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Article

On a class of p-valent functions with negative coefficients defined by opoola differential operator

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Abstract: Using opoola differential operator, we defined a subclass $S_p^n(\lambda,\alpha,\gamma,\delta)$ of the class of multivalent or p-valent functions. Several properties of the class were studied, such as coefficient inequalities, hadamard product, radii of close-to-convex, star-likeness, convexity, extreme points, the integral mean inequalities for the fractional derivatives, and further growth and distortion theorem are given using fractional calculus techniques.

Keywords: Multivalent functions; Opoola differential operator; Coefficient inequalities; Closure property.

MSC: 30C45; 30C50; 30C55.

1. Introduction

Let A denote the class of all functions, $f(z)$ normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$.

Definition 1. [1] For $f(z) \in A$, Opoola introduced the following operator:

$$\begin{aligned} &D^0(\mu,\beta,t)f(z) = f(z),\\ &D^1(\mu,\beta,t)f(z) = zD_t f(z) = tzf'(z) - z(\beta-\mu)t + [1+(\beta-\mu-1)t]f(z),\\ &D^n(\mu,\beta,t)f(z) = zD_t(D^{n-1}(\mu,\beta,t)f(z)), \qquad n \in N. \end{aligned} \tag{2}$$

If $f(z)$ is given by (1), then from (2), we see that

$$D^n(\mu,\beta,t)f(z) = z + \sum_{k=2}^{\infty} [1+(k+\beta-\mu-1)\,t]^n \, a_k z^k, \tag{3}$$

$(0 \le \mu \le \beta, t \ge 0$ and $n \in N_0 = N \cup 0)$.

Remark 1.
1. When $\beta = \mu$, $t = 1$,$D^n(\mu,\beta,t)f(z) = D^n f(z)$ by Salagean [2],
2. When $\beta = \mu$,$D^n(\mu,\beta,t)f(z) = D^n_\lambda f(z)$ by Al-Oboudi [3].

Definition 2. Let A_p denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \qquad (p = 1,2,\ldots), \tag{4}$$

which are analytic and multivalent in the open unit disc $U = (z \in C : |z| < 1)$. We define the following differential operator for the functions $f(z) \in A_p$

$$D^0(\mu,\beta,t,p)f(z) = f(z),$$
$$D^1(\mu,\beta,t,p)f(z) = zD_{t,p}f(z) = \frac{t}{p}zf'(z) - z^p(\beta-\mu)t + [1+(\beta-\mu-1)t]f(z),$$
$$D^n(\mu,\beta,t,p)f(z) = zD_{t,p}(D^{n-1}(\mu,\beta,t,p)f(z)), \qquad n \in N. \tag{5}$$

If $f(z)$ is given by (4), then from (5), we see that

$$D^n(\mu,\beta,t,p)f(z) = z^p + \sum_{k=p+1}^{\infty}\left[1+\left(\frac{k}{p}+\beta-\mu-1\right)t\right]^n a_k z^k\,, \tag{6}$$

$(0 \leq \mu \leq \beta, t \geq 0$ and $n \in N_0 = N \cup 0)$.

Let T_p denote the subclass of A_p consisting of functions of the form

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, \qquad (a_k \geq 0, p = 1,2,...)\,. \tag{7}$$

If $f(z)$ is given by Eq. (7), then from Eq. (5), we get

$$D^n(\mu,\beta,t,p)f(z) = z^p - \sum_{k=p+1}^{\infty}\left[1+\left(\frac{k}{p}+\beta-\mu-1\right)t\right]^n a_k z^k, \tag{8}$$

$(n \in N_0, a_k \geq 0, p = 1,2,..,0 \leq \mu \leq \beta, t \geq 0, n \in N_0 = N \cup 0)\,.$

Remark 2. When $\beta = \mu$ in (8),$D^n(\mu,\beta,t,p)f(z) = D^n_{\delta,p}f(z)$ defined by Bulut in [4]. Now, from (8), it follows that $D^n(\mu,\beta,t,p)f(z)$ can be written in terms of Convolution as

$$D^n(\mu,\beta,t,p)f(z) = (f * g)(z)\,,$$

where $f(z)$ is as in (7), while

$$g(z) = z^p - \sum_{k=p+1}^{\infty}\left[1+\left(\frac{k}{p}+\beta-\mu-1\right)t\right]^n z^k\,.$$

Definition 3. A function $f(z) \in T_p$ is in the class $S^n_p(\lambda,\alpha,\gamma,\delta)$ if

$$\left|\frac{(D^n(\mu,\beta,t,p)f(z))' - pz^{p-1}}{\lambda(D^n(\mu,\beta,t,p)f(z))' + (\alpha-\gamma)}\right| < \delta, \qquad (z \in U, n \in N_0)\,, \tag{9}$$

for some $0 \leq \lambda < 1, 0 \leq \gamma < 1, 0 < \alpha \leq 1, 0 < \delta < 1$, $D^n(\mu,\beta,t,p)f(z)$ as defined in (8).

Remark 3. When $\mu = \beta$ in (9),the class $S^n_p(\lambda,\alpha,\gamma,\delta)$ reduces to the class $R^n_p(\alpha,\beta,\gamma,\mu)$ studied by Bulut in [4].

Definition 4. [5,6] The fractional integral of order l is defined, for function $f(z)$ by

$$D_z^{-l}f(z) = \frac{1}{\Gamma(l)}\int_0^z \frac{f(t)}{(z-t)^{1-l}}d(t), \qquad (l > 0)\,, \tag{10}$$

where $f(z)$ is an analytic function in a simply connected region of z-plane containing the origin,and the multiplicity of $(z-t)^{l-1}$ is removed by requiring $log(z-t)$ to be real when $(z-t) > 0$.

Definition 5. [5,6] The fractional derivative of order l is defined, for function $f(z)$ by

$$D_z^l f(z) = \frac{1}{\Gamma(1-l)} \frac{d}{d(z)} \int_0^z \frac{f(t)}{(z-t)^l} d(t), \qquad (0 \le l < 1), \tag{11}$$

where f is an analytic function in a simply connected region of z-plane containing the origin,and the multiplicity of $(z-t)^{-l}$ is removed by requiring $log(z-t)$ to be real when $(z-t) > 0$.

Definition 6. [5,6] Under the hypothesis of Definition 4, the fractional derivative of order $p+l$ is defined for functions $f(z)$, by

$$D_z^{p+l} f(z) = \frac{d^p}{d(z)^p} D_z^l f(z), \qquad (0 \le l < 1, p \in N_0). \tag{12}$$

It readily follows from (9) and (10) that

$$D_z^{-l} z^k = \frac{\Gamma(k+1)}{\Gamma(k+l+1)} z^{k+l}, \qquad (l > 0, k \in N), \tag{13}$$

and

$$D_z^{l} z^k = \frac{\Gamma(k+1)}{\Gamma(k-l+1)} z^{k-l}, \qquad (0 \le l < 1, k \in N). \tag{14}$$

Lemma 1. [7] *If $f(z)$ and $g(z)$ are analytic in U with $f(z) \prec g(z)$, then for $\sigma > 0$ and $z = re^{i\theta}$, $(0 < r < 1)$, then*

$$\int_0^{2\pi} |f(z)|^\sigma \, d\theta \le \int_0^{2\pi} |g(z)|^\sigma \, d\theta .$$

In this work, several properties of the class $S_p^n(\lambda, \alpha, \gamma, \delta)$ are studied, such as coefficient inequalities, hadamard product, radii of close-to-convex, star-likeness, convexity, extreme points, the integral mean inequalities for the fractional derivatives, and further growth and distortion theorem are given using fractional calculus techniques. For more research on classes of multivalent or p-valent functions, see [7–18]

2. Main results

Theorem 1. *A function $f(z) \in T_p$ is in the class $S_p^n(\lambda, \alpha, \gamma, \delta)$ if and only if*

$$\sum_{k=p+1}^{\infty} k[1 + (\frac{k}{p} + \beta - \mu - 1)t]^n (1 + \delta\lambda) a_k \le \delta(\lambda p + \alpha - \gamma), \tag{15}$$

for some $0 \le \lambda < 1$, $0 \le \gamma < 1$,$0 < \alpha \le 1$,$0 < \delta < 1$. The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{\delta(\lambda p + \alpha - \gamma)}{k[1 + (\frac{k}{p} + \beta - \mu - 1)t]^n (1 + \delta\lambda)} z^k, \qquad (k \ge p+1).$$

Proof. Suppose that $f(z) \in S_p^n(\lambda, \alpha, \gamma, \delta)$, then we have from (9) that

$$\left| \frac{(D^n(\mu, \beta, t, p) f(z))' - pz^{p-1}}{\lambda (D^n(\mu, \beta, t, p) f(z))' + (\alpha - \gamma)} \right| < \delta .$$

By substitution, we have

$$\left| \frac{pz^{p-1} - \sum_{k=p+1}^{\infty} k[1 + (\frac{k}{p} + \beta - \mu - 1)t]^n a_k z^{k-1} - pz^{p-1}}{\lambda (pz^{p-1} - \sum_{k=p+1}^{\infty} k[1 + (\frac{k}{p} + \beta - \mu - 1)t]^n a_k z^{k-1}) + (\alpha - \gamma)} \right| < \delta,$$

$$\left|\frac{\sum_{k=p+1}^{\infty} k[1+(\frac{k}{p}+\beta-\mu-1)t]^n a_k z^{k-1}}{\lambda(pz^{p-1}-\sum_{k=p+1}^{\infty} k[1+(\frac{k}{p}+\beta-\mu-1)t]^n a_k z^{k-1})+(\alpha-\gamma)}\right| < \delta\,.$$

Since $\Re z \leq |z|$, then

$$\Re\left\{\frac{\sum_{k=p+1}^{\infty} k[1+(\frac{k}{p}+\beta-\mu-1)t]^n a_k z^{k-1}}{\lambda(pz^{p-1}-\sum_{k=p+1}^{\infty} k[1+(\frac{k}{p}+\beta-\mu-1)t]^n a_k z^{k-1})+(\alpha-\gamma)}\right\} < \delta\,.$$

If we choose z real and let $z \to 1^-$, then we get

$$\begin{aligned}
&\sum_{k=p+1}^{\infty} k[1+(\frac{k}{p}+\beta-\mu-1)t]^n a_k \leq \delta\left[\lambda(p-\sum_{k=p+1}^{\infty} k[1+(\frac{k}{p}+\beta-\mu-1)t]^n a_k)+(\alpha-\gamma)\right],\\
&\Rightarrow \sum_{k=p+1}^{\infty} k[1+(\frac{k}{p}+\beta-\mu-1)t]^n a_k \leq \delta\lambda p-\delta\lambda\sum_{k=p+1}^{\infty} k[1+(\frac{k}{p}+\beta-\mu-1)t]^n a_k+\delta(\alpha-\gamma),\\
&\Rightarrow \sum_{k=p+1}^{\infty} k[1+(\frac{k}{p}+\beta-\mu-1)t]^n a_k+\delta\lambda\sum_{k=p+1}^{\infty} k[1+(\frac{k}{p}+\beta-\mu-1)t]^n a_k \leq \delta(\lambda p+\alpha-\gamma),\\
&\Rightarrow \sum_{k=p+1}^{\infty} k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)a_k \leq \delta(\lambda p+\alpha-\gamma).
\end{aligned}$$

Conversely, suppose that the inequality (15) holds true and that $z \in \partial U : \{z \in C : |z| = 1\}$ and suppose that

$$\begin{aligned}
&\left|(D^n(\mu,\beta,t,p)f(z))'-pz^{p-1}\right|-\delta\left|\lambda(D^n(\mu,\beta,t,p)f(z))'+(\alpha-\gamma)\right|\\
&\leq \left|(D^n(\mu,\beta,t,p)f(z))'-pz^{p-1}-\delta(\lambda(D^n(\mu,\beta,t,p)f(z))'+(\alpha-\gamma))\right|\\
&= \Bigg|-\sum_{k=p+1}^{\infty} k[1+(\frac{k}{p}+\beta-\mu-1)t]^n a_k z^{k-1}-\delta\lambda pz^{p-1}\\
&+\delta\lambda\sum_{k=p+1}^{\infty} k[1+(\frac{k}{p}+\beta-\mu-1)t]^n a_k z^{k-1}-\delta(\alpha-\gamma)\Bigg|\\
&= \left|\sum_{k=p+1}^{\infty}(\delta\lambda-1)k[1+(\frac{k}{p}+\beta-\mu-1)t]^n a_k z^{k-1}-\delta(\lambda pz^{p-1}+\alpha-\gamma)\right|\\
&\leq \sum_{k=p+1}^{\infty}(\delta\lambda-1)k[1+(\frac{k}{p}+\beta-\mu-1)t]^n a_k-\delta(\lambda p+\alpha-\gamma)\\
&\leq \sum_{k=p+1}^{\infty} k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(\delta\lambda+1)a_k-\delta(\lambda p+\alpha-\gamma) \leq 0.
\end{aligned}$$

Since by maximum modulus theorem ,that the maximum modulus of an analytic function cannot be attained inside the domain but on the boundary, implies

$$\left|(D^n(\mu,\beta,t,p)f(z))'-pz^{p-1}\right|-\delta\left|\lambda(D^n(\mu,\beta,t,p)f(z))'+(\alpha-\gamma)\right| < 0\,,$$

i.e.,

$$\left|(D^n(\mu,\beta,t,p)f(z))'-pz^{p-1}\right| < \delta\left|\lambda(D^n(\mu,\beta,t,p)f(z))'+(\alpha-\gamma)\right|\,.$$

So,

$$\frac{\left|(D^n(\mu,\beta,t,p)f(z))'-pz^{p-1}\right|}{\left|\lambda(D^n(\mu,\beta,t,p)f(z))'+(\alpha-\gamma)\right|} < \delta\,,$$

implies

$$\left|\frac{(D^n(\mu,\beta,t,p)f(z))'-pz^{p-1}}{\lambda(D^n(\mu,\beta,t,p)f(z))'+(\alpha-\gamma)}\right| < \delta.$$

Hence, we have that $f(z) \in S_p^n(\lambda, \alpha, \gamma, \delta)$.

Corollary 1. *If $f(z) \in S_p^n(\lambda, \alpha, \gamma, \delta)$, then*

$$a_{p+1} \leq \frac{\delta(\lambda p + \alpha - \gamma)p^n}{(p+1)[p + (p(\beta - \mu) + 1)t]^n(1 + \delta\lambda)}.$$

□

Theorem 2. *The class $S_p^n(\lambda, \alpha, \gamma, \delta)$ is a class of convex functions.*

Proof. Let the functions

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, \qquad (a_k \geq 0, p = 1, 2, ...), \tag{16}$$

$$g(z) = z^p - \sum_{k=p+1}^{\infty} b_k z^k, \qquad (b_k \geq 0, p = 1, 2, ...), \tag{17}$$

be in the class $S_p^n(\lambda, \alpha, \gamma, \delta)$, then for $0 \leq j \leq 1$

$$h(z) = (1 - j)f(z) + jg(z) = z^p - \sum_{k=p+1}^{\infty} c_k z^k,$$

where $c_k = (1 - j)a_k + jb_k \geq 0$, then making use of (15), we see that

$$\begin{aligned}
&\sum_{k=p+1}^{\infty} k[1 + (\frac{k}{p} + \beta - \mu - 1)t]^n(1 + \delta\lambda)c_k \\
&= \sum_{k=p+1}^{\infty} k[1 + (\frac{k}{p} + \beta - \mu - 1)t]^n(1 + \delta\lambda)a_k + \sum_{k=p+1}^{\infty} k[1 + (\frac{k}{p} + \beta - \mu - 1)t]^n(1 + \delta\lambda)b_k \\
&< (1 - j)\delta(\lambda p + \alpha - \gamma) + j\delta(\lambda p + \alpha - \gamma) \\
&= \delta(\lambda p + \alpha - \gamma) - j\delta(\lambda p + \alpha - \gamma) + j\delta(\lambda p + \alpha - \gamma) \\
&= \delta(\lambda p + \alpha - \gamma),
\end{aligned}$$

implies $h(z) \in S_p^n(\lambda, \alpha, \gamma, \delta)$, which completes the proof. □

Theorem 3. *If each of the functions $f(z)$ and $g(z)$ is in the class $S_p^n(\lambda, \alpha, \gamma, \delta)$, then $(f * g)(z) \in S_p^n(\lambda, \alpha, \gamma, \Omega)$, where $\Omega \geq \frac{\delta^2(\lambda p + \alpha - \gamma)}{k[1 + (\frac{k}{p} + \beta - \mu - 1)t]^n(1 + \delta\lambda)^2 - \delta^2\lambda(\lambda p + \alpha - \gamma)}$.*

Proof. From (15), we have

$$\sum_{k=p+1}^{\infty} \frac{k[1 + (\frac{k}{p} + \beta - \mu - 1)t]^n(1 + \delta\lambda)}{\delta(\lambda p + \alpha - \gamma)} a_k \leq 1, \tag{18}$$

and

$$\sum_{k=p+1}^{\infty} \frac{k[1 + (\frac{k}{p} + \beta - \mu - 1)t]^n(1 + \delta\lambda)}{\delta(\lambda p + \alpha - \gamma)} b_k \leq 1. \tag{19}$$

We need to find the smallest Ω such that

$$\sum_{k=p+1}^{\infty} \frac{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\Omega\lambda)}{\Omega(\lambda p+\alpha-\gamma)} a_k b_k \leq 1\,. \tag{20}$$

From (18) and (19), we find by means of Cauchy-Schwarz inequalities that

$$\sum_{k=p+1}^{\infty} \frac{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)}{\delta(\lambda p+\alpha-\gamma)} \sqrt{a_k b_k} \leq 1\,. \tag{21}$$

Thus, it is enough to show that

$$\frac{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\Omega\lambda)}{\Omega(\lambda p+\alpha-\gamma)} a_k b_k \leq \frac{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)}{\delta(\lambda p+\alpha-\gamma)} \sqrt{a_k b_k}\,.$$

That is

$$\sqrt{a_k b_k} \leq \frac{\Omega(1+\delta\lambda)}{\delta(1+\Omega\lambda)}\,. \tag{22}$$

On the other hand, from (21), we have

$$\sqrt{a_k b_k} \leq \frac{\delta(\lambda p+\alpha-\gamma)}{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)}\,. \tag{23}$$

Therefore, in view of (22) and (23), it is enough to show that

$$\frac{\delta(\lambda p+\alpha-\gamma)}{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)} \leq \frac{\Omega(1+\delta\lambda)}{\delta(1+\Omega\lambda)},$$

i.e.,

$$\delta(\lambda p+\alpha-\gamma)\delta(1+\Omega\lambda) \leq kM(1+\delta\lambda)\Omega(1+\delta\lambda)\,,$$

where $M=[1+(\frac{k}{p}+\beta-\mu-1)t]^n$. So,

$$\delta^2[(\lambda p+\alpha-\gamma)+\Omega\lambda(\lambda p+\alpha-\gamma)] \leq k\Omega M(1+\delta\lambda)^2\,,$$

implies

$$\delta^2(\lambda p+\alpha-\gamma)+\delta^2\Omega\lambda(\lambda p+\alpha-\gamma) \leq k\Omega M(1+\delta\lambda)^2\,,$$

implies

$$\delta^2(\lambda p+\alpha-\gamma) \leq k\Omega M(1+\delta\lambda)^2-\delta^2\Omega\lambda(\lambda p+\alpha-\gamma)\,.$$

Also

$$\Omega\left[kM(1+\delta\lambda)^2-\delta^2\lambda(\lambda p+\alpha-\gamma)\right] \geq \delta^2(\lambda p+\alpha-\gamma)\,,$$

implies

$$\Omega \geq \frac{\delta^2(\lambda p+\alpha-\gamma)}{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)^2-\delta^2\lambda(\lambda p+\alpha-\gamma)}\,.$$

□

Theorem 4. *If $f(z)\in S_p^n(\lambda,\alpha,\gamma,\delta)$, then $f(z)$ is p-valently close-to-convex of order ρ in $|z|<r_1(\lambda,\alpha,\gamma,\delta,\rho)$, where*

$$r_1(\lambda,\alpha,\gamma,\delta,\rho)=\inf_k\left\{\frac{[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)(p-\rho)}{\delta(\lambda p+\alpha-\gamma)}\right\}^{\frac{1}{k-p}}, \qquad (k\geq p+1)\,.$$

Proof. Let $f(z) \in S_p^n(\lambda,\alpha,\gamma,\delta)$, then

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| < p-\rho\,,$$

implies

$$\left|\frac{pz^{p-1} - \sum_{k=p+1}^{\infty} ka_k z^{k-1} - pz^{p-1}}{z^{p-1}}\right| = \left|\sum_{k=p+1}^{\infty} ka_k z^{k-p}\right| \leq \sum_{k=p+1}^{\infty} ka_k \left|z\right|^{k-p} < p-\rho\,. \tag{24}$$

Since

$$\sum_{k=p+1}^{\infty} k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)a_k \leq \delta(\lambda p+\alpha-\gamma)\,, \tag{25}$$

hence, (24) is true if

$$\frac{k\left|z\right|^{k-p}}{p-\rho} < \frac{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)}{\delta(\lambda p+\alpha-\gamma)}\,. \tag{26}$$

Solving (26) for $|z|$, we obtain

$$|z| < \left\{\frac{[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)(p-\rho)}{\delta(\lambda p+\alpha-\gamma)}\right\}^{\frac{1}{k-p}}, \qquad (k \geq p+1).$$

Hence, the proof. □

Theorem 5. *If $f(z) \in S_p^n(\lambda,\alpha,\gamma,\delta)$,then $f(z)$ is p-valently starlike of order ρ in $|z| < r_2(\lambda,\alpha,\gamma,\delta,\rho)$, where*

$$r_2(\lambda,\alpha,\gamma,\delta,\rho) = \inf_k \left\{\frac{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)(p-\rho)}{\delta(\lambda p+\alpha-\gamma)(k-p)}\right\}^{\frac{1}{k-p}}, \qquad (k \geq p+1).$$

Proof. Let $f(z) \in S_p^n(\lambda,\alpha,\gamma,\delta)$, then

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p-\rho\,.$$

The inequality

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)} - p\right| &= \left|\frac{z(pz^{p-1} - \sum_{k=p+1}^{\infty} ka_k z^{k-1}) - p(z^p - \sum_{k=p+1}^{\infty} a_k z^k)}{z^p - \sum_{k=p+1}^{\infty} a_k z^k}\right| \\ &= \left|\frac{pz^p - \sum_{k=p+1}^{\infty} ka_k z^k - pz^p + p\sum_{k=p+1}^{\infty} a_k z^k)}{z^p - \sum_{k=p+1}^{\infty} a_k z^k}\right|\,. \end{aligned} \tag{27}$$

Since

$$\left|\frac{-\sum_{k=p+1}^{\infty}(k-p)ka_k z^k}{z^p - \sum_{k=p+1}^{\infty} a_k z^k}\right| = \left|\frac{\sum_{k=p+1}^{\infty}(k-p)a_k z^{k-p}}{1-\sum_{k=p+1}^{\infty} a_k z^{k-p}}\right| \leq \frac{\sum_{k=p+1}^{\infty}(k-p)a_k\left|z\right|^{k-p}}{1-\sum_{k=p+1}^{\infty} a_k\left|z\right|^{k-p}} < p-\rho\,, \tag{28}$$

i.e.,

$$\begin{aligned}\sum_{k=p+1}^{\infty} ka_k\left|z\right|^{k-p} - \sum_{k=p+1}^{\infty} pa_k\left|z\right|^{k-p} &< (p-\rho)(1-\sum_{k=p+1}^{\infty} a_k\left|z\right|^{k-p}) \\ &= \sum_{k=p+1}^{\infty} ka_k\left|z\right|^{k-p} - \sum_{k=p+1}^{\infty} pa_k\left|z\right|^{k-p}\end{aligned}$$

$$
\begin{aligned}
&< p - \sum_{k=p+1}^{\infty} p a_k |z|^{k-p} - \rho + \sum_{k=p+1}^{\infty} \rho a_k |z|^{k-p} \\
&= \sum_{k=p+1}^{\infty} k a_k |z|^{k-p} - \sum_{k=p+1}^{\infty} \rho a_k |z|^{k-p} \\
&< p - \rho \\
&= \sum_{k=p+1}^{\infty} \frac{(k-\rho) a_k |z|^{k-p}}{p-\rho} \\
&< 1.
\end{aligned}
$$

Since

$$\sum_{k=p+1}^{\infty} k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)a_k \leq \delta(\lambda p+\alpha-\gamma)\,.$$

This holds true if

$$\frac{(k-\rho)\,|z|^{k-p}}{p-\rho} < \frac{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)}{\delta(\lambda p+\alpha-\gamma)}\,,$$

$$|z| < \left\{\frac{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)(p-\rho)}{(k-p)\delta(\lambda p+\alpha-\gamma)}\right\}^{\frac{1}{k-p}}, \qquad (k \geq p+1),$$

hence, the proof. □

Theorem 6. *If $f(z) \in S_p^n(\lambda,\alpha,\gamma,\delta)$, then $f(z)$ is p-valently convex of order ρ in $|z| < r_3(\lambda,\alpha,\gamma,\delta,\rho)$, where*

$$r_3(\lambda,\alpha,\gamma,\delta,\rho) = \inf_k \left\{\frac{[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)p(p-\rho)}{\delta(\lambda p+\alpha-\gamma)(k-p)}\right\}^{\frac{1}{k-p}}, \qquad (k \geq p+1).$$

Proof. Let $f(z) \in S_p^n(\lambda,\alpha,\gamma,\delta)$, then

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| < p-\rho\,.$$

The inequality

$$
\begin{aligned}
&\left|1+\frac{zf''(z)}{f'(z)}-p\right| \\
&= \left|\frac{pz^{p-1}-\sum_{k=p+1}^{\infty} k a_k z^{k-1} + z(p(p-1)z^{p-2}-\sum_{k=p+1}^{\infty} k(k-1)a_k z^{k-2}) - p(pz^{p-1}-\sum_{k=p+1}^{\infty} k a_k z^{k-1})}{pz^{p-1}-\sum_{k=p+1}^{\infty} k a_k z^{k-1}}\right| \\
&= \left|\frac{pz^{p-1}-\sum_{k=p+1}^{\infty} k a_k z^{k-1} + p(p-1)z^{p-1}-\sum_{k=p+1}^{\infty} k(k-1)a_k z^{k-1}) - p^2 z^{p-1}+\sum_{k=p+1}^{\infty} pk a_k z^{k-1})}{pz^{p-1}-\sum_{k=p+1}^{\infty} k a_k z^{k-1}}\right| \\
&= \left|\frac{-\sum_{k=p+1}^{\infty} k(k-p)a_k z^{k-p}}{p-\sum_{k=p+1}^{\infty} k a_k z^{k-p}}\right| \\
&\leq \frac{\sum_{k=p+1}^{\infty} k(k-p)a_k |z|^{k-p}}{p-\sum_{k=p+1}^{\infty} k a_k |z|^{k-p}} \\
&< p-\rho.
\end{aligned}
$$

So,

$$\sum_{k=p+1}^{\infty} k(k-p)a_k |z|^{k-p} < (p-\rho)(p-\sum_{k=p+1}^{\infty} k a_k |z|^{k-p})\,,$$

implies

$$\sum_{k=p+1}^{\infty} k(k-p)a_k |z|^{k-p} < p(p-\rho) - (p-\rho) \sum_{k=p+1}^{\infty} ka_k |z|^{k-p}) \,,$$

implies

$$\sum_{k=p+1}^{\infty} k(k-\rho)a_k |z|^{k-p} < p(p-\rho) \,.$$

Since $\sum\limits_{k=p+1}^{\infty} k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)a_k \leq \delta(\lambda p+\alpha-\gamma)$. This is true if

$$\frac{k(k-\rho)\,|z|^{k-p}}{p(p-\rho)} < \frac{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)}{\delta(\lambda p+\alpha-\gamma)} \,,$$

$$|z| < \left\{ \frac{[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)p(p-\rho)}{(k-p)\delta(\lambda p+\alpha-\gamma)} \right\}^{\frac{1}{k-p}} \,, \qquad (k \geq p+1),$$

hence, the proof. □

Theorem 7. *Let*

$$f_p(z) = z^p, f_k(z) = z^p - \frac{\delta(\lambda p+\alpha-\gamma)}{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)} z^k, \qquad (k \geq p+1), \tag{29}$$

then, $f(z) \in S_p^n(\lambda,\alpha,\gamma,\delta)$ if and only if it can be expressed in the form

$$f(z) = \lambda_p f_p(z) + \sum_{k=p+1}^{\infty} \lambda_k f_k(z) \,,$$

where $\lambda_p \geq 0$ and λ_p=$1 - \sum\limits_{k=p+1}^{\infty} \lambda_k$.

Proof. Assume that $f(z) = \lambda_p f_p(z) + \sum_{k=p+1}^{\infty} \lambda_k f_k(z)$, then

$$f(z) = (1 - \sum_{k=p+1}^{\infty} \lambda_k) z^p + \sum_{k=p+1}^{\infty} \lambda_k \left\{ z^p - \frac{\delta(\lambda p+\alpha-\gamma)}{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)} z^k \right\} \,, \tag{30}$$

implies

$$f(z) = z^p - \sum_{k=p+1}^{\infty} \lambda_k \left\{ \frac{\delta(\lambda p+\alpha-\gamma)}{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)} \right\} z^k \,.$$

Thus,

$$\sum_{k=p+1}^{\infty} k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)\lambda_k \frac{\delta(\lambda p+\alpha-\gamma)}{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)}$$
$$= \delta(\lambda p+\alpha-\gamma) \sum_{k=p+1}^{\infty} \lambda_k = \delta(\lambda p+\alpha-\gamma)(1-\lambda_p) \leq \delta(\lambda p+\alpha-\gamma) \,,$$

which shows that $f(z)$ satisfies condition (15) and therefore, $f \in S_p^n(\lambda,\alpha,\gamma,\delta)$.

Conversely, suppose that $f(z) \in S_p^n(\lambda,\alpha,\gamma,\delta)$, since

$$a_k \leq \frac{\delta(\lambda p+\alpha-\gamma)}{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)}, \qquad (k \geq p+1),$$

we may set

$$\lambda_k = \frac{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)}{\delta(\lambda p+\alpha-\gamma)}a_k,$$
$$\lambda_p = 1-\sum_{k=p+1}^{\infty}\lambda_k\,,$$

then we obtain from $f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k$,

$$f(z) = (\lambda_p + \sum_{k=p+1}^{\infty}\lambda_k)z^p - \sum_{k=p+1}^{\infty}\lambda_k \frac{\delta(\lambda p+\alpha-\gamma)}{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)}z^k\,,$$

i.e.,

$$f(z) = \lambda_p z^p + \sum_{k=p+1}^{\infty}\lambda_k(z^p - \frac{\delta(\lambda p+\alpha-\gamma)}{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)}z^k)\,,$$

implies

$$f(z) = \lambda_p z^p + \sum_{k=p+1}^{\infty}\lambda_k f_k(z)\,.$$

This completes the proof. □

Corollary 2. *The extreme points of $S_p^n(\lambda,\alpha,\gamma,\delta)$ are given by;*

$$f_p(z) = z^p,$$
$$f_k(z) = z^p - \frac{\delta(\lambda p+\alpha-\gamma)}{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)}z^k, \qquad (k \geq p+1).$$

Theorem 8. *Let $f(z) \in S_p^n(\lambda,\alpha,\gamma,\delta)$ and suppose that*

$$\sum_{j=p+1}^{\infty}(j-q)_{q+1}a_j \leq \frac{\delta(\lambda p+\alpha-\gamma)\Gamma(k+1)\Gamma(2+p-l-q)}{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)\Gamma(k+1-l-q)\Gamma(p+1-q)}\,, \tag{31}$$

for some $0 \leq q \leq j$, $0 \leq l < 1$, $(j-q)_{q+1}$ denotes the pochhammer symbol defined by $(j-q)_{q+1} = (j-q)(j-q+1)...j$. Also, let the function

$$f_k(z) = z^p - \frac{\delta(\lambda p+\alpha-\gamma)}{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)}z^k(k \geq p+1)\,. \tag{32}$$

If there exists an analytic function $w(z)$ defined by

$$(w(z))^{k-p} = \frac{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)}{\delta(\lambda p+\alpha-\gamma)}\frac{\Gamma(k+1-l-q)}{\Gamma(k+1)}\sum_{j=p+1}^{\infty}(j-q)_{q+1\psi(j)a_j z^{j-p}}\,, \tag{33}$$

with $(k \geq q)$

$$\Psi(j) = \frac{\Gamma(j-q)}{\Gamma(j+1-l-q)}, \qquad (0 \leq l < 1, j \geq p+1),$$

then, for $\sigma > 0$ and $z = re^{i\theta}$, $(0 < r < 1)$,

$$\int_0^{2\pi}\left|D_z^{q+l}f(z)\right|^{\sigma}d\theta \leq \int_0^{2\pi}\left|D_z^{q+l}f_k(z)\right|^{\sigma}d\theta\,. \tag{34}$$

Proof. Let $f(z) = z^p - \sum_{j=p+1}^{\infty} a_j z^j$. By means of (12) and Definition 6, we have

$$D_z^{q+l} f(z) = \frac{\Gamma(p+1) z^{p-l-q}}{\Gamma(p+1-l-q)} - \sum_{j=p+1}^{\infty} \frac{\Gamma(j+1)}{\Gamma(j+1-l-q)} a_j z^{j-l-q}$$

$$= \frac{\Gamma(p+1) z^{p-l-q}}{\Gamma(p+1-l-q)} [1 - \sum_{j=p+1}^{\infty} \frac{\Gamma(j+1)\Gamma(p+1-l-q)}{\Gamma(p+1)\Gamma(j+1-l-q)} a_j z^{j-p}] \tag{35}$$

$$= \frac{\Gamma(p+1) z^{p-l-q}}{\Gamma(p+1-l-q)} [1 - \sum_{j=p+1}^{\infty} \frac{\Gamma(p+1-l-q)}{\Gamma(p+1)} (j-q)_{q+1} \Psi(j) a_j z^{j-p}] , \tag{36}$$

where $\Psi(j) = \frac{\Gamma(j-q)}{\Gamma(j+1-l-q)}$, $\quad (0 \le l < 1, j \ge p+1)$. Since ψ is a decreasing function of j, we get

$$0 < \Psi(j) \le \Psi(p+1) = \frac{\Gamma(p+1-q)}{\Gamma(2+p-l-q)} .$$

Similarly, from (32), (14), and Definition 6, we have

$$D_z^{q+l} f_k(z) = \frac{\Gamma(p+1) z^{p-l-q}}{\Gamma(p+1-l-q)} - \frac{\delta(\lambda p + \alpha - \gamma)\Gamma(k+1)}{k[1 + (\frac{k}{p} + \beta - \mu - 1)t]^n (1 + \delta\lambda)\Gamma(k+1-l-q)} z^{k-l-q}$$

$$= \frac{\Gamma(p+1) z^{p-l-q}}{\Gamma(p+1-l-q)} [1 - \frac{\delta(\lambda p + \alpha - \gamma)\Gamma(k+1)\Gamma(p+1-l-q)}{k[1 + (\frac{k}{p} + \beta - \mu - 1)t]^n (1 + \delta\lambda)\Gamma(p+1)\Gamma(k+1-l-q)} z^{k-p}] . \tag{37}$$

For some $\sigma > 0$ and $z = re^{i\theta}$, $(0 < r < 1)$, we show that

$$\int_0^{2\pi} \left| 1 - \sum_{j=p+1}^{\infty} \frac{\Gamma(p+1-l-q)}{\Gamma(p+1)} (j-q)_{q+1} \psi(j) a_j z^{j-p} \right|^{\sigma} d(\theta)$$

$$\le \int_0^{2\pi} \left| 1 - \frac{\delta(\lambda p + \alpha - \gamma)\Gamma(k+1)\Gamma(p+1-l-q)}{k[1 + (\frac{k}{p} + \beta - \mu - 1)t]^n (1 + \delta\lambda)\Gamma(p+1)\Gamma(k+1-l-q)} z^{k-p} \right|^{\sigma} d(\theta) , \tag{38}$$

so, by applying Lemma 1, it is enough to show that

$$1 - \sum_{j=p+1}^{\infty} \frac{\Gamma(p+1-l-q)}{\Gamma(p+1)} (j-q)_{q+1} \psi(j) a_j z^{j-p}$$

$$\prec 1 - \frac{\delta(\lambda p + \alpha - \gamma)\Gamma(k+1)\Gamma(p+1-l-q)}{k[1 + (\frac{k}{p} + \beta - \mu - 1)t]^n (1 + \delta\lambda)\Gamma(p+1)\Gamma(k+1-l-q)} z^{k-p} . \tag{39}$$

If the above subordination holds true, then we have an analytic function $w(z)$ with $w(0) = 0$, $|w(z)| < 1$, such that

$$1 - \sum_{j=p+1}^{\infty} \frac{\Gamma(p+1-l-q)}{\Gamma(p+1)} (j-q)_{q+1} \psi(j) a_j z^{j-p}$$

$$= 1 - \frac{\delta(\lambda p + \alpha - \gamma)\Gamma(k+1)\Gamma(p+1-l-q)}{k[1 + (\frac{k}{p} + \beta - \mu - 1)t]^n (1 + \delta\lambda)\Gamma(p+1)\Gamma(k+1-l-q)} w(z)^{k-p} . \tag{40}$$

By the condition of the Theorem, we define the function $w(z)$ by

$$(w(z))^{k-p} = \frac{k[1 + (\frac{k}{p} + \beta - \mu - 1)t]^n (1 + \delta\lambda)}{\delta(\lambda p + \alpha - \gamma)} \frac{\Gamma(k+1-l-q)}{\Gamma(k+1)} \sum_{j=p+1}^{\infty} (j-q)_{q+1} \psi(j) a_j z^{j-p} , \tag{41}$$

which readily yields $w(0) = 0$. For such a function $w(z)$, we have

$$
\begin{aligned}
|(w(z))|^{k-p} &\leq \frac{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)}{\delta(\lambda p+\alpha-\gamma)} \frac{\Gamma(k+1-l-q)}{\Gamma(k+1)} \sum_{j=p+1}^{\infty} (j-q)_{q+1}\psi(j)a_j\,|z|^{j-p} \\
&\leq |z| \frac{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)}{\delta(\lambda p+\alpha-\gamma)} \frac{\Gamma(k+1-l-q)}{\Gamma(k+1)} \psi(p+1) \sum_{j=p+1}^{\infty} (j-q)_{q+1}a_j \\
&= |z| \frac{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)}{\delta(\lambda p+\alpha-\gamma)} \frac{\Gamma(k+1-l-q)\Gamma(p+1-q)}{\Gamma(k+1)\Gamma(2+p-l-q)} \sum_{j=p+1}^{\infty} (j-q)_{q+1}a_j \\
&\leq |z| < 1. \qquad (42)
\end{aligned}
$$

By means of the hypothesis of the theorem, the result is proved. □

As a special case $q = 0$, we have following results from Theorem 8.

Corollary 3. *Let $f(z) \in S_p^n(\lambda,\alpha,\gamma,\delta)$ and suppose that*

$$\sum_{j=p+1}^{\infty} ja_j \leq \frac{\delta(\lambda p+\alpha-\gamma)\Gamma(k+1)\Gamma(2+p-l)}{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)\Gamma(k+1-l)\Gamma(p+1)}, \qquad (j \geq p+1)\,, \qquad (43)$$

if there exists an analytic function $w(z)$ defined by

$$(w(z))^{k-p} = \frac{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)}{\delta(\lambda p+\alpha-\gamma)} \frac{\Gamma(k+1-l)}{\Gamma(k+1)} \sum_{j=p+1}^{\infty} j\Psi(j)a_j z^{j-p}\,, \qquad (44)$$

with

$$\psi(j) = \frac{\Gamma(j)}{\Gamma(j+1-l)}, \qquad (0 \leq l < 1, j \geq p+1)\,,$$

then, for $\sigma > 0$ and $z = re^{i\theta}$, $(0 < r < 1)$

$$\int_0^{2\pi} \left|D_z^l f(z)\right|^{\sigma} d\theta \leq \int_0^{2\pi} \left|D_z^l f_k(z)\right|^{\sigma} d\theta\,. \qquad (45)$$

Letting $q = 1$, we have the following from Theorem 8.

Corollary 4. *Let $f(z) \in S_p^n(\lambda,\alpha,\gamma,\delta)$ and suppose that*

$$\sum_{j=p+1}^{\infty} j(j-1)a_j \leq \frac{\delta(\lambda p+\alpha-\gamma)\Gamma(k+1)\Gamma(p+1-l)}{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)\Gamma(k-l)\Gamma(p)}, \qquad (j \geq p+1)\,, \qquad (46)$$

if there exists an analytic function $w(z)$ define by

$$(w(z))^{k-p} = \frac{k[1+(\frac{k}{p}+\beta-\mu-1)t]^n(1+\delta\lambda)}{\delta(\lambda p+\alpha-\gamma)} \frac{\Gamma(k-l)}{\Gamma(k+1)} \sum_{j=p+1}^{\infty} j(j-1)\Psi(j)a_j z^{j-p}\,, \qquad (47)$$

with

$$\psi(j) = \frac{\Gamma(j-1)}{\Gamma(j-l)}, \qquad (0 \leq l < 1, j \geq p+1)\,,$$

then, for $\sigma > 0$ *and* $z = re^{i\theta}$, $(0 < r < 1)$

$$\int_0^{2\pi} \left| D_z^{1+l} f(z) \right|^{\sigma} d\theta \leq \int_0^{2\pi} \left| D_z^{1+l} f_k(z) \right|^{\sigma} d\theta, \qquad (0 \leq l < 1). \tag{48}$$

Theorem 9. *If* $f(z) \in S_p^n(\lambda, \alpha, \gamma, \delta)$, *then we have*

$$\left| D_z^{-l} f(z) \right| \leq \frac{\Gamma(p+1)}{\Gamma(p+l+1)} |z|^{p+l} \left[1 + \frac{\delta(\lambda p + \alpha - \gamma) p^n}{[p + (p(\beta - \mu) + 1)t]^n (1 + \delta\lambda)(p + l + 1)} |z| \right],$$

and

$$\left| D_z^{-l} f(z) \right| \geq \frac{\Gamma(p+1)}{\Gamma(p+l+1)} |z|^{p+l} \left[1 - \frac{\delta(\lambda p + \alpha - \gamma) p^n}{[p + (p(\beta - \mu) + 1)t]^n (1 + \delta\lambda)(p + l + 1)} |z| \right]. \tag{49}$$

Proof. Suppose that $f(z) \in S_p^n(\lambda, \alpha, \gamma, \delta)$, using Theorem 1, we find that

$$\sum_{k=p+1}^{\infty} k[1 + (\frac{k}{p} + \beta - \mu - 1)t]^n (1 + \delta\lambda) a_k \leq \delta(\lambda p + \alpha - \gamma),$$

implies

$$\frac{(p+1)[p + (p(\beta - \mu) + 1)t]^n (1 + \delta\lambda)}{p^n} \sum_{k=p+1}^{\infty} a_k \leq \delta(\lambda p + \alpha - \gamma), \tag{50}$$

i.e.,

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{\delta(\lambda p + \alpha - \gamma) p^n}{(p+1)[p + (p(\beta - \mu) + 1)t]^n (1 + \delta\lambda)}. \tag{51}$$

From (7), if $f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k$,

$$D_z^{-l} f(z) = \frac{\Gamma(p+1)}{\Gamma(p+l+1)} z^{p+l} - \sum_{k=p+1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+l+1)} a_k z^{k+l},$$

implies

$$\frac{\Gamma(p+l+1)}{\Gamma(p+1)} z^{-l} D_z^{-l} f(z) = z^p - \sum_{k=p+1}^{\infty} \frac{\Gamma(k+1)\Gamma(p+l+1)}{\Gamma(p+1)\Gamma(k+l+1)} a_k z^k = z^p - \sum_{k=p+1}^{\infty} \Psi(k) a_k z^k, \tag{52}$$

where

$$\Psi(k) = \frac{\Gamma(k+1)\Gamma(p+l+1)}{\Gamma(p+1)\Gamma(k+l+1)}. \tag{53}$$

Clearly, ψ is a decreasing function of k and we get

$$0 < \Psi(k) \leq \Psi(p+1) = \frac{p+1}{p+l+1}.$$

Using (51) and (53), we obtain,

$$\begin{aligned} \left| \frac{\Gamma(p+l+1)}{\Gamma(p+1)} z^{-l} D_z^{-l} f(z) \right| &\leq |z|^p + \psi(p+1) |z|^{p+1} \sum_{k=p+1}^{\infty} a_k \\ &\leq |z|^p + \frac{\delta(\lambda p + \alpha - \gamma) p^n}{[p + (p(\beta - \mu) + 1)t]^n (1 + \delta\lambda)(p + l + 1)} |z|^{p+1}, \end{aligned}$$

which is equivalent to assertion (49) and

$$\left|\frac{\Gamma(p+l+1)}{\Gamma(p+1)}z^{-l}D_z^{-l}f(z)\right| \geq |z|^p - \psi(p+1)\,|z|^{p+1}\sum_{k=p+1}^{\infty} a_k$$
$$\geq |z|^p - \frac{\delta(\lambda p+\alpha-\gamma)p^n}{[p+(p(\beta-\mu)+1)t]^n(1+\delta\lambda)(p+l+1)}\,|z|^{p+1}\,,$$

which completes the proof. □

Theorem 10. *If $f(z) \in S_p^n(\lambda,\alpha,\gamma,\delta)$, then we have*

$$\left|D_z^l f(z)\right| \leq \frac{\Gamma(p+1)}{\Gamma(p-l+1)}\,|z|^{p-l}\left[1+\frac{\delta(\lambda p+\alpha-\gamma)p^n}{[p+(p(\beta-\mu)+1)t]^n(1+\delta\lambda)(p-l+1)}\,|z|\right],$$

and

$$\left|D_z^l f(z)\right| \geq \frac{\Gamma(p+1)}{\Gamma(p-l+1)}\,|z|^{p-l}\left[1-\frac{\delta(\lambda p+\alpha-\gamma)p^n}{[p+(p(\beta-\mu)+1)t]^n(1+\delta\lambda)(p-l+1)}\,|z|\right]. \tag{54}$$

Proof. If $f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k$, then

$$D_z^l f(z) = \frac{\Gamma(p+1)}{\Gamma(p-l+1)}z^{p-l} - \sum_{k=p+1}^{\infty}\frac{\Gamma(k+1)}{\Gamma(k-l+1)}a_k z^{k-l},$$

implies

$$\frac{\Gamma(p-l+1)}{\Gamma(p+1)}z^l D_z^l f(z) = z^p - \sum_{k=p+1}^{\infty}\frac{\Gamma(k+1)\Gamma(p-l+1)}{\Gamma(p+1)\Gamma(k-l+1)}a_k z^k = z^p - \sum_{k=p+1}^{\infty}\Psi(k)a_k z^k, \tag{55}$$

where

$$\Psi(k) = \frac{\Gamma(p-l+1)\Gamma(k+1)}{\Gamma(p+1)\Gamma(k-l+1)}. \tag{56}$$

Clearly, Ψ is a decreasing function of k and we get

$$0 < \Psi(k) \leq \Psi(p+1) = \frac{p+1}{p-l+1}.$$

Using (51) and (56), we obtain,

$$\left|\frac{\Gamma(p-l+1)}{\Gamma(p+1)}z^l D_z^l f(z)\right| \leq |z|^p + \psi(p+1)\,|z|^{p+1}\sum_{k=p+1}^{\infty} a_k$$
$$\leq |z|^p + \frac{\delta(\lambda p+\alpha-\gamma)p^n}{[p+(p(\beta-\mu)+1)t]^n(1+\delta\lambda)(p-l+1)}\,|z|^{p+1}\,,$$

which is equivalent to assertion (54).

Similarly,

$$\left|\frac{\Gamma(p-l+1)}{\Gamma(p+1)}z^l D_z^l f(z)\right| \geq |z|^p - \Psi(p+1)\,|z|^{p+1}\sum_{k=p+1}^{\infty} a_k$$
$$\geq |z|^p - \frac{\delta(\lambda p+\alpha-\gamma)p^n}{[p+(p(\beta-\mu)+1)t]^n(1+\delta\lambda)(p-l+1)}\,|z|^{p+1}\,,$$

which completes the proof. □

Corollary 5. *If $f(z) \in S_p^n(\lambda, \alpha, \gamma, \delta)$, then we have*

$$|z|^p - \frac{\delta(\lambda p + \alpha - \gamma)p^n}{[p + (p(\beta - \mu) + 1)t]^n (1 + \delta\lambda)(p + 1)} |z|^{p+1} \leq |f(z)| \leq |z|^p + \frac{\delta(\lambda p + \alpha - \gamma)p^n}{[p + (p(\beta - \mu) + 1)t]^n (1 + \delta\lambda)(p + 1)} |z|^{p+1} . \tag{57}$$

Proof. From Definition 4, we have

$$\lim_{l \to 0} D_z^{-l} f(z) = f(z) .$$

Therefore, setting $l = 0$ in (49), we obtain

$$\left| D_z^0 f(z) \right| \leq \frac{\Gamma(p+1)}{\Gamma(p+0+1)} |z|^{p+0} \left[1 + \frac{\delta(\lambda p + \alpha - \gamma)p^n}{[p + (p(\beta - \mu) + 1)t]^n (1 + \delta\lambda)(p + 0 + 1)} |z| \right] ,$$

and

$$\left| D_z^0 f(z) \right| \geq \frac{\Gamma(p+1)}{\Gamma(p+0+1)} |z|^{p+0} \left[1 - \frac{\delta(\lambda p + \alpha - \gamma)p^n}{[p + (p(\beta - \mu) + 1)t]^n (1 + \delta\lambda)(p + 0 + 1)} |z| \right] ,$$

i.e.,

$$|z|^p - \frac{\delta(\lambda p + \alpha - \gamma)p^n}{[p + (p(\beta - \mu) + 1)t]^n (1 + \delta\lambda)(p + 1)} |z|^{p+1} \leq |f(z)| \leq |z|^p + \frac{\delta(\lambda p + \alpha - \gamma)p^n}{[p + (p(\beta - \mu) + 1)t]^n (1 + \delta\lambda)(p + 1)} |z|^{p+1} ,$$

which is (57). □

Corollary 6. *If $f(z) \in S_p^n(\lambda, \alpha, \gamma, \delta)$, then we have*

$$p\,|z|^{p-1} - \frac{\delta(\lambda p + \alpha - \gamma)p^n}{[p + (p(\beta - \mu) + 1)t]^n (1 + \delta\lambda)} |z| \leq \left| f'(z) \right| \leq p\,|z|^{p-1} + \frac{\delta(\lambda p + \alpha - \gamma)p^n}{[p + (p(\beta - \mu) + 1)t]^n (1 + \delta\lambda)} |z|^p . \tag{58}$$

Proof. From Definition 5, we have

$$\lim_{l \to 1} D_z^l f(z) = f'(z) .$$

Therefore, setting $l = 1$ in (54), we obtain

$$\left| D_z^1 f(z) \right| \leq \frac{\Gamma(p+1)}{\Gamma(p)} |z|^{p-1} \left[1 + \frac{\delta(\lambda p + \alpha - \gamma)p^n}{[p + (p(\beta - \mu) + 1)t]^n (1 + \delta\lambda)(p)} |z| \right] ,$$

and

$$\left| D_z^1 f(z) \right| \geq \frac{\Gamma(p+1)}{\Gamma(p)} |z|^{p-1} \left[1 - \frac{\delta(\lambda p + \alpha - \gamma)p^n}{[p + (p(\beta - \mu) + 1)t]^n (1 + \delta\lambda)(p)} |z| \right] ,$$

i.e.,

$$p\,|z|^{p-1} - \frac{\delta(\lambda p + \alpha - \gamma)p^n}{[p + (p(\beta - \mu) + 1)t]^n (1 + \delta\lambda)} |z|^p \leq \left| f'(z) \right| \leq p\,|z|^{p-1} + \frac{\delta(\lambda p + \alpha - \gamma)p^n}{[p + (p(\beta - \mu) + 1)t]^n (1 + \delta\lambda)} |z|^p ,$$

which is (58). □

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