{c}_1 \int_{t_0}^t e^{-(1-\lambda)(t-s)} \|A^\kappa e^{-(t-s)A} U_1(\cdot,s)\|_{L^p(\Omega)} ds \\ &= \alpha \tilde{c}_1 \int_{t_0}^t e^{-(1-\lambda)(t-s)} \|A^{\kappa-\delta} e^{(t-s)A} A^\delta U_1(\cdot,s)\|_{L^p(\Omega)} ds \\ &\leq \alpha \tilde{c}_1 K_1(p,\kappa-\delta) \int_{t_0}^t (t-s)^{-(\kappa-\delta)} e^{-(1-\lambda)(t-s)} \|A^\delta U_1(\cdot,s)\|_{L^p(\Omega)} ds \\ &\leq \alpha \tilde{c}_1 K_1(p,\kappa-\delta) 2\tilde{c}_2 J(J+1) \int_{t_0}^t (t-s)^{-(\kappa-\delta)} e^{-(1-\lambda)(t-s)} ds \\ &\leq \alpha \tilde{c}_1 K_1(p,\kappa-\delta) 2\tilde{c}_2 J(J+1) \int_0^\infty \sigma^{-(\kappa-\gamma)} e^{-(1-\lambda)\sigma} d\sigma \\ &\leq \bar{C} J(J+1), \qquad \forall\ t > t_0, \end{aligned} \tag{46}$$

with

$$\bar{C}_1 := 2\alpha \tilde{c}_1 \tilde{c}_2 \alpha K_1(p,\kappa-\delta) \int_0^\infty \sigma^{-(\delta-\kappa)} e^{-(1-\lambda)\sigma} d\sigma$$

being finite, because $\kappa + \lambda < 1$ by (40). Together with (45) and (46) this proves

$$\limsup_{t\to\infty} \|\Delta u_2(\cdot,t)\|_{L^\infty(\Omega)} \leq \bar{C} J(J+1).$$

Also according to a variation-of-constants formula associated with the third equation in (9), we can write

$$\begin{aligned} U_3(\cdot,t) &= \eta e^{(t-t_0)(\triangle-1)} U_3(.,t_0) + \gamma \int_{t_0}^t e^{(t-s)(\triangle-1)} U_1(\cdot,s) ds \\ &= \eta e^{-(1-\lambda)(t-t_0)} e^{-(t-t_0)A} U_3(.,t_0) + \gamma \int_{t_0}^t e^{-(1-\lambda)(t-s)} e^{-(t-s)A} U_1(.,s) ds, \quad \forall\, t > t_0, \end{aligned}$$

and hence use (42) to estimate

$$\begin{aligned} \|U_3(\cdot,t)\|_{W^{2,\infty}(\Omega)} &\leq \tilde{c}_1 \|A^\kappa U_3(\cdot,t)\|_{L^p(\Omega)} \\ &\leq \eta \tilde{c}_1 e^{-(1-\lambda)(t-t_0)} \|A^\kappa e^{-(t-t_0)A} U_3(\cdot,t_0)\|_{L^p(\Omega)} \\ &\quad + \gamma \tilde{c}_1 \int_{t_0}^t e^{-(1-\lambda)(t-s)} \|A^\kappa e^{-(t-s)A} U_1(\cdot,s)\|_{L^p(\Omega)} ds, \ \forall\, t > t_0. \end{aligned} \tag{47}$$

By (26), we have

$$\begin{aligned} &\eta \tilde{c}_1 e^{-(1-\lambda)(t-t_0)} \|A^\kappa e^{-(t-t_0)A} U_3(\cdot,t_0)\|_{L^p(\Omega)} ds \\ &\leq \eta \tilde{c}_1 e^{-(1-\lambda)(t-t_0)} \|A^\kappa e^{-(t-s)A} U_3(\cdot,t_0)\|_{L^p(\Omega)} ds \ \to 0 \ \ as\ t \to \infty, \end{aligned} \tag{48}$$

whereas (26) combined with (43) shows that

$$
\begin{aligned}
\gamma\tilde{c}_1 \int_{t_0}^{t} & e^{-(1-\lambda)(t-s)} \|A^{\kappa} e^{-(t-s)A} U_1(\cdot,s)\|_{L^p(\Omega)} ds \\
&= \gamma\tilde{c}_1 \int_{t_0}^{t} e^{-(1-\lambda)(t-s)} \|A^{\kappa-\delta} e^{(t-s)A} A^{\delta} U_1(\cdot,s)\|_{L^p(\Omega)} ds \\
&\leq \gamma\tilde{c}_1 K_1(p,\kappa-\delta) \int_{t_0}^{t} (t-s)^{-(\kappa-\delta)} e^{-(1-\lambda)(t-s)} \|A^{\delta} U_1(\cdot,s)\|_{L^p(\Omega)} ds \\
&\leq \gamma\tilde{c}_1 K_1(p,\kappa-\delta) 2\tilde{c}_2 J(J+1) \int_{t_0}^{t} (t-s)^{-(\kappa-\delta)} e^{-(1-\lambda)(t-s)} ds \\
&\leq \gamma\tilde{c}_1 K_1(p,\kappa-\delta) 2\tilde{c}_2 J(J+1) \int_{0}^{\infty} \sigma^{-(\kappa-\gamma)} e^{-(1-\lambda)\sigma} d\sigma \\
&\leq \bar{C} J(J+1), \quad \forall\, t > t_0
\end{aligned} \tag{49}
$$

with

$$\bar{C}_1 := 2\gamma\tilde{c}_1\tilde{c}_2\alpha K_1(p,\kappa-\delta) \int_0^{\infty} \sigma^{-(\delta-\kappa)} e^{-(1-\lambda)\sigma} d\sigma$$

being finite, because $\kappa + \lambda < 1$ by (40). By (45) and (46),

$$\limsup_{t\to\infty} \|\Delta u_3(\cdot,t)\|_{L^\infty(\Omega)} \leq \bar{C} J(J+1)$$

and thereby completes the proof. □

6. Refined pointwise inequalities for u_1

Lemma 5. *Suppose that $\mu > \frac{n(\chi\alpha-\xi\gamma)}{4}$ and (5) hold. Then there exists $C_1 > 0$ with the property that any solution (u_1, u_2, u_3) of (1) satisfies*

$$\liminf_{t\to\infty} \big(\inf_{x\in\Omega} u_1(x,t)\big) \geq 1 + C_1 J(J+1)\,, \tag{50}$$

with J is mentioned in Lemma 1 and $C_1 = \frac{2c_2(\xi-\chi)}{\mu}$.

Proof. In accordance with Lemma 4, we fix $c_1 > 0$ such that for any solution we have

$$\limsup_{t\to\infty} \|\Delta u_2(\cdot,t)\|_{L^\infty(\Omega)} \leq C_1 J(J+1),$$

$$\limsup_{t\to\infty} \|\Delta u_3(\cdot,t)\|_{L^\infty(\Omega)} \leq C_1 J(J+1),$$

whence we can pick $t_0 = t_0(u_1,u_2,u_3) > 0$ such that

$$\|\Delta u_2(\cdot,t)\|_{L^\infty(\Omega)} \leq 2c_1 J(J+1), \;\; \forall\, t \geq t_0, \tag{51}$$

$$\|\Delta u_3(\cdot,t)\|_{L^\infty(\Omega)} \leq 2c_1 J(J+1), \;\; \forall\, t \geq t_0. \tag{52}$$

Therefore, in the first equation in (1) we can estimate

$$
\begin{aligned}
u_{1t} &= \Delta u_1 - \chi \nabla u_1 \nabla u_2 - \chi u_1 \Delta u_2 + \xi \nabla u_1 \nabla u_3 + \xi u_1 \Delta u_3 - \mu u_1^2 + \mu u_1 \\
&\geq \Delta u_1 - \chi \nabla u_1 \nabla u_2 + \xi \nabla u_1 \nabla u_3 - 2\chi c_1 J(J+1) u_1 + 2\xi c_1 J(J+1) u_1 - \mu u_1^2 + \mu u_1\,,
\end{aligned}
$$

for all $x \in \Omega$, $t > t_0$. Thus, if we let $y \in C^1([t_0,\infty))$ denote the solution of

$$
\begin{cases}
y'(t) = \ell \cdot y(t) - \mu y^2(t), \;\; t > t_0 \\
y(t_0) = \tilde{c}_3 := \inf_{x\in\Omega} u_1(x,t_0),
\end{cases} \tag{53}
$$

where $\ell := \mu + 2c_1 J(J+1)(\xi - \chi)$, then the comparison principle asserts that

$$u_1(x,t) \geq y(t), \quad \forall\, x \in \Omega \text{ and } t \geq t_0. \tag{54}$$

As u_1 is strictly positive in $\bar{\Omega} \times (0, \infty)$ by the strong maximum principle, $\tilde{c}_3$ must be positive, so that e.g. explicitly solving the Bernoulli-type initial-value problem (53) shows that

$$y(t) \to \frac{(\ell)_+}{\mu} \quad as\ t \to \infty.$$

Therefore,

$$\liminf_{t\to\infty} (\inf_{x\in\Omega} u_1(x,t)) \geq \liminf_{t\to\infty} y(t) \geq \frac{\mu + 2c_1 J(J+1)(\xi-\chi)}{\mu},$$

we put $C_1 = \frac{2c_2(\xi-\chi)}{\mu}$, this compleat the prove. □

Corollary 1. *Let $\mu > \frac{n(\chi\alpha - \xi\gamma)}{4}$. Assume (5) holds. Then one can find $C \geq 0$ with the property that if (u_1, u_2, u_3) is the solution of (1), we have*

$$\limsup_{t\to\infty} \|U_1(\cdot,t)\|_{L^\infty(\Omega)} \leq CJ(J+1), \tag{55}$$

where $U_1 = u_1 - 1$ and J is mentioned in Lemma 1.

Proof. By Lemma 5, there exists $c_1 \geq 1$ such that

$$\limsup_{t\to\infty} \|U_{-1}(\cdot,t)\|_{L^\infty(\Omega)} \leq c_1 J(J+1)$$

On the other hand, Lemma 1 says that

$$\limsup_{t\to\infty} \|U_{+1}(\cdot,t)\|_{L^\infty(\Omega)} \leq c_1 J(J+1).$$

Thus, (55) holds if we let $C = c_1$. □

7. Exponential decay: Proof of main result

In this section we prove that U_1 actually must converge to zero, uniformly with respect to $x \in \Omega$, at an exponential rate.

Lemma 6. *Let $\epsilon \in (0,1)$, and (5) hold. Then there exists $\theta_0 = \theta_0(\epsilon) \in (0,1)$ such that if $\theta = \frac{n(\chi\alpha-\xi\gamma)}{4\mu} \leq \theta_0$, for each solution of (1), one can find $C > 0$ such that*

$$\|U_1(.,t)\|_{L^\infty(\Omega)} \leq Ce^{-\epsilon t}, \quad \forall\, t > 0. \tag{56}$$

Proof. We fix an arbitrary $p > n$ and then recall known smoothing estimates for $(e^{\tau\Delta})_{\tau\geq 0}$, which in conjunction with the Poincaré inequality yield positive constants c_1, c_2 and c_3 such that

$$\|\nabla e^{\tau\Delta}\varphi\|_{L^p(\Omega)} \leq c_1 \|\nabla\varphi\|_{L^p(\Omega)} \text{ for all } \tau > 0 \text{ and any } \varphi \in W^{1,p}(\Omega), \tag{57}$$

and

$$\|\nabla e^{\tau\Delta}\varphi\|_{L^p(\Omega)} \leq c_2(1 + \tau^{-\frac{1}{2}})\|\varphi\|_{L^\infty(\Omega)} \text{ for all } \tau > 0 \text{ and each } \varphi \in L^\infty(\Omega), \tag{58}$$

as well as

$$\|e^{\tau\Delta}\nabla\cdot\varphi\|_{L^\infty(\Omega)} \leq c_3(1 + \tau^{-\frac{1}{2}-\frac{n}{2p}})\|\varphi\|_{L^p(\Omega)}, \tag{59}$$

for all $\tau > 0$ and all $\varphi \in C^1(\bar{\Omega}, \mathbb{R}^n)$ fulfilling $\varphi \cdot \nu = 0$ on $\partial\Omega$ [21,25,26]. We furthermore note that since $\delta < 1$ and $\frac{1}{2} + \frac{n}{2p} < 1$, the number

$$c_4 := \int_0^\infty (1 + \sigma^{-\frac{1}{2}})e^{-(1-\epsilon)\sigma} d\sigma,$$

and

$$c_5 := \int_0^\infty (1+\sigma^{-\frac{1}{2}-\frac{n}{2p}})e^{\epsilon\sigma}d\sigma\,,$$

are finite. Next, applying Lemma 1, Lemma 2 and Corollary 1 we obtain $c_6 > 0$ and $c_7 > 0$ such that whenever $\frac{n(\chi\alpha-\xi\gamma)}{4\mu} < 1$ and (u_{10}, u_{20}, u_{30}) satisfies (5), then with $\theta = \frac{n(\chi\alpha-\xi\gamma)}{4\mu}$, we have

$$\limsup_{t\to\infty} \|u_1(.,t)\|_{L^\infty(\Omega)} \le c_6 J$$

and

$$\limsup_{t\to\infty} \|\nabla u_2(\cdot,t)\|_{L^p(\Omega)} \le c_7 J,$$

$$\limsup_{t\to\infty} \|\nabla u_3(\cdot,t)\|_{L^p(\Omega)} \le c_7 J$$

as well as

$$\limsup_{t\to\infty} \|U_1(\cdot,t)\|_{L^\infty(\Omega)} \le c_8 J(J+1),$$

whence

$$\|u_1(\cdot,t)\|_{L^\infty(\Omega)} \le 2c_6 J, \ \ \forall\, t \ge t_0, \tag{60}$$

$$\|\nabla u_2(\cdot,t)\|_{L^p(\Omega)} \le 2c_7 J, \ \ \forall\, t \ge t_0, \tag{61}$$

$$\|\nabla u_3(\cdot,t)\|_{L^p(\Omega)} \le 2c_7 J, \ \ \forall\, t \ge t_0, \tag{62}$$

as well as

$$\|U_1(\cdot,t)\|_{L^\infty(\Omega)} \le 2c_8 J(J+1), \ \ \forall\, t \ge \ t_0 \tag{63}$$

with some suitably large $t_0 = t_0(u_1, u_2, u_3) > 0$.

We now fix $\theta_0 \in (0,1)$ small enough such that

$$2\chi\alpha c_2 c_3 c_4 c_5 c_6 J|\Omega|^{\frac{1}{p}} \le \frac{1}{20}, \tag{64}$$

$$2\xi\gamma c_2 c_3 c_4 c_5 c_6 J|\Omega|^{\frac{1}{p}} \le \frac{1}{10}, \tag{65}$$

$$\frac{\mu}{1-\epsilon} \le \frac{1}{10}, \tag{66}$$

$$\frac{2\mu c_8 J(J+1)}{1-\epsilon} \le \frac{1}{10}, \tag{67}$$

and henceforth suppose that $\mu > 0$, $\chi > 0$ and $\xi > 0$ are fixed numbers such that $\theta = \frac{n(\chi\alpha-\xi\gamma)}{4\mu}$ satisfies $\theta < \theta_0$

We then choose a large number $M > 0$ fulfilling

$$4c_8 J(J+1) \le \frac{M}{10}, \tag{68}$$

$$4\chi\beta c_2 c_3 c_5 c_6 c_7 J^2 \le \frac{M}{20}, \tag{69}$$

$$4\xi\eta c_2 c_3 c_5 c_6 c_7 J^2 \le \frac{M}{20}, \tag{70}$$

and let (u_1, u_2, u_3) solve (1) with some (u_{10}, u_{20}, u_{30}) satisfying (5). Then with $t_0 := t_0(u_1, u_2, u_3)$ as introduced above and (U_1, U_2, U_3) as in (8), we consider the set

$$S := \{T_0 \ge t_0 | \|U_1(\cdot,t)\|_{L^\infty(\Omega)} \le Me^{-\epsilon(t-t_0)}, \ \ \forall\, t \in [t_0, T_0]\}$$

and note that S is not empty, because (63) and (68) imply that $\|U_1(.,t)\|_{L^\infty(\Omega)} \leq \frac{M}{10}$. In particular, $T := \sup S \in (t_0, \infty]$ is well-defined, and in order to prove the lemma it is sufficient to make sure that actually

$$T = \infty. \tag{71}$$

To verify this, we first use (9) to represent ∇U_2 and ∇U_3 according to

$$\nabla U_2(.,t) = \beta \nabla e^{(t-t_0)(\triangle-1)} U_2(.,t_0) + \alpha \int_{t_0}^{t} \nabla e^{(t-s)(\triangle-1)} U_1(.,s)ds \equiv I_8 + I_9,\ for\ all\ t > t_0, \tag{72}$$

and use (57), (61) and the fact that $\epsilon < 1$ to estimate

$$\begin{aligned} I_8 &= \beta \|\nabla e^{(t-t_0)(\triangle-1)U_2(.,t_0)}\|_{L^p(\Omega)} \\ &= \beta e^{-(t-t_0)} \|\nabla e^{(t-t_0)\triangle} U_2(.,t_0)\|_{L^p(\Omega)} \\ &\leq \ \beta e^{-(t-t_0)} c_1 \|\nabla U_2(.,t_0)\|_{L^p(\Omega)} \\ &\leq \ 2\beta c_1 c_7 J e^{-(t-t_0)} \\ &\leq \ 2\beta c_1 c_7 J e^{-\epsilon(t-t_0)}, \quad \forall\, t > t_0. \end{aligned}$$

Furthermore, (58) along with the Hölder inequality and the definitions of T and c_4 entails that

$$\begin{aligned} I_9 &= \alpha \| \textstyle\int_{t_0}^{t} \nabla e^{(t-s)(\triangle-1)} U_1(.,s) ds\|_{L^p(\Omega)} \\ &\leq \alpha c_2 \textstyle\int_{t_0}^{t} (1 + (t-s)^{-\frac{1}{2}}) e^{-(t-s)} \|U_1(\cdot,t)\|_{L^p(\Omega)} ds \\ &\leq \alpha c_2 |\Omega|^{\frac{1}{p}} \textstyle\int_{t_0}^{t} (1 + (t-s)^{-\frac{1}{2}}) e^{-(t-s)} \|U_1(.,t)\|_{L^\infty(\Omega)} ds \\ &\leq \alpha c_2 |\Omega|^{\frac{1}{2}} \textstyle\int_{t_0}^{t} (1 + (t-s)^{-\frac{1}{2}}) e^{-(t-s)} M e^{-\epsilon(s-t_0)} \\ &\leq \alpha c_2 |\Omega|^{\frac{1}{p}} M \left(\textstyle\int_{0}^{t-t_0} (1 + \sigma^{\frac{1}{2}}) e^{-(1-\epsilon)\sigma} d\sigma \right) e^{-\epsilon(t-t_0)} \\ &\leq \alpha c_2 c_4 |\Omega|^{\frac{1}{p}} M e^{-\epsilon(t-t_0)}, \quad \forall\, t \in (t_0, T), \end{aligned}$$

whence (72) shows that

$$\|\nabla U_2(\cdot,t)\|_{L^p(\Omega)} \leq \left\{ 2\beta c_1 c_7 J + \alpha c_2 c_4 |\Omega|^{\frac{1}{p}} M \right\} e^{-\epsilon(t-t_0)}, \ \ \forall\, t \in (t_0, T), \tag{73}$$

and from the third equation in (9) we write

$$\nabla U_3(\cdot,t) = \eta \nabla e^{(t-t_0)(\triangle-1)} U_3(\cdot,t_0) + \gamma \textstyle\int_{t_0}^{t} \nabla e^{(t-s)(\triangle-1)} U_1(\cdot,s) ds \equiv I_{10} + I_{11}, \ \ \forall\, t > t_0 \tag{74}$$

and use (57), (62) and the fact that $\epsilon < 1$ to estimate

$$\begin{aligned} I_{10} &= \eta \|\nabla e^{(t-t_0)(\triangle-1)U_3(\cdot,t_0)}\|_{L^p(\Omega)} \\ &= \eta e^{-(t-t_0)} \|\nabla e^{(t-t_0)\triangle} U_3(\cdot,t_0)\|_{L^p(\Omega)} \\ &\leq \ \eta e^{-(t-t_0)} c_1 \|\nabla U_3(\cdot,t_0)\|_{L^p(\Omega)} \\ &\leq \ 2\eta c_1 c_7 J e^{-(t-t_0)} \\ &\leq \ 2\eta c_1 c_7 J e^{-\epsilon(t-t_0)}, \quad \forall\, t > t_0. \end{aligned}$$

Furthermore, (58) along with the Hölder inequality and the definitions of T and c_4 entails that

$$\begin{aligned} I_{11} &= \gamma \| \textstyle\int_{t_0}^{t} \nabla e^{(t-s)(\triangle-1)} U_1(.,s) ds\|_{L^p(\Omega)} \\ &\leq \gamma c_2 \textstyle\int_{t_0}^{t} (1 + (t-s)^{-\frac{1}{2}}) e^{-(t-s)} \|U_1(\cdot,t)\|_{L^p(\Omega)} ds \\ &\leq \gamma c_2 |\Omega|^{\frac{1}{p}} \textstyle\int_{t_0}^{t} (1 + (t-s)^{-\frac{1}{2}}) e^{-(t-s)} \|U_1(.,t)\|_{L^\infty(\Omega)} ds \\ &\leq \gamma c_2 |\Omega|^{\frac{1}{2}} \textstyle\int_{t_0}^{t} (1 + (t-s)^{-\frac{1}{2}}) e^{-(t-s)} M e^{-\epsilon(s-t_0)} \\ &\leq \gamma c_2 |\Omega|^{\frac{1}{p}} M \left(\textstyle\int_{0}^{t-t_0} (1 + \sigma^{\frac{1}{2}}) e^{-(1-\epsilon)\sigma} d\sigma \right) e^{-\epsilon(t-t_0)} \\ &\leq \gamma c_2 c_4 |\Omega|^{\frac{1}{p}} M e^{-\epsilon(t-t_0)}, \quad \forall\, t \in (t_0, T). \end{aligned}$$

Whence (72) shows that

$$\|\nabla U_3(\cdot,t)\|_{L^p(\Omega)} \le \left\{2\eta c_1 c_7 J + \gamma c_2 c_4 |\Omega|^{\frac{1}{p}} M\right\} e^{-\epsilon(t-t_0)}, \quad \forall\, t \in (t_0,T). \tag{75}$$

We next write

$$U_1(\cdot,t) = e^{(t-t_0)(\Delta-1)}U_1(\cdot,t_0) - \chi \int_{t_0}^t e^{(t-s)} e^{(t-s)(\Delta-1)} \nabla.(u_1(\cdot,s)\nabla U_2)ds \\ + \xi \int_{t_0}^t e^{(t-s)} e^{(t-s)(\Delta-1)} \nabla.(u_1(.,s)\nabla U_3)ds - \mu \int_{t_0}^t e^{(t-s)(\Delta-1)} U_1(\cdot,s)ds - \mu \int_{t_0}^t e^{(t-s)(\Delta-1)} U_1^2(\cdot,s)ds,$$

and thus obtain that

$$\begin{aligned}\|U_1(\cdot,t)\|_{L^\infty(\Omega)} =& e^{-(t-t_0)}\|e^{(t-t_0)\Delta}U_1(.,t_0)\|_{L^\infty(\Omega)} + \chi \int_{t_0}^t e^{-(t-s)} \|e^{(t-s)\Delta}\nabla.(u_1(.,s)\nabla U_2(.,s))\|_{L^\infty(\Omega)}ds \\ &+ \xi \int_{t_0}^t e^{-(t-s)} \|e^{(t-s)\Delta}\nabla.(u_1(.,s)\nabla U_3(.,s))\|_{L^\infty(\Omega)}ds + \mu \int_{t_0}^t e^{-(t-s)} \|e^{(t-s)\Delta}U_1(.,s)\|_{L^\infty(\Omega)}ds \\ &+ \mu \int_{t_0}^t e^{-(t-s)} \|e^{(t-s)\Delta}U_1^2(.,s)\|_{L^\infty(\Omega)}ds \\ \le& I_{12} + I_{13} + I_{14} + I_{15} + I_{16}, \quad \forall\, t > t_0.\end{aligned} \tag{76}$$

Here the maximum principle together with (63) and (68) ensures that

$$\begin{aligned} I_{12} &= e^{-(t-t_0)}\|e^{(t-t_0)\Delta}U_1(.,t_0)\|_{L^\infty(\Omega)} \\ &\le e^{-(t-t_0)}\|U_1(.,t_0)\|_{L^\infty(\Omega)} \\ &\le e^{-(t-t_0)} 2c_8 J(J+1) \\ &\le 4c_8 J(J+1) e^{-\epsilon(t-t_0)} \\ &\le \frac{M}{10} e^{-\epsilon(t-t_0)}, \quad \forall\, t > t_0, \end{aligned} \tag{77}$$

again because $\epsilon < 1$. We next recall (59) and (60) and employ the estimate (73) to see that

$$\begin{aligned} I_{13} &= \chi \int_{t_0}^t e^{-(t-s)} \|e^{(t-s)\Delta}\nabla.(u_1(.,s)\nabla U_2(\cdot,s))\|_{L^\infty(\Omega)}ds \\ &\le c_3\chi \int_{t_0}^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right) e^{-(t-s)} \|u_1(\cdot,s)\nabla U_2(\cdot,s)\|_{L^p(\Omega)}ds \\ &\le c_3\chi \int_{t_0}^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right) e^{-(t-s)} \|u_1(\cdot,s)\|_{L^\infty(\Omega)}\|\nabla U_2(\cdot,s)\|_{L^p(\Omega)}ds \\ &\le c_3\chi \int_{t_0}^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right) e^{-(t-s)} \cdot 2c_6 J\left\{2\beta c_1 c_7 J + \alpha c_2 c_4 |\Omega|^{\frac{1}{p}} M\right\} e^{-\epsilon(t-t_0)}ds \end{aligned}$$

for all $t \in (t_0,T)$, in view of the definition of c_5, the restrictions (64) and (69) thus imply that

$$\chi \int_{t_0}^t e^{-(t-s)} \|e^{(t-s)\Delta}\nabla.(u_1(.,s)\nabla U_2(.,s))\|_{L^\infty(\Omega)}ds \le 2\chi c_3 c_5 c_6 J\left\{2\beta c_1 c_7 J + \alpha c_2 c_4 |\Omega|^{\frac{1}{p}} M\right\} e^{-\epsilon(t-t_0)}ds \\ \le \left\{4\chi c_1 c_3 c_5 c_6 c_7 \beta J^2 + 2\chi\alpha c_2 c_3 c_4 c_5 c_6 |\Omega|^{\frac{1}{p}} M\right\} e^{-\epsilon(t-t_0)}ds \le \left\{\frac{M}{20} + \frac{M}{20}\right\} e^{-\epsilon(t-t_0)} \le \frac{M}{10} e^{-\epsilon(t-t_0)}, \quad \forall\, t \in (t_0,T), \tag{78}$$

also again because $\epsilon < 1$. We next recall (59) and (60) and employ the estimate (75) to see that

$$\begin{aligned} I_{14} &= \xi \int_{t_0}^t e^{-(t-s)} \|e^{(t-s)\Delta}\nabla.(u_1(.,s)\nabla U_3(.,s))\|_{L^\infty(\Omega)}ds \\ &\le c_3\xi \int_{t_0}^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right) e^{-(t-s)} \|u_1(.,s)\nabla U_3(.,s)\|_{L^p(\Omega)}ds \\ &\le c_3\xi \int_{t_0}^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right) e^{-(t-s)} \|u_1(.,s)\|_{L^\infty(\Omega)}\|\nabla U_3(.,s)\|_{L^p(\Omega)}ds \\ &\le c_3\xi \int_{t_0}^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right) e^{-(t-s)}.2c_6 J\left\{2\eta c_1 c_7 J + \gamma c_2 c_4 |\Omega|^{\frac{1}{p}} M\right\} e^{-\epsilon(t-t_0)}ds \end{aligned}$$

for all $t \in (t_0, T)$, in view of the definition of c_5, the restrictions (65) and (70) thus imply that

$$\xi \int_{t_0}^{t} e^{-(t-s)} \|e^{(t-s)\Delta} \nabla.(u_1(.,s)\nabla U_3(.,s))\|_{L^\infty(\Omega)} ds \leq 2\xi c_3 c_5 c_6 J \left\{ 2\eta c_1 c_7 J + \gamma c_2 c_4 |\Omega|^{\frac{1}{p}} M \right\} e^{-\epsilon(t-t_0)} ds$$
$$\leq \left\{ 4\xi c_1 c_3 c_5 c_6 c_7 \eta J^2 + 2\xi \gamma c_2 c_3 c_4 c_5 c_6 |\Omega|^{\frac{1}{p}} M \right\} e^{-\epsilon(t-t_0)} ds \leq \left\{ \frac{M}{20} + \frac{M}{20} \right\} e^{-\epsilon(t-t_0)} \leq \frac{M}{10} e^{-\epsilon(t-t_0)}, \quad \forall\, t \in (t_0, T). \tag{79}$$

Now we estimate I_{15} with the definition of T to find that

$$\begin{aligned} I_{15} := \mu \int_{t_0}^{t} e^{-(t-s)} \|e^{(t-s)\Delta} U_1(.,s)\|_{L^\infty(\Omega)} ds \\ &\leq \mu \int_{t_0}^{t} e^{-(t-s)} \|U_1(.,s)\|_{L^\infty(\Omega)} ds \\ &\leq \mu \int_{t_0}^{t} e^{-(t-s)} .M e^{-\epsilon(s-t_0)} ds \\ &\leq \mu.M \int_{t_0}^{t} e^{-(1-\epsilon)(t-s)} e^{-\epsilon(s-t_0)} ds \\ &\leq \mu.M \int_{0}^{t-t_0} e^{-(1-\epsilon)(t-s)} e^{-\epsilon(t-t_0)} d\sigma \\ &\leq \mu.M \int_{0}^{\infty} e^{-(1-\epsilon)\sigma} e^{-\epsilon(t-t_0)} d\sigma \\ &\leq \frac{\mu.M}{1-\epsilon} e^{-\epsilon(t-t_0)} \\ &\leq \frac{M}{10} e^{-\epsilon(t-t_0)}, \quad \forall\, t \in (t_0, T). \end{aligned} \tag{80}$$

Finally, to treat the fifth summand on the right of (70), we combine (63)and (67) with the definition of T to find that

$$\begin{aligned} I_{16} := \mu \int_{t_0}^{t} e^{-(t-s)} \|e^{(t-s)\Delta} U_1^2(.,s)\|_{L^\infty(\Omega)} ds \\ &\leq \mu \int_{t_0}^{t} e^{-(t-s)} \|U_1^2(.,s)\|_{L^\infty(\Omega)} ds \\ &\leq \mu \int_{t_0}^{t} e^{-(t-s)} \|U_1(.,s)\|_{L^\infty(\Omega)} \|U_1(.,s)\|_{L^\infty(\Omega)} ds \\ &\leq \mu \int_{t_0}^{t} e^{-(t-s)} 2c_8 J(J+1).M e^{-\epsilon(s-t_0)} ds \\ &\leq 2\mu c_8 J(J+1).M \left(\int_{0}^{t-t_0} e^{-(1-\epsilon)\sigma} d\sigma \right) e^{-\epsilon(t-t_0)} \\ &\leq 2\mu c_8 J(J+1).M \left(\int_{0}^{\infty} e^{-(1-\epsilon)\sigma} d\sigma \right) e^{-\epsilon(t-t_0)} \\ &\leq \frac{2\mu c_8 J(J+1)}{1-\epsilon} .M e^{-\epsilon(t-t_0)} \\ &\leq \frac{M}{10} e^{-\epsilon(t-t_0)}, \quad \forall\, t \in (t_0, T). \end{aligned} \tag{81}$$

In conjunction with (77),(78),(79), (80) and (81), this yields

$$\|U_1(\cdot,t)\|_{L^\infty(\Omega)} \leq 5\tfrac{M}{10} e^{-\epsilon(t-t_0)} = \tfrac{M}{2} e^{-\epsilon(t-t_0)}, \quad \forall\, t \in (t_0, T),$$

which by continuity of U_1 implies that indeed T cannot be finite. This shows (71) and hence proves the lemma. □

Now our main result can be obtained by combining Lemma 6 with a straightforward consequence thereof for the asymptotic of U_2 ,U_3.

Proof. (Proof of Theorem 1) We fix any $\epsilon \in (0,1)$ and let $\theta_0 = \theta_0(\epsilon)$ be as thereupon provided by Lemma 6. Then given u_{10} , u_{20} and u_{30} fulfilling (5), we apply Lemma 6 to find $c_1 > 0$ such that with (U_1, U_2, U_3) as in (8) we have

$$\|U_1(\cdot,t)\|_{L^\infty(\Omega)} \leq c_1 e^{-\epsilon t}, \quad \forall\, t > 0. \tag{82}$$

Now from (9) we writing U_2 in the form

$$U_2\cdot,t) = e^{t(\Delta-1)}(u_{20} - \frac{\alpha}{\beta}) + \int_0^t e^{(t-s)(\Delta-1)} U_1(\cdot,t)ds, \quad t > 0.$$

From the maximum principle we infer that

$$\begin{aligned}\|U_2(\cdot,t)\|_{L^\infty(\Omega)} &= e^{-t}\|e^{t\Delta}(u_{20} - \frac{\alpha}{\beta})\|_{L^\infty(\Omega)} + \int_0^t e^{-(t-s)}\|e^{(t-s)\Delta}U_1(\cdot,s)\|_{L^\infty(\Omega)}ds,\\ &\leq e^{-t}\|(u_{20} - \frac{\alpha}{\beta})\|_{L^\infty(\Omega)} + \int_0^t e^{-(t-s)}\|U_1(\cdot,s)\|_{L^\infty(\Omega)}ds, \quad \forall\, t > 0.\end{aligned}$$

Let $c_2 = \|u_{20} - \frac{\alpha}{\beta}\|_{L^\infty(\Omega)}$, form (82), we therefore obtain

$$\|U_2(\cdot,t)\|_{L^\infty(\Omega)} \leq c_2 e^{-t} + c_1 \int_0^t e^{-(t-s)} e^{-\epsilon s} ds = c_2 e^{-t} + \frac{c_1}{1-\epsilon}(e^{-\epsilon t} - e^{-t}), \quad \forall\, t > 0. \tag{83}$$

We also from (9) writing U_3 in the form

$$U_3(\cdot,t) = e^{t(\Delta-1)}(u_{30} - \frac{\gamma}{\eta}) + \int_0^t e^{(t-s)(\Delta-1)} U_1(\cdot,t)ds, \quad t > 0.$$

From the maximum principle, we have

$$\begin{aligned}\|U_3(\cdot,t)\|_{L^\infty(\Omega)} &= e^{-t}\|e^{t\Delta}(u_{30} - \frac{\gamma}{\eta})\|_{L^\infty(\Omega)} + \int_0^t e^{-(t-s)}\|e^{(t-s)\Delta}U_1(\cdot,s)\|_{L^\infty(\Omega)}ds,\\ &\leq e^{-t}\|(u_{30} - \frac{\gamma}{\eta})\|_{L^\infty(\Omega)} + \int_0^t e^{-(t-s)}\|U_1(\cdot,s)\|_{L^\infty(\Omega)}ds, \quad \forall\, t > 0.\end{aligned}$$

Let $c_3 = \|u_{30} - \frac{\gamma}{\eta}\|_{L^\infty(\Omega)}$, form (82), we have

$$\|U_3(\cdot,t)\|_{L^\infty(\Omega)} \leq c_3 e^{-t} + c_1 \int_0^t e^{-(t-s)} e^{-\epsilon s} ds = c_3 e^{-t} + \frac{c_1}{1-\epsilon}(e^{-\epsilon t} - e^{-t}), \; \forall\, t > 0. \tag{84}$$

In light of the definitions of U_1 ,U_2 and U_3, (82),(83) and (84) establish (7). □

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