Open Journal of Mathematical Analysis

PSR Press

Article

On generalized Tetranacci numbers: Closed forms of the sum formulas $\sum_{k=0}^{n} kx^kW_k$ and $\sum_{k=1}^{n} kx^kW_{-k}$

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Communicated by: Absar Ul Haq
Received: 20 May 2022; Accepted: 28 September 2022; Published: 30 December 2022.

Abstract: In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} kx^kW_k$ and $\sum_{k=1}^{n} kx^kW_{-k}$ for generalized Tetranacci numbers are presented. As special cases, we give summation formulas of Tetranacci, Tetranacci-Lucas, and other fourth-order recurrence sequences.

Keywords: Tetranacci numbers; Tetranacci-Lucas numbers; fourth order Pell numbers; sum formulas; summing formulas.

MSC: 11B39; 11B83.

1. Introduction

There have been so many studies of the sequences of numbers in the literature which are defined recursively. Two of these types of sequences are the sequences of Tetranacci and Tetranacci-Lucas which are special cases of generalized Tetranacci numbers. A generalized Tetranacci sequence

$$\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3; r, s, t, u)\}_{n\geq 0}$$

is defined by the fourth-order recurrence relations

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}, \tag{1}$$

with the initial values W_0, W_1, W_2, W_3 are arbitrary complex (or real) numbers not all being zero and r, s, t, u are complex numbers.

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [1–6].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{t}{u}W_{-(n-1)} - \frac{s}{u}W_{-(n-2)} - \frac{r}{u}W_{-(n-3)} + \frac{1}{u}W_{-(n-4)},$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1) holds for all integer n.

For some specific values of W_0, W_1, W_2, W_3 and r, s, t, u, it is worth presenting these special Tetranacci numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s, t, u and initial values.

In literature, for example, the following names and notations (see Table 1) are used for the special case of r, s, t, u and initial values.

Table 1. A few special cases of generalized Tetranacci sequences

No	Sequences (Numbers)	Notation	OEIS [7]	Ref.
1	Tetranacci	$\{M_n\} = \{W_n(0,1,1,2;1,1,1,1)\}$	A000078	[8]
2	Tetranacci-Lucas	$\{R_n\} = \{W_n(4,1,3,7;1,1,1,1)\}$	A073817	[8]
3	fourth order Pell	$\{P_n^{(4)}\} = \{W_n(0,1,2,5;2,1,1,1)\}$	A103142	[9]
4	fourth order Pell-Lucas	$\{Q_n^{(4)}\} = \{W_n(4,2,6,17;2,1,1,1)\}$	A331413	[9]
5	modified fourth order Pell	$\{E_n^{(4)}\} = \{W_n(0,1,1,3;2,1,1,1)\}$	A190139	[9]
6	fourth order Jacobsthal	$\{J_n^{(4)}\} = \{W_n(0,1,1,1;1,1,1,2)\}$	A007909	[10]
7	fourth order Jacobsthal-Lucas	$\{j_n^{(4)}\} = \{W_n(2,1,5,10;1,1,1,2)\}$	A226309	[10]
8	modified fourth order Jacobsthal	$\{K_n^{(4)}\} = \{W_n(3,1,3,10;1,1,1,2)\}$		[10]
9	fourth-order Jacobsthal Perrin	$\{Q_n^{(4)}\} = \{W_n(3,0,2,8;1,1,1,2)\}$		[10]
10	adjusted fourth-order Jacobsthal	$\{S_n^{(4)}\} = \{W_n(0,1,1,2;1,1,1,2)\}$		[10]
11	modified fourth-order Jacobsthal-Lucas	$\{R_n^{(4)}\} = \{W_n(4,1,3,7;1,1,1,2)\}$		[10]
12	4-primes	$\{G_n\} = \{W_n(0,0,1,2;2,3,5,7)\}$		[11]
13	Lucas 4-primes	$\{H_n\} = \{W_n(4,2,10,41;2,3,5,7)\}$		[11]
14	modified 4-primes	$\{E_n\} = \{W_n(0,0,1,1;2,3,5,7)\}$		[11]

Here OEIS stands for On-line Encyclopedia of Integer Sequences. For easy writing, from now on, we drop the superscripts from the sequences, for example we write J_n for $J_n^{(4)}$.

We present some works on sum formulas of the numbers in the following Table 2.

Table 2. A few special studies of sum formulas

Name of sequence	Papers which deal with sum formulas
Pell and Pell-Lucas	[12–16]
Generalized Fibonacci	[17–23]
Generalized Tribonacci	[24–27]
Generalized Tetranacci	[6,24,28,29]
Generalized Pentanacci	[24,30,31]
Generalized Hexanacci	[32,33]

The following theorem present some linea sum formulas of generalized Tetranacci numbers with positive subscripts.

Theorem 1. [34, Theorem 1] *For $n \geq 0$ we have the following formulas:*

(a) *If $rx + sx^2 + tx^3 + ux^4 - 1 \neq 0$, then*

$$\sum_{k=0}^{n} x^k W_k = \frac{\Theta_1(x)}{rx + sx^2 + tx^3 + ux^4 - 1}.$$

(b) *If $r^2x + 2ux^2 - s^2x^2 + t^2x^3 - u^2x^4 + 2sx + 2rtx^2 - 2sux^3 - 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{\Theta_2(x)}{r^2x + 2ux^2 - s^2x^2 + t^2x^3 - u^2x^4 + 2sx + 2rtx^2 - 2sux^3 - 1}.$$

(c) *If $r^2x + 2ux^2 - s^2x^2 + t^2x^3 - u^2x^4 + 2sx + 2rtx^2 - 2sux^3 - 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{\Theta_3(x)}{r^2x + 2ux^2 - s^2x^2 + t^2x^3 - u^2x^4 + 2sx + 2rtx^2 - 2sux^3 - 1}$$

where $\Theta_1(x) = x^{n+3}W_{n+3} - x^{n+2}(rx-1)W_{n+2} - x^{n+1}\left(sx^2 + rx - 1\right)W_{n+1} + ux^{n+4}W_n - x^3W_3 + x^2(rx-1)W_2 + x(sx^2 + rx - 1)W_1 + (tx^3 + sx^2 + rx - 1)W_0,$

$\Theta_2(x) = x^{n+1}\left(-ux^2-sx+1\right)W_{2n+2} + x^{n+2}(t+rs+rux)W_{2n+1} + x^{n+2}(u+t^2x-u^2x^2+rt-sux)W_{2n} + ux^{n+2}(r+tx)W_{2n-1} - x^2(r+tx)W_3 + x(r^2x+ux^2+sx+rtx^2-1)W_2 - x^2(t+rux-stx)W_1 + (r^2x+ux^2-s^2x^2+t^2x^3+2sx+2rtx^2-sux^3-1)W_0,$
$\Theta_3(x) = x^{n+1}(r+tx)W_{2n+2} + x^{n+1}(s-s^2x+t^2x^2-u^2x^3+ux-2sux^2+rtx)W_{2n+1} + x^{n+1}(t+rux-stx)W_{2n} - ux^{n+1}(ux^2+sx-1)W_{2n-1} + x(ux^2+sx-1)W_3 - x^2(t+rs+rux)W_2 + (r^2x+ux^2-s^2x^2+2sx+rtx^2-sux^3-1)W_1 - ux^2(r+tx)W_0.$

The following theorem present some linear sum formulas of generalized Tetranacci numbers with negative subscripts.

Theorem 2. [34, Theorem 8] *Let x be a real or complex numbers. For $n \geq 1$ we have the following formulas:*

(a) *If $rx^3+sx^2+tx+u-x^4 \neq 0$, then*

$$\sum_{k=1}^{n} x^k W_{-k} = \frac{\Theta_4(x)}{rx^3+sx^2+tx+u-x^4}.$$

(b) *If $2sx^3+t^2x+2ux^2+r^2x^3-s^2x^2-u^2-x^4+2rtx^2-2sux \neq 0$ then*

$$\sum_{k=1}^{n} x^k W_{-2k} = \frac{x\Theta_5(x)}{2sx^3+t^2x+2ux^2+r^2x^3-s^2x^2-u^2-x^4+2rtx^2-2sux}.$$

(c) *If $2sx^3+t^2x+2ux^2+r^2x^3-s^2x^2-u^2-x^4+2rtx^2-2sux \neq 0$ then*

$$\sum_{k=1}^{n} x^k W_{-2k+1} = \frac{x\Theta_6(x)}{2sx^3+t^2x+2ux^2+r^2x^3-s^2x^2-u^2-x^4+2rtx^2-2sux}$$

where $\Theta_4(x) = -x^{n+1}W_{-n+3} + x^{n+1}(r-x)W_{-n+2} + x^{n+1}(s+rx-x^2)W_{-n+1} + x^{n+1}(t+rx^2+sx-x^3)W_{-n} + xW_3 - x(r-x)W_2 + x(-s-rx+x^2)W_1 + x(-t-rx^2-sx+x^3)W_0,$
$\Theta_5(x) = x^n(u+sx-x^2)W_{-2n+2} - x^n(ru+tx+rsx)W_{-2n+1} + x^n(2sx^2-s^2x+r^2x^2-su+ux-x^3+rtx)W_{-2n} - ux^n(t+rx)W_{-2n-1} + (t+rx)W_3 + (-u-r^2x-rt-sx+x^2)W_2 + (ru-st+tx)W_1 - (2sx^2-s^2x+r^2x^2-su+ux+t^2-x^3+2rtx)W_0,$
$\Theta_6(x) = -x^{n+1}(t+rx)W_{-2n+2} + x^{n+1}(u+r^2x+rt+sx-x^2)W_{-2n+1} - x^{n+1}(ru-st+tx)W_{-2n} + ux^n(u+sx-x^2)W_{-2n-1} + (-u-sx+x^2)W_3 + (ru+tx+rsx)W_2 + (-2sx^2+s^2x-r^2x^2+su-ux+x^3-rtx)W_1 + u(t+rx)W_0.$

In this work, we investigate linear summation formulas of generalized Tetranacci numbers.

2. Linear sum formulas of generalized Tetranacci numbers with positive subscripts

The following theorem present some linear sum formulas of generalized Tetranacci numbers with positive subscripts.

Theorem 3. *Let x be a real or complex non-zero numbers. For $n \geq 0$ we have the following formulas:*

(a) *If $sx^2+tx^3+ux^4+rx-1 \neq 0$ then*

$$\sum_{k=0}^{n} kx^k W_k = \frac{\Omega_1}{(sx^2+tx^3+ux^4+rx-1)^2}$$

where $\Omega_1 = x^{n+3}(n(sx^2+tx^3+ux^4+rx-1)+sx^2+2rx-ux^4-3)W_{n+3} + x^{n+2}(n(1-rx)(sx^2+tx^3+ux^4+rx-1)-2+4rx-tx^3-2ux^4-2r^2x^2-rsx^3+rux^5)W_{n+2} + x^{n+1}(-n(sx^2+rx-1)(sx^2+tx^3+ux^4+rx-1)-1+2sx^2-2tx^3-3ux^4-r^2x^2-s^2x^4+2rx-2rsx^3+rtx^4+2rux^5+sux^6)W_{n+1} + ux^{n+4}(n(sx^2+tx^3+ux^4+rx-1)-4+2sx^2+tx^3+3rx)W_n + x^3(-sx^2+ux^4-2rx+3)W_3 + x^2(tx^3+2ux^4+2r^2x^2-4rx+rsx^3-rux^5+2)W_2 + x(-2sx^2+2tx^3+3ux^4+r^2x^2+s^2x^4-2rx+2rsx^3-rtx^4-2rux^5-sux^6+1)W_1 - ux^4(2sx^2+tx^3+3rx-4)W_0.$

(b) *If $r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2-2sux^3-1 \neq 0$ then*

$$\sum_{k=0}^{n} kx^k W_{2k} = \frac{\Omega_2}{(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2-2sux^3-1)^2}$$

where $\Omega_2 = x^{n+1}(-n(ux^2+sx-1)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2-2sux^3-1)-1-s^2x^2-2t^2x^3+u^2x^4-u^3x^6+2sx-2rtx^2-r^2sx^2-2r^2ux^3+st^2x^4-s^2ux^4-2su^2x^5-2rtux^4+ux^2)W_{2n+2}+x^{n+2}(n(t+rs+rux)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2-2sux^3-1)+2rs^2x-t^3x^3-2rs-2t+r^3sx+r^2tx+2ru^2x^3+2r^3ux^2+ru^3x^5+2tu^2x^4-3rux+2stx+4rsux^2+2stux^3-rst^2x^3+rs^2ux^3+2rsu^2x^4+2r^2tux^3)W_{2n+1}+u\,x^{n+2}(n(r+tx)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2-2sux^3-1)+r^3x-2r-3tx+4stx^2+2tux^3+2r^2tx^2+rt^2x^3-s^2tx^3+2ru^2x^4+tu^2x^5+2rsx+2rsux^3)W_{2n-1}+x^{n+2}(n(u+t^2x-u^2x^2+rt-sux)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2-2sux^3-1)+4u^2x^2-3t^2x-2u-2u^3x^4-2rt+2r^2t^2x^2-3r^2u^2x^3-s^2t^2x^3+2s^2u^2x^4+r^3tx+r^2ux+rt^3x^3+4st^2x^2-4s^2ux^2-6su^2x^3+s^3ux^3+t^2ux^3+su^3x^5+5sux-2r^2sux^2-2rtu^2x^4+2rstx)W_{2n}+x^2(2r-r^3x+3tx-4stx^2-2tux^3-2r^2tx^2-rt^2x^3+s^2tx^3-2ru^2x^4-tu^2x^5-2rsx-2rsux^3)W_3+x(-2r^2x-ux^2+r^4x^2+s^2x^2+2t^2x^3-u^2x^4+u^3x^6-2sx+r^2t^2x^4+2r^2u^2x^5-rtx^2+3r^2sx^2+2r^3tx^3+2r^2ux^3-st^2x^4+s^2ux^4+2su^2x^5+4rstx^3+4rtux^4-rs^2tx^4+2r^2sux^4+rtu^2x^6+1)W_2+x^2(2t+t^3x^3-r^2tx+4s^2tx^2-2ru^2x^3-2r^3ux^2-s^3tx^3-ru^3x^5-2tu^2x^4+3rux-5stx-4rsux^2+2r^2stx^2+2rst^2x^3+rs^2ux^3-2r^2tux^3+stu^2x^5)W_1+ux^2(-r^2x-4ux^2+4s^2x^2-s^3x^3+t^2x^3+2u^2x^4-5sx+6sux^3+2r^2sx^2+3r^2ux^3-2s^2ux^4-su^2x^5+t^2ux^5+2rstx^3+4rtux^4+2)W_0.$

(c) *If* $r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2-2sux^3-1\neq 0$ *then*

$$\sum_{k=0}^{n} kx^kW_{2k+1} = \frac{\Omega_3}{(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2-2sux^3-1)^2}$$

where $\Omega_3 = +x^{n+1}(n(r+tx)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2-2sux^3-1)-t^3x^4-2tx-r-2rux^2+2stx^2+rs^2x^2-r^2tx^2-2rt^2x^3+3ru^2x^4+2tu^2x^5+4rsux^3+2stux^4)W_{2n+2}+x^{n+1}(n(s-s^2x+t^2x^2-u^2x^3+ux-2sux^2+rtx)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2-2sux^3-1)+2s^2x-s-s^3x^2-3t^2x^2+4u^2x^3-2u^3x^5-r^2s^2x^2+2r^2t^2x^3-3r^2u^2x^4+6sux^2+r^3tx^2+r^2ux^2+rt^3x^4+2st^2x^3-4s^2ux^3-5su^2x^4+t^2ux^4-2rtx-4r^2sux^3-2rtu^2x^5-2ux-2rstux^4)W_{2n+1}+x^{n+1}(n(t+rux-stx)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2-2sux^3-1)-2t^3x^3-2tux^2-t-2rt^2x^2-s^2tx^2+r^3ux^2+st^3x^4+2ru^3x^5+3tu^2x^4-2rux+2stx+2rsux^2+4stux^3-r^2stx^2+2rsu^2x^4-rt^2ux^4-2s^2tux^4-2stu^2x^5)W_{2n}+ux^{n+1}(-n(ux^2+sx-1)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2-2sux^3-1)-1-s^2x^2-2t^2x^3+u^2x^4-u^3x^6+2sx-r^2sx^2-2r^2ux^3+st^2x^4-s^2ux^4-2su^2x^5-2rtux^4+ux^2-2rtx^2)W_{2n-1}+x(-ux^2+s^2x^2+2t^2x^3-u^2x^4+u^3x^6-2sx+2rtx^2+r^2sx^2+2r^2ux^3-st^2x^4+s^2ux^4+2su^2x^5+2rtux^4+1)W_3+x^2(2t+t^3x^3+2rs-2rs^2x-r^3sx-r^2tx-2ru^2x^3-2r^3ux^2-ru^3x^5-2tu^2x^4+3rux-2stx-4rsux^2-2stux^3+rst^2x^3-rs^2ux^3-2rsu^2x^4-2r^2tux^3)W_2+x^2(2u+3t^2x-4u^2x^2+2u^3x^4+2rt-2r^2t^2x^2+3r^2u^2x^3+s^2t^2x^3-2s^2u^2x^4-r^3tx-r^2ux-rt^3x^3-4st^2x^2+4s^2ux^2+6su^2x^3-s^3ux^3-t^2ux^3-su^3x^5-5sux+2r^2sux^2+2rtu^2x^4-2rstx)W_1+ux^2(2r-r^3x+3tx-4stx^2-2tux^3-2r^2tx^2-rt^2x^3+s^2tx^3-2ru^2x^4-tu^2x^5-2rsx-2rsux^3)W_0.$

Proof. **(a)** Using the recurrence relation

$$W_n = rW_{n-1}+sW_{n-2}+tW_{n-3}+uW_{n-4}\,,$$

i.e.,

$$uW_{n-4} = W_n-rW_{n-1}-sW_{n-2}-tW_{n-3}\,,$$

we obtain

$$\begin{array}{rcl}
u\times 0\times x^0W_0 &=& 0\times x^0W_4-r\times 0\times x^0W_3-s\times 0\times x^0W_2-t\times 0\times x^0W_1,\\
u\times 1\times x^1W_1 &=& 1\times x^1W_5-r\times 1\times x^1W_4-s\times 1\times x^1W_3-t\times 1\times x^1W_2,\\
u\times 2\times x^2W_2 &=& 2\times x^2W_6-r\times 2\times x^2W_5-s\times 2\times x^2W_4-t\times 2\times x^2W_3,\\
u\times 3\times x^3W_3 &=& 3\times x^3W_7-r\times 3\times x^3W_6-s\times 3\times x^3W_5-t\times 3\times x^3W_4,\\
&&\vdots\\
u(n-4)x^{n-4}W_{n-4} &=& (n-4)x^{n-4}W_n-r(n-4)x^{n-4}W_{n-1}-s(n-4)x^{n-4}W_{n-2}-t(n-4)x^{n-4}W_{n-3},\\
u(n-3)x^{n-3}W_{n-3} &=& (n-3)x^{n-3}W_{n+1}-r(n-3)x^{n-3}W_n-s(n-3)x^{n-3}W_{n-1}-t(n-3)x^{n-3}W_{n-2},\\
u(n-2)x^{n-2}W_{n-2} &=& (n-2)x^{n-2}W_{n+2}-r(n-2)x^{n-2}W_{n+1}-s(n-2)x^{n-2}W_n-t(n-2)x^{n-2}W_{n-1},\\
u(n-1)x^{n-1}W_{n-1} &=& (n-1)x^{n-1}W_{n+3}-r(n-1)x^{n-1}W_{n+2}-s(n-1)x^{n-1}W_{n+1}-t(n-1)x^{n-1}W_n,
\end{array}$$

$$u\times n\times x^{n}W_{n} = u\times n\times x^{n}W_{n+4}-ru\times n\times x^{n}W_{n+3}-su\times n\times x^{n}W_{n+2}-tu\times n\times x^{n}W_{n+1}.$$

If we add the equations side by side we get
$u\sum_{k=0}^{n}kx^{k}W_{k}=(nx^{n}W_{n+4}+(n-1)x^{n-1}W_{n+3}+(n-2)x^{n-2}W_{n+2}+(n-3)x^{n-3}W_{n+1}-(-1)x^{-1}W_{3}-(-2)x^{-2}W_{2}-(-3)x^{-3}W_{1}-(-4)x^{-4}W_{0}+\sum_{k=0}^{n}kx^{k-4}W_{k}-4\sum_{k=0}^{n}x^{k-4}W_{k})-r(nx^{n}W_{n+3}+(n-1)x^{n-1}W_{n+2}+(n-2)x^{n-2}W_{n+1}-(-1)x^{-1}W_{2}-(-2)x^{-2}W_{1}-(-3)x^{-3}W_{0}+\sum_{k=0}^{n}kx^{k-3}W_{k}-3\sum_{k=0}^{n}x^{k-3}W_{k})-s(nx^{n}W_{n+2}+(n-1)x^{n-1}W_{n+1}-(-1)x^{-1}W_{1}-(-2)x^{-2}W_{0}+\sum_{k=0}^{n}kx^{k-2}W_{k}-2\sum_{k=0}^{n}x^{k-2}W_{k})-t(nx^{n}W_{n+1}-(-1)x^{-1}W_{0}+\sum_{k=0}^{n}kx^{k-1}W_{k}-\sum_{k=0}^{n}x^{k-1}W_{k}).$

Then if we denote $\sum_{k=0}^{n}x^{k}W_{k}$ and $\sum_{k=0}^{n}kx^{k}W_{k}$ as

$$A = \sum_{k=0}^{n}x^{k}W_{k},$$
$$a = \sum_{k=0}^{n}kx^{k}W_{k},$$

and use

$$W_{n+4}=rW_{n+3}+sW_{n+2}+tW_{n+1}+uW_{n},$$

we obtain
$ua = (nx^{n}(rW_{n+3}+sW_{n+2}+tW_{n+1}+uW_{n})+(n-1)x^{n-1}W_{n+3}+(n-2)x^{n-2}W_{n+2}+(n-3)x^{n-3}W_{n+1}-(-1)x^{-1}W_{3}-(-2)x^{-2}W_{2}-(-3)x^{-3}W_{1}-(-4)x^{-4}W_{0}+x^{-4}a-4x^{-4}A)-r(nx^{n}W_{n+3}+(n-1)x^{n-1}W_{n+2}+(n-2)x^{n-2}W_{n+1}-(-1)x^{-1}W_{2}-(-2)x^{-2}W_{1}-(-3)x^{-3}W_{0}+x^{-3}a-3x^{-3}A)-s(nx^{n}W_{n+2}+(n-1)x^{n-1}W_{n+1}-(-1)x^{-1}W_{1}-(-2)x^{-2}W_{0}+x^{-2}a-2x^{-2}A)-t(nx^{n}W_{n+1}-(-1)x^{-1}W_{0}+x^{-1}a-x^{-1}A).$

Using Theorem 1 (a) and solving the last equation for a, we get (a).

(b) and (c) Using the recurrence relation

$$W_{n}=rW_{n-1}+sW_{n-2}+tW_{n-3}+uW_{n-4}$$

i.e.,

$$rW_{n-1}=W_{n}-sW_{n-2}-tW_{n-3}-uW_{n-4},$$

we obtain

$$\begin{aligned}
r\times 1\times x^{1}W_{3} &= 1\times x^{1}W_{4}-s\times 1\times x^{1}W_{2}-t\times 1\times x^{1}W_{1}-u\times 1\times x^{1}W_{0},\\
r\times 2\times x^{2}W_{5} &= 2\times x^{2}W_{6}-s\times 2\times x^{2}W_{4}-t\times 2\times x^{2}W_{3}-u\times 2\times x^{2}W_{2},\\
r\times 3\times x^{3}W_{7} &= 3\times x^{3}W_{8}-s\times 3\times x^{3}W_{6}-t\times 3\times x^{3}W_{5}-u\times 3\times x^{3}W_{4},\\
r\times 4\times rx^{4}W_{9} &= 4\times rx^{4}W_{10}-s\times 4\times rx^{4}W_{8}-t\times 4\times rx^{4}W_{7}-u\times 4\times rx^{4}W_{6},\\
&\vdots\\
r(n-1)x^{n-1}W_{2n-1} &= (n-1)x^{n-1}W_{2n}-s(n-1)x^{n-1}W_{2n-2}-t(n-1)x^{n-1}W_{2n-3}-u(n-1)x^{n-1}W_{2n-4},\\
rnx^{n}W_{2n+1} &= nx^{n}W_{2n+2}-snx^{n}W_{2n}-tnx^{n}W_{2n-1}-unx^{n}W_{2n-2}.
\end{aligned}$$

Now, if we add the above equations side by side, we get

$$r(-0\times x^{0}W_{1}+\sum_{k=0}^{n}kx^{k}W_{2k+1})=(nx^{n}W_{2n+2}-0\times x^{0}W_{2}-(-1)x^{-1}W_{0}+\sum_{k=0}^{n}(k-1)x^{k-1}W_{2k})-s(-0\times x^{0}W_{0}$$
$$+\sum_{k=0}^{n}kx^{k}W_{2k})-t(-(n+1)x^{n+1}W_{2n+1}+\sum_{k=0}^{n}(k+1)x^{k+1}W_{2k+1})-u(-(n+1)x^{n+1}W_{2n}+\sum_{k=0}^{n}(k+1)x^{k+1}W_{2k}),$$

and so

$$r(-0\times x^0W_1+\sum_{k=0}^{n}kx^kW_{2k+1})=(nx^nW_{2n+2}-0\times x^0W_2-(-1)x^{-1}W_0+x^{-1}\sum_{k=0}^{n}kx^kW_{2k}-x^{-1}\sum_{k=0}^{n}x^kW_{2k})$$
$$-s(-0\times x^0W_0+\sum_{k=0}^{n}kx^kW_{2k})-t(-(n+1)x^{n+1}W_{2n+1}+x^1\sum_{k=0}^{n}kx^kW_{2k+1}+x^1\sum_{k=0}^{n}x^kW_{2k+1})$$
$$-u(-(n+1)x^{n+1}W_{2n}+x^1\sum_{k=0}^{n}kx^kW_{2k}+x^1\sum_{k=0}^{n}x^kW_{2k}). \tag{2}$$

Similarly, using the recurrence relation

$$W_n=rW_{n-1}+sW_{n-2}+tW_{n-3}+uW_{n-4},$$

i.e.,

$$rW_{n-1}=W_n-sW_{n-2}-tW_{n-3}-uW_{n-4},$$

we write the following obvious equations;

$$\begin{array}{rcl}
r\times 1\times x^1W_2 &=& 1\times x^1W_3-s\times 1\times x^1W_1-t\times 1\times x^1W_0-u\times 1\times x^1W_{-1},\\
r\times 2\times x^2W_4 &=& 2\times x^2W_5-s\times 2\times x^2W_3-t\times 2\times x^2W_2-u\times 2\times x^2W_1,\\
r\times 3\times x^3W_6 &=& 3\times x^3W_7-s\times 3\times x^3W_5-t\times 3\times x^3W_4-u\times 3\times x^3W_3,\\
r\times 8\times x^4W_8 &=& 4\times x^4W_9-s\times 8\times x^4W_7-t\times 8\times x^4W_6-u\times 8\times x^4W_5,\\
&\vdots&\\
r(n-1)x^{n-1}W_{2n-2} &=& (n-1)x^{n-1}W_{2n-1}-s(n-1)x^{n-1}W_{2n-3}-t(n-1)x^{n-1}W_{2n-4}-u(n-1)x^{n-1}W_{2n-5},\\
rnx^nW_{2n} &=& nx^nW_{2n+1}-snx^nW_{2n-1}-tnx^nW_{2n-2}-unx^nW_{2n-3},\\
r(n+1)x^{n+1}W_{2n+2} &=& (n+1)x^{n+1}W_{2n+3}-s(n+1)x^{n+1}W_{2n+1}-t(n+1)x^{n+1}W_{2n}-u(n+1)x^{n+1}W_{2n-1}.
\end{array}$$

Now, if we add the above equations side by side, we obtain

$$r(-0\times x^0W_0+\sum_{k=0}^{n}kx^kW_{2k})=(-0\times x^0W_1+\sum_{k=0}^{n}kx^kW_{2k+1})-s(-(n+1)x^{n+1}W_{2n+1}+\sum_{k=0}^{n}(k+1)x^{k+1}W_{2k+1})$$
$$-t(-(n+1)x^{n+1}W_{2n}+\sum_{k=0}^{n}(k+1)x^{k+1}W_{2k})-u(-(n+2)x^{n+2}W_{2n+1}-(n+1)x^{n+1}W_{2n-1}+1\times x^1W_{-1}$$
$$+\sum_{k=0}^{n}(k+2)x^{k+2}W_{2k+1}).$$

Since

$$W_{-1}=-\frac{t}{u}W_0-\frac{s}{u}W_1-\frac{r}{u}W_2+\frac{1}{u}W_3,$$

we have

$$r(-0\times x^0W_0+\sum_{k=0}^{n}kx^kW_{2k})=(-0\times x^0W_1+\sum_{k=0}^{n}kx^kW_{2k+1})-s(-(n+1)x^{n+1}W_{2n+1}+x^1\sum_{k=0}^{n}kx^kW_{2k+1}$$
$$+x^1\sum_{k=0}^{n}x^kW_{2k+1})-t(-(n+1)x^{n+1}W_{2n}+x^1\sum_{k=0}^{n}kx^kW_{2k}+x^1\sum_{k=0}^{n}x^kW_{2k})-u(-(n+2)x^{n+2}W_{2n+1}$$
$$-(n+1)x^{n+1}W_{2n-1}+1\times x^1(-\frac{t}{u}W_0-\frac{s}{u}W_1-\frac{r}{u}W_2+\frac{1}{u}W_3)+x^2\sum_{k=0}^{n}kx^kW_{2k+1}+2x^2\sum_{k=0}^{n}x^kW_{2k+1}). \tag{3}$$

Then, solving the system (2)-(3) (using Theorem 1 (b) and (c)), the required result of (b) and (c) follow. In fact, if we denote

$$a=\sum_{k=0}^{n}kx^kW_{2k},$$

$$\begin{aligned} b &= \sum_{k=0}^{n} kx^k W_{2k+1}, \\ f &= \sum_{k=0}^{n} x^k W_{2k}, \\ g &= \sum_{k=0}^{n} x^k W_{2k+1}, \end{aligned}$$

(2) and (3) can be written as follows:

$$\begin{aligned} r(-0\times x^0W_1+b) &= (nx^nW_{2n+2}-0\times x^0W_2-(-1)x^{-1}W_0+x^{-1}a-x^{-1}f)-s(-0\times x^0W_0+a) \\ &\quad -t(-(n+1)x^{n+1}W_{2n+1}+x^1b+x^1g)-u(-(n+1)x^{n+1}W_{2n}+x^1a+x^1f), \\ r(-0\times x^0W_0+a) &= (-0\times x^0W_1+b)-s(-(n+1)x^{n+1}W_{2n+1}+x^1b+x^1g)-t(-(n+1)x^{n+1}W_{2n}+x^1a+x^1f) \\ &\quad -u(-(n+2)x^{n+2}W_{2n+1}-(n+1)x^{n+1}W_{2n-1}+1\times x^1(-\frac{t}{u}W_0-\frac{s}{u}W_1-\frac{r}{u}W_2+\frac{1}{u}W_3)+x^2b+2x^2g). \end{aligned}$$

Using Theorem 1 (b) and (c) and solving the last two simultaneous equations with respect to a and b, we get (b) and (c).

□

Remark 1. Note that the proof of Theorem 3 can be done by taking the derivative of the formulas in Theorem 1. In fact, since

$$\begin{aligned} \sum_{k=0}^{n} x^k W_k &= \frac{\Theta_1(x)}{rx+sx^2+tx^3+ux^4-1}, \\ \sum_{k=0}^{n} x^k W_{2k} &= \frac{\Theta_2(x)}{r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2-2sux^3-1}, \\ \sum_{k=0}^{n} x^k W_{2k+1} &= \frac{\Theta_3(x)}{r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2-2sux^3-1}, \end{aligned}$$

by taking the derivative of the both sides of the above formulas with respect to x, we get

$$\begin{aligned} \sum_{k=0}^{n} kx^{k-1} W_k &= \frac{(rx+sx^2+tx^3+ux^4-1)\Theta_1'(x)-(4ux^3+3tx^2+2sx+r)\Theta_1(x)}{(rx+sx^2+tx^3+ux^4-1)^2}, \\ \sum_{k=0}^{n} kx^{k-1} W_{2k} &= \frac{\begin{array}{c}(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2-2sux^3-1)\Theta_2'(x) \\ -(r^2+4rtx-2s^2x-6sux^2+2s+3t^2x^2-4u^2x^3+4ux)\Theta_2(x)\end{array}}{(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2-2sux^3-1)^2}, \\ \sum_{k=0}^{n} kx^{k-1} W_{2k+1} &= \frac{\begin{array}{c}(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2-2sux^3-1)\Theta_3'(x) \\ -(r^2+4rtx-2s^2x-6sux^2+2s+3t^2x^2-4u^2x^3+4ux)\Theta_3(x)\end{array}}{(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2-2sux^3-1)^2}, \end{aligned}$$

i.e.,

$$\begin{aligned} \sum_{k=0}^{n} kx^{k} W_k &= x\frac{(rx+sx^2+tx^3+ux^4-1)\Theta_1'(x)-(4ux^3+3tx^2+2sx+r)\Theta_1(x)}{(rx+sx^2+tx^3+ux^4-1)^2}, \\ \sum_{k=0}^{n} kx^{k} W_{2k} &= x\frac{\begin{array}{c}(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2-2sux^3-1)\Theta_2'(x) \\ -(r^2+4rtx-2s^2x-6sux^2+2s+3t^2x^2-4u^2x^3+4ux)\Theta_2(x)\end{array}}{(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2-2sux^3-1)^2}, \\ \sum_{k=0}^{n} kx^{k} W_{2k+1} &= x\frac{\begin{array}{c}(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2-2sux^3-1)\Theta_3'(x) \\ -(r^2+4rtx-2s^2x-6sux^2+2s+3t^2x^2-4u^2x^3+4ux)\Theta_3(x)\end{array}}{(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2-2sux^3-1)^2}, \end{aligned}$$

where $\Theta_1^{'}(x)$, $\Theta_2^{'}(x)$ and $\Theta_3^{'}(x)$ denotes the derivatives of $\Theta_1(x)$, $\Theta_2(x)$ and $\Theta_1(x)$ respectively.

3. Special Cases

In this section, for the special cases of x, we present the closed form solutions (identities) of the sums $\sum_{k=0}^{n} kx^kW_k$, $\sum_{k=0}^{n} kx^kW_{2k}$ and $\sum_{k=0}^{n} kx^kW_{2k+1}$ for the specific case of sequence $\{W_n\}$.

3.1. The case $x = 1$

In this subsection we consider the special case $x = 1$.

The case $x = 1$ of Theorem 3 is given in Soykan [34].

We only consider the case $x = 1, r = 1, s = 1, t = 1, u = 2$ (which is not considered in [34]).

Observe that setting $x = 1, r = 1, s = 1, t = 1, u = 2$ (i.e., for the generalized fourth order Jacobsthal sequence case) in Theorem 3 (b), (c) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule (twice) however provides the evaluation of the sum formulas.

Theorem 4. *If $r = 1, s = 1, t = 1, u = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} kW_k = \frac{1}{16}((4n-2)W_{n+3} - 4W_{n+2} - (4n+2)W_{n+1} + 2(4n+2)W_n + 2W_3 + 4W_2 + 2W_1 - 4W_0)$.

(b) $\sum_{k=0}^{n} kW_{2k} = \frac{1}{288}(4(6n^2+8n-169)W_{2n+2} + 8\left(-6n^2+4n+159\right)W_{2n+1} + 4(6n^2+44n-151)W_{2n} + 8(-6n^2+4n+159)W_{2n-1} - 636W_3 + 1312W_2 - 636W_1 + 1240W_0)$.

(c) $\sum_{k=0}^{n} kW_{2k+1} = \frac{1}{288}(4(-6n^2+16n+149)W_{2n+2} + 16(3n^2+13n-80)W_{2n+1} + 4(-6n^2+16n+149)W_{2n} + 8(6n^2+8n-169)W_{2n-1} + 676W_3 + 604W_1 - 1272W_2 - 1272W_0)$.

Proof. **(a)** We use Theorem 3 (a). If we set $x = 1, r = 1, s = 1, t = 1, u = 2$ in Theorem 3 (a) we get (a).

(b) We use Theorem 3 (b). If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 3 (b) then we have

$$\sum_{k=0}^{n} kx^kW_{2k} = \frac{g_1(x)}{(4x^4+3x^3-5x^2-3x+1)^2},$$

where $g_1(x) = -x^{n+1}(2x^2-2x+6x^3+x^4+8x^5+8x^6-n(2x^2+x-1)(4x^4+3x^3-5x^2-3x+1)+1)W_{2n+2} + x^{n+2}(12x^2+16x^3+16x^4+8x^5-n(2x+2)(4x^4+3x^3-5x^2-3x+1)-4)W_{2n+1} + x^{n+2}(12x+10x^2-32x^3-16x^4+8x^5+n(4x^2+x-3)(4x^4+3x^3-5x^2-3x+1)-6)W_{2n} + 2x^{n+2}(6x^2+8x^3+8x^4+4x^5-n(x+1)(4x^4+3x^3-5x^2-3x+1)-2)W_{2n-1} - x^2(4x^5+8x^4+8x^3+6x^2-2)W_3 + x(12x^6+16x^5+9x^4+12x^3+2x^2-4x+1)W_2 - x^2(4x^5+8x^4+8x^3+6x^2-2)W_1 - 2x^2(2x^5-12x^4-20x^3+2x^2+6x-2)W_0$. For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (b) using

$$\sum_{k=0}^{n} kW_{2k} = \left.\frac{\frac{d^2}{dx^2}(g_1(x))}{\frac{d^2}{dx^2}((4x^4+3x^3-5x^2-3x+1)^2)}\right|_{x=1} = \frac{1}{288}(4(6n^2+8n-169)W_{2n+2} + 8\left(-6n^2+4n+159\right)W_{2n+1} + 4(6n^2+44n-151)W_{2n} + 8(-6n^2+4n+159)W_{2n-1} - 636W_3 + 1312W_2 - 636W_1 + 1240W_0).$$

(c) We use Theorem 3 (c). If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 3 (c) then we have

$$\sum_{k=0}^{n} kx^kW_{2k+1} = \frac{g_2(x)}{(4x^4+3x^3-5x^2-3x+1)^2},$$

where $g_2(x) = -x^{n+1}(2x+2x^2-6x^3-15x^4-8x^5+n(x+1)(4x^4+3x^3-5x^2-3x+1)+1)W_{2n+2} - x^{n+1}(4x-10x^2-4x^3+33x^4+24x^5-n(4x^3+3x^2-2x-1)(4x^4+3x^3-5x^2-3x+1)+1)W_{2n+1} - x^{n+1}W_{2n}(2x+2x^2-6x^3-15x^4-8x^5+n(x+1)(4x^4+3x^3-5x^2-3x+1)+1) - 2x^{n+1}(2x^2-2x+6x^3+x^4+8x^5+8x^6-n(2x^2+x-1)(4x^4+3x^3-5x^2-3x+1)+1)W_{2n-1} + x(8x^6+8x^5+x^4+6x^3+2x^2-2x+1)W_3 - x^2(8x^5+16x^4+16x^3+12x^2-4)W_2 - x^2(8x^5-16x^4-32x^3+10x^2+12x-6)W_1 - 2x^2(4x^5+8x^4+8x^3+6x^2-2)W_0$. For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (c) using

$$\sum_{k=0}^{n} kW_{2k+1} = \left.\frac{\frac{d^2}{dx^2}(g_2(x))}{\frac{d^2}{dx^2}((4x^4+3x^3-5x^2-3x+1)^2)}\right|_{x=1} = \frac{1}{288}(4(-6n^2+16n+149)W_{2n+2} + 16(3n^2+13n-80)W_{2n+1} + 4(-6n^2+16n+149)W_{2n} + 8(6n^2+8n-169)W_{2n-1} + 676W_3 + 604W_1 - 1272W_2 - 1272W_0).$$

□

Taking $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1$ in the last theorem, we have the following corollary which presents linear sum formulas of the fourth-order Jacobsthal numbers.

Corollary 1. *For $n \geq 0$, fourth order Jacobsthal numbers have the following property:*

(a) $\sum_{k=0}^{n} kJ_k = \frac{1}{16}((4n-2)J_{n+3} - 4J_{n+2} - (4n+2)J_{n+1} + 2(4n+2)J_n + 8)$.

(b) $\sum_{k=0}^{n} kJ_{2k} = \frac{1}{288}(4(6n^2+8n-169)J_{2n+2} + 8\left(-6n^2+4n+159\right)J_{2n+1} + 4(6n^2+44n-151)J_{2n} + 8(-6n^2+4n+159)J_{2n-1} + 40)$.

(c) $\sum_{k=0}^{n} kJ_{2k+1} = \frac{1}{288}(4(-6n^2+16n+149)J_{2n+2} + 16(3n^2+13n-80)J_{2n+1} + 4(-6n^2+16n+149)J_{2n} + 8(6n^2+8n-169)J_{2n-1} + 8)$.

From the last theorem, we have the following corollary which gives linear sum formula of the fourth order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10$).

Corollary 2. *For $n \geq 0$, fourth order Jacobsthal-Lucas numbers have the following property:*

(a) $\sum_{k=0}^{n} kj_k = \frac{1}{16}((4n-2)j_{n+3} - 4j_{n+2} - (4n+2)j_{n+1} + 2(4n+2)j_n + 34)$.

(b) $\sum_{k=0}^{n} kj_{2k} = \frac{1}{288}(4(6n^2+8n-169)j_{2n+2} + 8\left(-6n^2+4n+159\right)j_{2n+1} + 4(6n^2+44n-151)j_{2n} + 8(-6n^2+4n+159)j_{2n-1} + 2044)$.

(c) $\sum_{k=0}^{n} kj_{2k+1} = \frac{1}{288}(4(-6n^2+16n+149)j_{2n+2} + 16(3n^2+13n-80)j_{2n+1} + 4(-6n^2+16n+149)j_{2n} + 8(6n^2+8n-169)j_{2n-1} - 1540)$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10$ in the last theorem, we have the following corollary which presents linear sums formula of the modified fourth order Jacobsthal numbers.

Corollary 3. *For $n \geq 0$, modified fourth order Jacobsthal numbers have the following property:*

(a) $\sum_{k=0}^{n} kK_k = \frac{1}{16}((4n-2)K_{n+3} - 4K_{n+2} - (4n+2)K_{n+1} + 2(4n+2)K_n + 22)$.

(b) $\sum_{k=0}^{n} kK_{2k} = \frac{1}{288}(4(6n^2+8n-169)K_{2n+2} + 8\left(-6n^2+4n+159\right)K_{2n+1} + 4(6n^2+44n-151)K_{2n} + 8(-6n^2+4n+159)K_{2n-1} + 660)$.

(c) $\sum_{k=0}^{n} kK_{2k+1} = \frac{1}{288}(4(-6n^2+16n+149)K_{2n+2} + 16(3n^2+13n-80)K_{2n+1} + 4(-6n^2+16n+149)K_{2n} + 8(6n^2+8n-169)K_{2n-1} - 268)$.

From the last theorem, we have the following corollary which gives linear sums formula of the fourth-order Jacobsthal Perrin numbers (take $W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8$).

Corollary 4. *For $n \geq 0$, fourth-order Jacobsthal Perrin numbers have the following property:*

(a) $\sum_{k=0}^{n} kQ_k = \frac{1}{16}((4n-2)Q_{n+3} - 4Q_{n+2} - (4n+2)Q_{n+1} + 2(4n+2)Q_n + 12)$.

(b) $\sum_{k=0}^{n} kQ_{2k} = \frac{1}{288}(4(6n^2+8n-169)Q_{2n+2} + 8\left(-6n^2+4n+159\right)Q_{2n+1} + 4(6n^2+44n-151)Q_{2n} + 8(-6n^2+4n+159)Q_{2n-1} + 1256)$.

(c) $\sum_{k=0}^{n} kQ_{2k+1} = \frac{1}{288}(4(-6n^2+16n+149)Q_{2n+2} + 16(3n^2+13n-80)Q_{2n+1} + 4(-6n^2+16n+149)Q_{2n} + 8(6n^2+8n-169)Q_{2n-1} - 952)$.

Taking $W_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2$ in the theorem, we have the following corollary which presents linear sum formula of the adjusted fourth-order Jacobsthal numbers.

Corollary 5. *For $n \geq 0$, adjusted fourth-order Jacobsthal numbers have the following property:*

(a) $\sum\limits_{k=0}^{n} kS_k = \frac{1}{16}((4n-2)S_{n+3} - 4S_{n+2} - (4n+2)S_{n+1} + 2(4n+2)S_n + 10)$.

(b) $\sum\limits_{k=0}^{n} kS_{2k} = \frac{1}{288}(4(6n^2+8n-169)S_{2n+2} + 8\left(-6n^2+4n+159\right)S_{2n+1} + 4(6n^2+44n-151)S_{2n} + 8(-6n^2+4n+159)S_{2n-1} - 596)$.

(c) $\sum\limits_{k=0}^{n} kS_{2k+1} = \frac{1}{288}(4(-6n^2+16n+149)S_{2n+2} + 16(3n^2+13n-80)S_{2n+1} + 4(-6n^2+16n+149)S_{2n} + 8(6n^2+8n-169)S_{2n-1} + 684)$.

From the last theorem, we have the following corollary which gives linear sum formulas of the modified fourth-order Jacobsthal-Lucas numbers (take $W_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7$).

Corollary 6. *For $n \geq 0$, modified fourth-order Jacobsthal-Lucas numbers have the following property:*

(a) $\sum\limits_{k=0}^{n} kR_k = \frac{1}{16}((4n-2)R_{n+3} - 4R_{n+2} - (4n+2)R_{n+1} + 2(4n+2)R_n + 12)$.

(b) $\sum\limits_{k=0}^{n} kR_{2k} = \frac{1}{288}(4(6n^2+8n-169)R_{2n+2} + 8\left(-6n^2+4n+159\right)R_{2n+1} + 4(6n^2+44n-151)R_{2n} + 8(-6n^2+4n+159)R_{2n-1} + 3808)$.

(c) $\sum\limits_{k=0}^{n} kR_{2k+1} = \frac{1}{288}(4(-6n^2+16n+149)R_{2n+2} + 16(3n^2+13n-80)R_{2n+1} + 4(-6n^2+16n+149)R_{2n} + 8(6n^2+8n-169)R_{2n-1} - 3568)$.

3.2. The case $x = -1$

In this subsection we consider the special case $x = -1$ and we present the closed form solutions (identities) of the sums $\sum\limits_{k=0}^{n} k(-1)^k kW_k$, $\sum\limits_{k=0}^{n} k(-1)^k W_{2k}$ and $\sum\limits_{k=0}^{n} k(-1)^k W_{2k+1}$ for the specific case of the sequence $\{W_n\}$.

Taking $x = -1, r = s = t = u = 1$ in Theorem 3 (a), (b) and (c), we obtain the following proposition.

Proposition 1. *If $x = -1, r = s = t = u = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum\limits_{k=0}^{n} k(-1)^k W_k = (-1)^n((n+5)W_{n+3} - (2n+9)W_{n+2} + (n+2)W_{n+1} - (n+6)W_n) - 5W_3 + 9W_2 - 2W_1 + 6W_0$.

(b) $\sum\limits_{k=0}^{n} k(-1)^k W_{2k} = (-1)^n((n+2)W_{2n+2} - (n+3)W_{2n+1} - (n+2)W_{2n} + W_{2n-1}) - W_3 - W_2 + 4W_1 + 3W_0$.

(c) $\sum\limits_{k=0}^{n} k(-1)^k W_{2k+1} = (-1)^n(-W_{2n+2} + (n+3)W_{2n} + (n+2)W_{2n-1}) - 2W_3 + 3W_2 + 2W_1 - W_0$.

From the above proposition, we have the following corollary which gives linear sum formulas of Tetranacci numbers (take $W_n = M_n$ with $M_0 = 0, M_1 = 1, M_2 = 1, M_3 = 2$).

Corollary 7. *For $n \geq 0$, Tetranacci numbers have the following properties.*

(a) $\sum\limits_{k=0}^{n} k(-1)^k M_k = (-1)^n((n+5)M_{n+3} - (2n+9)M_{n+2} + (n+2)M_{n+1} - (n+6)M_n) - 3$.

(b) $\sum\limits_{k=0}^{n} k(-1)^k M_{2k} = (-1)^n((n+2)M_{2n+2} - (n+3)M_{2n+1} - (n+2)M_{2n} + M_{2n-1}) + 1$.

(c) $\sum\limits_{k=0}^{n} k(-1)^k M_{2k+1} = (-1)^n(-M_{2n+2} + (n+3)M_{2n} + (n+2)M_{2n-1}) + 1$.

Taking $W_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7$ in the above proposition, we have the following corollary which presents linear sum formulas of Tetranacci-Lucas numbers.

Corollary 8. *For $n \geq 0$, Tetranacci-Lucas numbers have the following properties.*

(a) $\sum\limits_{k=0}^{n} k(-1)^k R_k = (-1)^n ((n+5)R_{n+3} - (2n+9)R_{n+2} + (n+2)R_{n+1} - (n+6)R_n) + 14.$

(b) $\sum\limits_{k=0}^{n} k(-1)^k R_{2k} = (-1)^n ((n+2)R_{2n+2} - (n+3)R_{2n+1} - (n+2)R_{2n} + R_{2n-1}) + 6.$

(c) $\sum\limits_{k=0}^{n} k(-1)^k R_{2k+1} = (-1)^n (-R_{2n+2} + (n+3)R_{2n} + (n+2)R_{2n-1}) - 7.$

Taking $x = -1, r = 2, s = t = u = 1$ in Theorem 3 (a), (b) and (c), we obtain the following proposition.

Proposition 2. *If $x = -1, r = 2, s = t = u = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum\limits_{k=0}^{n} k(-1)^k W_k = \frac{1}{4}((-1)^n ((2n+7)W_{n+3} - (6n+19)W_{n+2} + (4n+6)W_{n+1} - (2n+9)W_n) - 7W_3 + 19W_2 - 6W_1 + 9W_0).$

(b) $\sum\limits_{k=0}^{n} k(-1)^k W_{2k} = \frac{1}{4}((-1)^n ((2n+3)W_{2n+2} - (2n+3)W_{2n+1} - (4n+10)W_{2n} - (2n+5)W_{2n-1}) + 5W_3 - 13W_2 - 2W_1 + 5W_0).$

(c) $\sum\limits_{k=0}^{n} k(-1)^k W_{2k+1} = \frac{1}{4}((-1)^n ((2n+3)W_{2n+2} - (2n+7)W_{2n+1} - 2W_{2n} + (2n+3)W_{2n-1}) - 3W_3 + 3W_2 + 10W_1 + 5W_0).$

From the last proposition, we have the following corollary which gives linear sum formulas of the fourth-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5$).

Corollary 9. *For $n \geq 0$, fourth-order Pell numbers have the following properties:*

(a) $\sum\limits_{k=0}^{n} k(-1)^k P_k = \frac{1}{4}((-1)^n ((2n+7)P_{n+3} - (6n+19)P_{n+2} + (4n+6)P_{n+1} - (2n+9)P_n) - 3).$

(b) $\sum\limits_{k=0}^{n} k(-1)^k P_{2k} = \frac{1}{4}((-1)^n ((2n+3)P_{2n+2} - (2n+3)P_{2n+1} - (4n+10)P_{2n} - (2n+5)P_{2n-1}) - 3).$

(c) $\sum\limits_{k=0}^{n} k(-1)^k P_{2k+1} = \frac{1}{4}((-1)^n ((2n+3)P_{2n+2} - (2n+7)P_{2n+1} - 2P_{2n} + (2n+3)P_{2n-1}) + 1).$

Taking $W_n = Q_n$ with $Q_0 = 4, Q_1 = 2, Q_2 = 6, Q_3 = 17$ in the last proposition, we have the following corollary which presents linear sum formulas of the fourth-order Pell-Lucas numbers.

Corollary 10. *For $n \geq 0$, fourth-order Pell-Lucas numbers have the following properties:*

(a) $\sum\limits_{k=0}^{n} k(-1)^k Q_k = \frac{1}{4}((-1)^n ((2n+7)Q_{n+3} - (6n+19)Q_{n+2} + (4n+6)Q_{n+1} - (2n+9)Q_n) + 19).$

(b) $\sum\limits_{k=0}^{n} k(-1)^k Q_{2k} = \frac{1}{4}((-1)^n ((2n+3)Q_{2n+2} - (2n+3)Q_{2n+1} - (4n+10)Q_{2n} - (2n+5)Q_{2n-1}) + 23).$

(c) $\sum\limits_{k=0}^{n} k(-1)^k Q_{2k+1} = \frac{1}{4}((-1)^n ((2n+3)Q_{2n+2} - (2n+7)Q_{2n+1} - 2Q_{2n} + (2n+3)Q_{2n-1}) + 7).$

Observe that setting $x = -1, r = 1, s = 1, t = 1, u = 2$ (i.e., for the generalized fourth order Jacobsthal case) in Theorem 3 (a), (b) and (c), makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule however provides the evaluation of the sum formulas.

Theorem 5. *If $r = 1, s = 1, t = 1, u = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum\limits_{k=0}^{n} k(-1)^k W_k = \frac{1}{36}((-1)^n (-(3n^2+5n-53)W_{n+3} + 2(3n^2+2n-54)W_{n+2} - (3n^2-13n-53)W_{n+1} + 2(3n^2+11n-45)W_n) - 53W_3 + 108W_2 - 53W_1 + 90W_0).$

(b) $\sum\limits_{k=0}^{n} k(-1)^k W_{2k} = \frac{1}{100}((-1)^n (-(15n^2-4n-159)W_{2n+2} + 2(5n^2+2n-54)W_{2n+1} + (35n^2+54n-350)W_{2n} + 2(5n^2+2n-54)W_{2n-1}) + 54W_3 - 213W_2 + 54W_1 + 296W_0).$

(c) $\sum\limits_{k=0}^{n} k(-1)^k W_{2k+1} = \frac{1}{100}((-1)^n (-(5n^2-8n-51)W_{2n+2} + (20n^2+58n-191)W_{2n+1} - (5n^2-8n-51)W_{2n} - 2(15n^2-4n-159)W_{2n-1}) - 159W_3 + 108W_2 + 350W_1 + 108W_0).$

Proof. **(a)** We use Theorem 3 (a). If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 3 (a) then we have

$$\sum_{k=0}^{n} x^k W_k = \frac{g_3(x)}{(2x-1)^2 (x+1)^2 (x^2+1)^2},$$

where $g_3(x) = x^{n+3}(2x + n(2x^4 + x^3 + x^2 + x - 1) + x^2 - 2x^4 - 3)W_{n+3} - x^{n+2}(2x^2 - 4x + 2x^3 + 4x^4 - 2x^5 + n(x - 1)(2x^4 + x^3 + x^2 + x - 1) + 2)W_{n+2} - x^{n+1}(4x^3 - x^2 - 2x + 6x^4 - 4x^5 - 2x^6 + n(x^2 + x - 1)(2x^4 + x^3 + x^2 + x - 1) + 1) W_{n+1} + 2x^{n+4}(3x + n(2x^4 + x^3 + x^2 + x - 1) + 2x^2 + x^3 - 4)W_n + x^3(2x^4 - x^2 - 2x + 3)W_3 + x^2(-2x^5 + 4x^4 + 2x^3 + 2x^2 - 4x + 2)W_2 - x(2x^6 + 4x^5 - 6x^4 - 4x^3 + x^2 + 2x - 1)W_1 - 2x^4(x^3 + 2x^2 + 3x - 4)W_0$.

For $x = -1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (b) using

$\sum\limits_{k=0}^{n} k(-1)^k W_k = \left.\frac{\frac{d^2}{dx^2}(g_3(x))}{\frac{d^2}{dx^2}\left((2x-1)^2(x+1)^2(x^2+1)^2\right)}\right|_{x=-1} = \frac{1}{36}((-1)^n(-(3n^2 + 5n - 53)W_{n+3} + 2(3n^2 + 2n - 54)W_{n+2} - (3n^2 - 13n - 53)W_{n+1} + 2(3n^2 + 11n - 45)W_n) - 53W_3 + 108W_2 - 53W_1 + 90W_0)$.

(b) We use Theorem 3 (b). If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 3 (b) then we have

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{g_4(x)}{(4x-1)^2(x-1)^2(x+1)^4},$$

where $g_4(x) = -x^{n+1}(2x^2 - 2x + 6x^3 + x^4 + 8x^5 + 8x^6 - n(2x^2 + x - 1)(4x^4 + 3x^3 - 5x^2 - 3x + 1) + 1)W_{2n+2} + x^{n+2}(12x^2 + 16x^3 + 16x^4 + 8x^5 - n(2x + 2)(4x^4 + 3x^3 - 5x^2 - 3x + 1) - 4)W_{2n+1} + x^{n+2}(12x + 10x^2 - 32x^3 - 16x^4 + 8x^5 + n(4x^2 + x - 3)\left((4x^4 + 3x^3 - 5x^2 - 3x + 1\right) - 6)W_{2n} + 2x^{n+2}(6x^2 + 8x^3 + 8x^4 + 4x^5 - n(x + 1)(4x^4 + 3x^3 - 5x^2 - 3x + 1) - 2)W_{2n-1} - x^2(4x^5 + 8x^4 + 8x^3 + 6x^2 - 2)W_3 + x(12x^6 + 16x^5 + 9x^4 + 12x^3 + 2x^2 - 4x + 1)W_2 - x^2(4x^5 + 8x^4 + 8x^3 + 6x^2 - 2)W_1 - 2x^2(2x^5 - 12x^4 - 20x^3 + 2x^2 + 6x - 2)W_0$.

For $x = -1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (b) using

$\sum\limits_{k=0}^{n} k(-1)^k W_{2k} = \left.\frac{\frac{d^4}{dx^4}(g_4(x))}{\frac{d^4}{dx^4}((4x-1)^2(x-1)^2(x+1)^4)}\right|_{x=-1} = \frac{1}{100}((-1)^n(-(15n^2 - 4n - 159)W_{2n+2} + 2(5n^2 + 2n - 54)W_{2n+1} + (35n^2 + 54n - 350)W_{2n} + 2(5n^2 + 2n - 54)W_{2n-1}) + 54W_3 - 213W_2 + 54W_1 + 296W_0)$.

(c) We use Theorem 3 (c). If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 3 (c) then we have

$$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{g_5(x)}{(4x-1)^2(x-1)^2(x+1)^4},$$

where $g_5(x) = -x^{n+1}(2x + 2x^2 - 6x^3 - 15x^4 - 8x^5 + n(x + 1)(4x^4 + 3x^3 - 5x^2 - 3x + 1) + 1)W_{2n+2} - x^{n+1}(4x - 10x^2 - 4x^3 + 33x^4 + 24x^5 - n(4x^3 + 3x^2 - 2x - 1)(4x^4 + 3x^3 - 5x^2 - 3x + 1) + 1)W_{2n+1} - x^{n+1}(2x + 2x^2 - 6x^3 - 15x^4 - 8x^5 + n(x + 1)(4x^4 + 3x^3 - 5x^2 - 3x + 1) + 1)W_{2n} - 2x^{n+1}(2x^2 - 2x + 6x^3 + x^4 + 8x^5 + 8x^6 - n(2x^2 + x - 1)(4x^4 + 3x^3 - 5x^2 - 3x + 1) + 1)W_{2n-1} + x(8x^6 + 8x^5 + x^4 + 6x^3 + 2x^2 - 2x + 1)W_3 - x^2(8x^5 + 16x^4 + 16x^3 + 12x^2 - 4)W_2 - x^2(8x^5 - 16x^4 - 32x^3 + 10x^2 + 12x - 6)W_1 - 2x^2(4x^5 + 8x^4 + 8x^3 + 6x^2 - 2)W_0$.

For $x = -1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (c) using

$\sum\limits_{k=0}^{n} k(-1)^k W_{2k+1} = \left.\frac{\frac{d^4}{dx^4}(g_5(x))}{\frac{d^4}{dx^4}((4x-1)^2(x-1)^2(x+1)^4)}\right|_{x=-1} = \frac{1}{100}((-1)^n(-(5n^2 - 8n - 51)W_{2n+2} + (20n^2 + 58n - 191)W_{2n+1} - (5n^2 - 8n - 51)W_{2n} - 2(15n^2 - 4n - 159)W_{2n-1}) - 159W_3 + 108W_2 + 350W_1 + 108W_0)$.

□

Taking $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1$ in the last theorem, we have the following corollary which presents linear sum formula of fourth-order Jacobsthal numbers.

Corollary 11. *For $n \geq 0$, fourth order Jacobsthal numbers have the following property:*

(a) $\sum\limits_{k=0}^{n} k(-1)^k J_k = \frac{1}{36}((-1)^n(-(3n^2 + 5n - 53)J_{n+3} + 2(3n^2 + 2n - 54)J_{n+2} - (3n^2 - 13n - 53)J_{n+1} + 2(3n^2 + 11n - 45)J_n) + 2)$.

(b) $\sum\limits_{k=0}^{n} k(-1)^k J_{2k} = \frac{1}{100}((-1)^n(-(15n^2-4n-159)J_{2n+2}+2(5n^2+2n-54)J_{2n+1}+(35n^2+54n-350)J_{2n}+2(5n^2+2n-54)J_{2n-1})-105)$.

(c) $\sum\limits_{k=0}^{n} k(-1)^k J_{2k+1} = \frac{1}{100}((-1)^n(-(5n^2-8n-51)J_{2n+2}+(20n^2+58n-191)J_{2n+1}-(5n^2-8n-51)J_{2n}-2(15n^2-4n-159)J_{2n-1})+299)$.

From the last theorem, we have the following corollary which gives linear sum formula of the fourth order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10$).

Corollary 12. *For $n \geq 0$, fourth order Jacobsthal-Lucas numbers have the following property:*

(a) $\sum\limits_{k=0}^{n} k(-1)^k j_k = \frac{1}{36}((-1)^n(-(3n^2+5n-53)j_{n+3}+2(3n^2+2n-54)j_{n+2}-(3n^2-13n-53)j_{n+1}+2(3n^2+11n-45)j_n)+137)$.

(b) $\sum\limits_{k=0}^{n} k(-1)^k j_{2k} = \frac{1}{100}((-1)^n(-(15n^2-4n-159)j_{2n+2}+2(5n^2+2n-54)j_{2n+1}+(35n^2+54n-350)j_{2n}+2(5n^2+2n-54)j_{2n-1})+121)$.

(c) $\sum\limits_{k=0}^{n} k(-1)^k j_{2k+1} = \frac{1}{100}((-1)^n(-(5n^2-8n-51)j_{2n+2}+(20n^2+58n-191)j_{2n+1}-(5n^2-8n-51)j_{2n}-2(15n^2-4n-159)j_{2n-1})-484)$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10$ in the last theorem, we have the following corollary which presents linear sum formula of the modified fourth order Jacobsthal numbers.

Corollary 13. *For $n \geq 0$,modified fourth order Jacobsthal numbers have the following property:*

(a) $\sum\limits_{k=0}^{n} k(-1)^k K_k = \frac{1}{36}((-1)^n(-(3n^2+5n-53)K_{n+3}+2(3n^2+2n-54)K_{n+2}-(3n^2-13n-53)K_{n+1}+2(3n^2+11n-45)K_n)+11)$.

(b) $\sum\limits_{k=0}^{n} k(-1)^k K_{2k} = \frac{1}{100}((-1)^n(-(15n^2-4n-159)K_{2n+2}+2(5n^2+2n-54)K_{2n+1}+(35n^2+54n-350)K_{2n}+2(5n^2+2n-54)K_{2n-1})+843)$.

(c) $\sum\limits_{k=0}^{n} k(-1)^k K_{2k+1} = \frac{1}{100}((-1)^n(-(5n^2-8n-51)K_{2n+2}+(20n^2+58n-191)K_{2n+1}-(5n^2-8n-51)K_{2n}-2(15n^2-4n-159)K_{2n-1})-592)$.

From the last theorem, we have the following corollary which gives linear sum formula of the fourth-order Jacobsthal Perrin numbers (take $W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8$).

Corollary 14. *For $n \geq 0$, fourth-order Jacobsthal Perrin numbers have the following property:*

(a) $\sum\limits_{k=0}^{n} k(-1)^k Q_k = \frac{1}{36}((-1)^n(-(3n^2+5n-53)Q_{n+3}+2(3n^2+2n-54)Q_{n+2}-(3n^2-13n-53)Q_{n+1}+2(3n^2+11n-45)Q_n)+62)$.

(b) $\sum\limits_{k=0}^{n} k(-1)^k Q_{2k} = \frac{1}{100}((-1)^n(-(15n^2-4n-159)Q_{2n+2}+2(5n^2+2n-54)Q_{2n+1}+(35n^2+54n-350)Q_{2n}+2(5n^2+2n-54)Q_{2n-1})+894)$.

(c) $\sum\limits_{k=0}^{n} k(-1)^k Q_{2k+1} = \frac{1}{100}((-1)^n(-(5n^2-8n-51)Q_{2n+2}+(20n^2+58n-191)Q_{2n+1}-(5n^2-8n-51)Q_{2n}-2(15n^2-4n-159)Q_{2n-1})-732)$.

Taking $W_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2$ in the theorem, we have the following corollary which presents linear sum formula of the adjusted fourth-order Jacobsthal numbers.

Corollary 15. *For $n \geq 0$, adjusted fourth-order Jacobsthal numbers have the following property:*

(a) $\sum\limits_{k=0}^{n} k(-1)^k S_k = \frac{1}{36}((-1)^n(-(3n^2+5n-53)S_{n+3}+2(3n^2+2n-54)S_{n+2}-(3n^2-13n-53)S_{n+1}+2(3n^2+11n-45)S_n)-51)$.

(b) $\sum_{k=0}^{n} k(-1)^k S_{2k} = \frac{1}{100}((-1)^n(-(15n^2-4n-159)S_{2n+2}+2(5n^2+2n-54)S_{2n+1}+(35n^2+54n-350)S_{2n}+2(5n^2+2n-54)S_{2n-1})-51)$.

(c) $\sum_{k=0}^{n} k(-1)^k S_{2k+1} = \frac{1}{100}((-1)^n(-(5n^2-8n-51)S_{2n+2}+(20n^2+58n-191)S_{2n+1}-(5n^2-8n-51)S_{2n}-2(15n^2-4n-159)S_{2n-1})+140)$.

From the last theorem, we have the following corollary which gives linear sum formula of the modified fourth-order Jacobsthal-Lucas numbers (take $W_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7$).

Corollary 16. *For $n \geq 0$, modified fourth-order Jacobsthal-Lucas numbers have the following property:*

(a) $\sum_{k=0}^{n} k(-1)^k R_k = \frac{1}{36}((-1)^n(-(3n^2+5n-53)R_{n+3}+2(3n^2+2n-54)R_{n+2}-(3n^2-13n-53)R_{n+1}+2(3n^2+11n-45)R_n)+260)$.

(b) $\sum_{k=0}^{n} k(-1)^k R_{2k} = \frac{1}{100}((-1)^n(-(15n^2-4n-159)R_{2n+2}+2(5n^2+2n-54)R_{2n+1}+(35n^2+54n-350)R_{2n}+2(5n^2+2n-54)R_{2n-1})+977)$.

(c) $\sum_{k=0}^{n} k(-1)^k R_{2k+1} = \frac{1}{100}((-1)^n(-(5n^2-8n-51)R_{2n+2}+(20n^2+58n-191)R_{2n+1}-(5n^2-8n-51)R_{2n}-2(15n^2-4n-159)R_{2n-1})-7)$.

Taking $x = -1, r = 2, s = 3, t = 5, u = 7$ in Theorem 3 (a), (b) and (c), we obtain the following proposition.

Proposition 3. *If $r = 2, s = 3, t = 5, u = 7$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} k(-1)^k W_k = \frac{1}{4}((-1)^n(-(2n-11)W_{n+3}+(6n-35)W_{n+2}+8W_{n+1}+7(2n-9)W_n)-11W_3+35W_2-8W_1+63W_0)$.

(b) $\sum_{k=0}^{n} k(-1)^k W_{2k} = \frac{1}{36}((-1)^n(-(6n-7)W_{2n+2}+(6n+5)W_{2n+1}+72(n-1)W_{2n}+7(6n-13)W_{2n-1})+13W_3-33W_2-44W_1+7W_0)$.

(c) $\sum_{k=0}^{n} k(-1)^k W_{2k+1} = \frac{1}{36}((-1)^n(-(6n-19)W_{2n+2}+3(18n-17)W_{2n+1}+4(3n-14)W_{2n}-7(6n-7)W_{2n-1})-7W_3-5W_2+72W_1+91W_0)$.

From the last proposition, we have the following corollary which gives linear sum formulas of 4-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 1, G_3 = 2$).

Corollary 17. *For $n \geq 0$, 4-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k G_k = \frac{1}{4}((-1)^n(-(2n-11)G_{n+3}+(6n-35)G_{n+2}+8G_{n+1}+7(2n-9)G_n)+13)$.

(b) $\sum_{k=0}^{n} k(-1)^k G_{2k} = \frac{1}{36}((-1)^n(-(6n-7)G_{2n+2}+(6n+5)G_{2n+1}+72(n-1)G_{2n}+7(6n-13)G_{2n-1})-7)$.

(c) $\sum_{k=0}^{n} k(-1)^k G_{2k+1} = \frac{1}{36}((-1)^n(-(6n-19)G_{2n+2}+3(18n-17)G_{2n+1}+4(3n-14)G_{2n}-7(6n-7)G_{2n-1})-19)$.

Taking $W_n = H_n$ with $H_0 = 4, H_1 = 2, H_2 = 10, H_3 = 41$ in the last proposition, we have the following corollary which presents linear sum formulas of Lucas 4-primes numbers.

Corollary 18. *For $n \geq 0$, Lucas 4-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k H_k = \frac{1}{4}((-1)^n(-(2n-11)H_{n+3}+(6n-35)H_{n+2}+8H_{n+1}+7(2n-9)H_n)+135)$.

(b) $\sum_{k=0}^{n} k(-1)^k H_{2k} = \frac{1}{36}((-1)^n(-(6n-7)H_{2n+2}+(6n+5)H_{2n+1}+72(n-1)H_{2n}+7(6n-13)H_{2n-1})+143)$.

(c) $\sum_{k=0}^{n} k(-1)^k H_{2k+1} = \frac{1}{36}((-1)^n(-(6n-19)H_{2n+2}+3(18n-17)H_{2n+1}+4(3n-14)H_{2n}-7(6n-7)H_{2n-1})+171)$.

From the last proposition, we have the following corollary which gives linear sum formulas of modified 4-primes numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 1, E_3 = 1$).

Corollary 19. *For $n \geq 0$, modified 4-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k E_k = \frac{1}{4}((-1)^n(-(2n-11)E_{n+3} + (6n-35)E_{n+2} + 8E_{n+1} + 7(2n-9)E_n) + 24)$.

(b) $\sum_{k=0}^{n} k(-1)^k E_{2k} = \frac{1}{36}((-1)^n(-(6n-7)E_{2n+2} + (6n+5)E_{2n+1} + 72(n-1)E_{2n} + 7(6n-13)E_{2n-1}) - 20)$.

(c) $\sum_{k=0}^{n} k(-1)^k E_{2k+1} = \frac{1}{36}((-1)^n(-(6n-19)E_{2n+2} + 3(18n-17)E_{2n+1} + 4(3n-14)E_{2n} - 7(6n-7)E_{2n-1}) - 12)$.

3.3. The case $x = i$

In this subsection we consider the special case $x = i$. Taking $x = i, r = s = t = u = 1$ in Theorem 3 (a), (b) and (c), we obtain the following proposition.

Proposition 4. *If $x = i, r = s = t = u = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} ki^k W_k = i^n(i(n+(5-2i))W_{n+3} + (1-i)(n+(\frac{9}{2}-\frac{5}{2}i))W_{n+2} + (-1-2i)(n+(4-2i))W_{n+1} - (n+(6-2i))W_n) - (2+5i)W_3 - (2-7i)W_2 + (8+6i)W_1 + (6-2i)W_0$.

(b) $\sum_{k=0}^{n} ki^k W_{2k} = \frac{1}{9-40i}((-13-6i)i^n(n+(\frac{8}{41}-\frac{10}{41}i))W_{2n+2} + (14-3i)i^n(n+(\frac{81}{205}+\frac{32}{205}i))W_{2n+1} + (15-12i)i^n(n+(\frac{106}{123}-\frac{10}{41}i))W_{2n} + (9+i)i^n(n+(\frac{57}{82}+\frac{21}{82}i))W_{2n-1} - (6+3i)W_3 + (10+i)W_2 + 2iW_1 - (4-17i)W_0)$.

(c) $\sum_{k=0}^{n} ki^k W_{2k+1} = \frac{1}{9-40i}((1-9i)i^n(n-(\frac{25}{82}-\frac{21}{82}i))W_{2n+2} + (2-18i)i^n(n+(\frac{57}{82}+\frac{21}{82}i))W_{2n+1} + (-4-5i)i^n(n-(\frac{33}{41}+\frac{10}{41}i))W_{2n} + (-13-6i)i^n(n+(\frac{8}{41}-\frac{10}{41}i))W_{2n-1} + (4-2i)W_3 - (6+i)W_2 - (10-14i)W_1 - (6+3i)W_0)$.

From the above proposition, we have the following corollary which gives linear sum formulas of Tetranacci numbers (take $W_n = M_n$ with $M_0 = 0, M_1 = 1, M_2 = 1, M_3 = 2$).

Corollary 20. *For $n \geq 0$, Tetranacci numbers have the following properties.*

(a) $\sum_{k=0}^{n} ki^k M_k = i^n(i(n+(5-2i))M_{n+3} + (1-i)(n+(\frac{9}{2}-\frac{5}{2}i))M_{n+2} + (-1-2i)(n+(4-2i))M_{n+1} - (n+(6-2i))M_n) + (2+3i)$.

(b) $\sum_{k=0}^{n} ki^k M_{2k} = \frac{1}{9-40i}((-13-6i)i^n(n+(\frac{8}{41}-\frac{10}{41}i))M_{2n+2} + (14-3i)i^n(n+(\frac{81}{205}+\frac{32}{205}i))M_{2n+1} + (15-12i)i^n(n+(\frac{106}{123}-\frac{10}{41}i))M_{2n} + (9+i)i^n(n+(\frac{57}{82}+\frac{21}{82}i))M_{2n-1} + (-2-3i))$.

(c) $\sum_{k=0}^{n} ki^k M_{2k+1} = \frac{1}{9-40i}((1-9i)i^n(n-(\frac{25}{82}-\frac{21}{82}i))M_{2n+2} + (2-18i)i^n(n+(\frac{57}{82}+\frac{21}{82}i))M_{2n+1} + (-4-5i)i^n(n-(\frac{33}{41}+\frac{10}{41}i))M_{2n} + (-13-6i)i^n(n+(\frac{8}{41}-\frac{10}{41}i))M_{2n-1} + (-8+9i))$.

Taking $M_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7$ in the above proposition, we have the following corollary which presents linear sum formulas of Tetranacci-Lucas numbers.

Corollary 21. *For $n \geq 0$, Tetranacci-Lucas numbers have the following properties.*

(a) $\sum_{k=0}^{n} ki^k R_k = i^n(i(n+(5-2i))R_{n+3} + (1-i)(n+(\frac{9}{2}-\frac{5}{2}i))R_{n+2} + (-1-2i)(n+(4-2i))R_{n+1} - (n+(6-2i))R_n) + (12-16i)$.

(b) $\sum_{k=0}^{n} ki^k R_{2k} = \frac{1}{9-40i}((-13-6i)i^n(n+(\frac{8}{41}-\frac{10}{41}i))R_{2n+2} + (14-3i)i^n(n+(\frac{81}{205}+\frac{32}{205}i))R_{2n+1} + (15-12i)i^n(n+(\frac{106}{123}-\frac{10}{41}i))R_{2n} + (9+i)i^n(n+(\frac{57}{82}+\frac{21}{82}i))R_{2n-1} + (-28+52i))$.

(c) $\sum_{k=0}^{n} ki^k R_{2k+1} = \frac{1}{9-40i}((1-9i)i^n(n-(\frac{25}{82}-\frac{21}{82}i))R_{2n+2} + (2-18i)i^n(n+(\frac{57}{82}+\frac{21}{82}i))R_{2n+1} + (-4-5i)i^n(n-(\frac{33}{41}+\frac{10}{41}i))R_{2n} + (-13-6i)i^n(n+(\frac{8}{41}-\frac{10}{41}i))R_{2n-1} + (-24-15i))$.

Corresponding sums of the other fourth order generalized Tetranacci numbers can be calculated similarly.

4. Linear sum formulas of generalized Tetranacci numbers with negative subscripts

The following Theorem present some linear sum formulas of generalized Tetranacci numbers with negative subscripts.

Theorem 6. *Let x be a real or complex non-zero numbers. For $n \geq 1$ we have the following formulas:*

(a) *If $u + rx^3 + sx^2 + tx - x^4 \neq 0$, then*

$$\sum_{k=1}^{n} kx^k W_{-k} = \frac{\Omega_4}{(u + rx^3 + sx^2 + tx - x^4)^2},$$

where $\Omega_4 = x^{n+1}(n(-u - rx^3 - sx^2 - tx + x^4) - u + 2rx^3 + sx^2 - 3x^4)W_{-n+3} + x^{n+1}(n(r - x)(u + rx^3 + sx^2 + tx - x^4) + 4rx^4 - tx^2 - 2r^2x^3 + ru - 2ux - 2x^5 - rsx^2)W_{-n+2} + x^{n+1}(n(s + rx - x^2)(u + rx^3 + sx^2 + tx - x^4) + 2rx^5 + 2sx^4 - 2tx^3 - 3ux^2 - r^2x^4 - s^2x^2 + su - x^6 - 2rsx^3 + rtx^2 + 2rux)W_{-n+1} + x^{n+1}(n(t + rx^2 + sx - x^3)(u + rx^3 + sx^2 + tx - x^4) - 4ux^3 + tu + 3rux^2 + 2sux)W_{-n} + x(u - 2rx^3 - sx^2 + 3x^4)W_3 + x(-4rx^4 + tx^2 + 2r^2x^3 - ru + 2ux + 2x^5 + rsx^2)W_2 + x(-2rx^5 - 2sx^4 + 2tx^3 + 3ux^2 + r^2x^4 + s^2x^2 - su + x^6 + 2rsx^3 - rtx^2 - 2rux)W_1 + ux(-t - 3rx^2 - 2sx + 4x^3)W_0.$

(b) *If $r^2x^3 + 2rtx^2 - s^2x^2 - 2sux + 2sx^3 + t^2x - u^2 + 2ux^2 - x^4 \neq 0$ then*

$$\sum_{k=1}^{n} kx^k W_{-2k} = \frac{\Omega_5}{(r^2x^3 + 2rtx^2 - s^2x^2 - 2sux + 2sx^3 + t^2x - u^2 + 2ux^2 - x^4)^2},$$

where $\Omega_5 = x^{n+1}(n(u + sx - x^2)(2sx^3 + t^2x + 2ux^2 + r^2x^3 - s^2x^2 - u^2 - x^4 + 2rtx^2 - 2sux) + 2sx^5 + ux^4 - s^2x^4 - 2t^2x^3 + u^2x^2 - u^3 - x^6 - 2rtx^4 - 2su^2x - r^2sx^4 + st^2x^2 - 2r^2ux^3 - s^2ux^2 - 2rtux^2)W_{-2n+2} + x^{n+1}(n(ru + tx + rsx)(-2sx^3 - t^2x - 2ux^2 - r^2x^3 + s^2x^2 + u^2 + x^4 - 2rtx^2 + 2sux) + ru^3 - 2tx^5 - t^3x^2 - 2rsx^5 - 3rux^4 + 2stx^4 + 2tu^2x + 2rs^2x^4 + r^3sx^4 + 2ru^2x^2 + r^2tx^4 + 2r^3ux^3 + 2rsu^2x + 4rsux^3 + 2stux^2 - rst^2x^2 + rs^2ux^2 + 2r^2tux^2)W_{-2n+1} + x^{n+1}(n(2sx^2 - s^2x + r^2x^2 - su + ux - x^3 + rtx)(2sx^3 + t^2x + 2ux^2 + r^2x^3 - s^2x^2 - u^2 - x^4 + 2rtx^2 - 2sux) + su^3 - 2u^3x - 2ux^5 - 3t^2x^4 + 4u^2x^3 + 2r^2t^2x^3 - 3r^2u^2x^2 - s^2t^2x^2 - 2rtx^5 + 5sux^4 + rt^3x^2 + 4st^2x^3 + r^3tx^4 - 6su^2x^2 + r^2ux^4 + 2s^2u^2x - 4s^2ux^3 + s^3ux^2 + t^2ux^2 + 2rstx^4 - 2rtu^2x - 2r^2sux^3)W_{-2n} + ux^{n+1}(n(t + rx)(-2sx^3 - t^2x - 2ux^2 - r^2x^3 + s^2x^2 + u^2 + x^4 - 2rtx^2 + 2sux) + tu^2 - 2rx^5 - 3tx^4 + r^3x^4 + 2rsx^4 + 2ru^2x + 4stx^3 + 2tux^2 + rt^2x^2 + 2r^2tx^3 - s^2tx^2 + 2rsux^2)W_{-2n-1} - x(tu^2 - 2rx^5 - 3tx^4 + r^3x^4 + 2rsx^4 + 2ru^2x + 4stx^3 + 2tux^2 + rt^2x^2 + 2r^2tx^3 - s^2tx^2 + 2rsux^2)W_3 + x(-2sx^5 - ux^4 - 2r^2x^5 + r^4x^4 + s^2x^4 + 2t^2x^3 - u^2x^2 + u^3 + x^6 + r^2t^2x^2 + rtu^2 - rtx^4 + 2su^2x + 3r^2sx^4 - st^2x^2 + 2r^2u^2x + 2r^3tx^3 + 2r^2ux^3 + s^2ux^2 + 4rstx^3 + 4rtux^2 - rs^2tx^2 + 2r^2sux^2)W_2 - x(ru^3 - 2tx^5 - t^3x^2 - stu^2 - 3rux^4 + 5stx^4 + 2tu^2x + 2ru^2x^2 + r^2tx^4 - 4s^2tx^3 + s^3tx^2 + 2r^3ux^3 + 4rsux^3 - 2rst^2x^2 - 2r^2stx^3 - rs^2ux^2 + 2r^2tux^2)W_1 + ux(-su^2 + t^2u - 5sx^4 + 2u^2x - 4ux^3 - r^2x^4 + 4s^2x^3 - s^3x^2 + t^2x^2 + 2x^5 + 6sux^2 - 2s^2ux + 2r^2sx^3 + 3r^2ux^2 + 2rstx^2 + 4rtux)W_0.$

(c) *If $r^2x^3 + 2rtx^2 - s^2x^2 - 2sux + 2sx^3 + t^2x - u^2 + 2ux^2 - x^4 \neq 0$ then*

$$\sum_{k=1}^{n} kx^k W_{-2k+1} = \frac{\Omega_6}{(r^2x^3 + 2rtx^2 - s^2x^2 - 2sux + 2sx^3 + t^2x - u^2 + 2ux^2 - x^4)^2},$$

where $\Omega_6 = x^{n+2}(n(t + rx)(-2sx^3 - t^2x - 2ux^2 - r^2x^3 + s^2x^2 + u^2 + x^4 - 2rtx^2 + 2sux) + 2tu^2 - rx^5 - t^3x - 2tx^4 + 3ru^2x - 2rux^3 + 2stx^3 + rs^2x^3 - 2rt^2x^2 - r^2tx^3 + 4rsux^2 + 2stux)W_{-2n+2} + x^{n+2}(n(u + r^2x + rt + sx - x^2)(2sx^3 + t^2x + 2ux^2 + r^2x^3 - s^2x^2 - u^2 - x^4 + 2rtx^2 - 2sux) + 2s^2x^4 - s^3x^3 - 3t^2x^3 + 4u^2x^2 - 2u^3 - r^2s^2x^3 + 2r^2t^2x^2 - 2rtu^2 + rt^3x - 2rtx^4 - 5su^2x + 6sux^3 + t^2ux - sx^5 - 2ux^4 + 2st^2x^2 - 3r^2u^2x + r^3tx^3 + r^2ux^3 - 4s^2ux^2 - 2rstux - 4r^2sux^2)W_{-2n+1} + x^{n+2}(n(ru - st + tx)(-2sx^3 - t^2x - 2ux^2 - r^2x^3 + s^2x^2 + u^2 + x^4 - 2rtx^2 + 2sux) + 2ru^3 - tx^5 - 2t^3x^2 - 2stu^2 + st^3x - 2rux^4 + 2stx^4 + 3tu^2x - 2tux^3 - 2rt^2x^3 - s^2tx^3 + r^3ux^3 + 2rsu^2x + 2rsux^3 - rt^2ux + 4stux^2 - 2s^2tux - r^2stx^3)W_{-2n} + ux^{n+1}(n(u + sx - x^2)(2sx^3 + t^2x + 2ux^2 + r^2x^3 - s^2x^2 - u^2 - x^4 + 2rtx^2 - 2sux) + 2sx^5 + ux^4 - s^2x^4 - 2t^2x^3 + u^2x^2 - u^3 - x^6 - 2rtx^4 - 2su^2x - r^2sx^4 + st^2x^2 - 2r^2ux^3 - s^2ux^2 - 2rtux^2)W_{-2n-1} + x(-2sx^5 - ux^4 + s^2x^4 + 2t^2x^3 - u^2x^2 + u^3 + x^6 + 2rtx^4 + 2su^2x + r^2sx^4 - st^2x^2 + 2r^2ux^3 + s^2ux^2 + 2rtux^2)W_3 - x(ru^3 - 2tx^5 - t^3x^2 - 2rsx^5 - 3rux^4 + 2stx^4 + 2tu^2x + 2rs^2x^4 + r^3sx^4 + 2ru^2x^2 + r^2tx^4 + 2r^3ux^3 + 2rsu^2x + 4rsux^3 + 2stux^2 - rst^2x^2 + rs^2ux^2 + 2r^2tux^2)W_2 - x(su^3 - 2u^3x - 2ux^5 - 3t^2x^4 + 4u^2x^3 + 2r^2t^2x^3 - 3r^2u^2x^2 - s^2t^2x^2 - 2rtx^5 + 5sux^4 + rt^3x^2 + 4st^2x^3 + r^3tx^4 - 6su^2x^2 + r^2ux^4 + 2s^2u^2x - 4s^2ux^3 + s^3ux^2 + t^2ux^2 + 2rstx^4 - 2rtu^2x - 2r^2sux^3) W_1 - ux(tu^2 - 2rx^5 - 3tx^4 + r^3x^4 + 2rsx^4 + 2ru^2x + 4stx^3 + 2tux^2 + rt^2x^2 + 2r^2tx^3 - s^2tx^2 + 2rsux^2)W_0.$

Proof. **(a)** Using the recurrence relation

$$W_{-n+4}=r\times W_{-n+3}+s\times W_{-n+2}+t\times W_{-n+1}+u\times W_{-n}$$

i.e.,

$$uW_{-n}=W_{-n+4}-rW_{-n+3}-sW_{-n+2}-tW_{-n+1}.$$

we obtain

$$\begin{array}{rcl}
unx^{n}W_{-n} &=& nx^{n}W_{-n+4}-rnx^{n}W_{-n+3}-snx^{n}W_{-n+2}-tnx^{n}W_{-n+1},\\
u(n-1)x^{n-1}W_{-n+1} &=& (n-1)x^{n-1}W_{-n+5}-r(n-1)x^{n-1}W_{-n+4}\\
&& -s(n-1)x^{n-1}W_{-n+3}-t(n-1)x^{n-1}W_{-n+2},\\
u(n-2)x^{n-2}W_{-n+2} &=& (n-2)x^{n-2}W_{-n+6}-r(n-2)x^{n-2}W_{-n+5}\\
&& -s(n-2)x^{n-2}W_{-n+4}-t(n-2)x^{n-2}W_{-n+3},\\
&& \vdots\\
u\times 5\times W_{-5} &=& 5\times W_{-1}-r\times 5\times W_{-2}-s\times 5\times W_{-3}-t\times 5\times W_{-4},\\
u\times 4\times x^{4}W_{-4} &=& 4\times x^{4}W_{0}-r\times 4\times x^{4}W_{-1}-s\times 4\times x^{4}W_{-2}-t\times 4\times x^{4}W_{-3},\\
u\times 3\times x^{3}W_{-3} &=& 3\times x^{3}W_{1}-r\times 3\times x^{3}W_{0}-s\times 3\times x^{3}W_{-1}-t\times 3\times x^{3}W_{-2},\\
u\times 2\times x^{2}W_{-2} &=& 2\times x^{2}W_{2}-r\times 2\times x^{2}W_{1}-s\times 2\times x^{2}W_{0}-t\times 2\times x^{2}W_{-1},\\
u\times 1\times x^{1}W_{-1} &=& 1\times x^{1}W_{3}-r\times 1\times x^{1}W_{2}-s\times 1\times x^{1}W_{1}-t\times 1\times x^{1}W_{0}.
\end{array}$$

If we add the above equations side by side (and using Theorem 2 (a)), we get (a)
(b) and (c) Using the recurrence relation

$$W_{-n+4}=rW_{-n+3}+sW_{-n+2}+tW_{-n+1}+uW_{-n},$$

i.e.,

$$tW_{-n+1}=W_{-n+4}-rW_{-n+3}-sW_{-n+2}-uW_{-n},$$

we obtain

$$\begin{array}{rcl}
tnx^{n}W_{-2n+1} &=& nx^{n}W_{-2n+4}-rnx^{n}W_{-2n+3}-snx^{n}W_{-2n+2}-unx^{n}W_{-2n},\\
t(n-1)x^{n-1}W_{-2n+3} &=& (n-1)x^{n-1}W_{-2n+6}-r(n-1)x^{n-1}W_{-2n+5}\\
&& -s(n-1)x^{n-1}W_{-2n+4}-u(n-1)x^{n-1}W_{-2n+2},\\
t(n-2)x^{n-2}W_{-2n+5} &=& (n-2)x^{n-2}W_{-2n+8}-r(n-2)x^{n-2}W_{-2n+7}\\
&& -s(n-2)x^{n-2}W_{-2n+6}-u(n-2)x^{n-2}W_{-2n+4},\\
&& \vdots\\
t\times 3\times x^{3}W_{-5} &=& 3\times x^{3}W_{-2}-r\times 3\times x^{3}W_{-3}-s\times 3\times x^{3}W_{-4}-u\times 3\times x^{3}W_{-6},\\
t\times 2\times x^{2}W_{-3} &=& 2\times x^{2}W_{0}-r\times 2\times x^{2}W_{-1}-s\times 2\times x^{2}W_{-2}-u\times 2\times x^{2}W_{-4},\\
t\times 1\times x^{1}W_{-1} &=& 1\times x^{1}W_{2}-r\times 1\times x^{1}W_{1}-s\times 1\times x^{1}W_{0}-u\times 1\times x^{1}W_{-2}.
\end{array}$$

If we add the equations side by side, we get

$$\begin{aligned}
t\sum_{k=1}^{n}kx^{k}W_{-2k+1} =&(-(n+1)x^{n+1}W_{-2n+2}-(n+2)x^{n+2}W_{-2n}+2\times x^{2}W_{0}+1\times x^{1}W_{2}+x^{2}\sum_{k=1}^{n}kx^{k}W_{-2k}\\
&+2x^{2}\sum_{k=1}^{n}x^{k}W_{-2k})-r(-(n+1)x^{n+1}W_{-2n+1}+1\times x^{1}W_{1}+x^{1}\sum_{k=1}^{n}kx^{k}W_{-2k+1}+x^{1}\sum_{k=1}^{n}x^{k}W_{-2k+1})\\
&-s(-(n+1)x^{n+1}W_{-2n}+1\times x^{1}W_{0}+x^{1}\sum_{k=1}^{n}kx^{k}W_{-2k}+x^{1}\sum_{k=1}^{n}x^{k}W_{-2k})-u(\sum_{k=1}^{n}kx^{k}W_{-2k}). \qquad (4)
\end{aligned}$$

Similarly, using the recurrence relation

$$W_{-n+4} = rW_{-n+3} + sW_{-n+2} + tW_{-n+1} + uW_{-n} ,$$

i.e.,

$$tW_{-n} = W_{-n+3} - rW_{-n+2} - sW_{-n+1} - uW_{-n-1} ,$$

we obtain

$$\begin{array}{rcl} tnx^nW_{-2n} & = & nx^nW_{-2n+3} - rnx^nW_{-2n+2} - snx^nW_{-2n+1} - unx^nW_{-2n-1}, \\ t(n-1)x^{n-1}W_{-2n+2} & = & (n-1)\times x^{n-1}W_{-2n+5} - r(n-1)x^{n-1}W_{-2n+4} \\ & & -s(n-1)x^{n-1}W_{-2n+3} - u(n-1)x^{n-1}W_{-2n+1}, \\ t(n-2)x^{n-2}W_{-2n+4} & = & (n-2)\times x^{n-2}W_{-2n+7} - r(n-2)x^{n-2}W_{-2n+6} \\ & & -s(n-2)x^{n-2}W_{-2n+5} - u(n-2)x^{n-2}W_{-2n+3}, \\ & & \vdots \\ t\times 3\times x^3W_{-6} & = & 3\times x^3W_{-3} - r\times 3\times x^3W_{-4} - s\times 3\times x^3W_{-5} - u\times 3\times x^3W_{-7}, \\ t\times 2\times x^2W_{-4} & = & 2\times x^2W_{-1} - r\times 2\times x^2W_{-2} - s\times 2\times x^2W_{-3} - u\times 2\times x^2W_{-5}, \\ t\times 1\times x^1W_{-2} & = & 1\times x^1W_1 - r\times 1\times x^1W_0 - s\times 1\times x^1W_{-1} - u\times 1\times x^1W_{-3}. \end{array}$$

If we add the equations side by side, we get

$$\begin{array}{rcl} t\sum_{k=1}^{n} kx^kW_{-2k} & = & (-(n+1)x^{n+1}W_{-2n+1} + 1\times x^1W_1 + x^1\sum_{k=1}^{n} kx^kW_{-2k+1} + x^1\sum_{k=1}^{n} x^kW_{-2k+1}) \\ & & -r(-(n+1)x^{n+1}W_{-2n} + 1\times x^1W_0 + x^1\sum_{k=1}^{n} kx^kW_{-2k} + x^1\sum_{k=1}^{n} x^kW_{-2k}) \\ & & -s(\sum_{k=1}^{n} kx^kW_{-2k+1}) - u(nx^nW_{-2n-1} + x^{-1}\sum_{k=1}^{n} kx^kW_{-2k+1} - x^{-1}\sum_{k=1}^{n} x^kW_{-2k+1}). \end{array} \quad (5)$$

Then, solving system (4)-(5) (using Theorem 2 (b) and (c)), the required result of (b) and (c) follow.
□

Remark 2. Note that the proof of Theorem 6 can be done by taking the derivative of the formulas in Theorem 2. In fact, since

$$\begin{array}{rcl} \sum_{k=1}^{n} x^kW_{-k} & = & \dfrac{\Theta_4(x)}{rx^3 + sx^2 + tx + u - x^4}, \\ \sum_{k=1}^{n} x^kW_{-2k} & = & \dfrac{x\Theta_5(x)}{2sx^3 + t^2x + 2ux^2 + r^2x^3 - s^2x^2 - u^2 - x^4 + 2rtx^2 - 2sux}, \\ \sum_{k=1}^{n} x^kW_{-2k+1} & = & \dfrac{x\Theta_6(x)}{2sx^3 + t^2x + 2ux^2 + r^2x^3 - s^2x^2 - u^2 - x^4 + 2rtx^2 - 2sux}, \end{array}$$

by taking the derivative of the both sides of the above formulas with respect to x, we get

$$\begin{array}{rcl} \sum_{k=1}^{n} kx^{k-1}W_{-k} & = & \dfrac{(rx^3 + sx^2 + tx + u - x^4)\Theta_4'(x) - (-4x^3 + 3rx^2 + 2sx + t)\Theta_4(x)}{(rx^3 + sx^2 + tx + u - x^4)^2}, \\ \sum_{k=1}^{n} kx^{k-1}W_{-2k} & = & \dfrac{\begin{array}{c}(2sx^3 + t^2x + 2ux^2 + r^2x^3 - s^2x^2 - u^2 - x^4 + 2rtx^2 - 2sux)(\Theta_5(x) + x\Theta_5'(x)) \\ -(3r^2x^2 + 4rtx - 2s^2x + 6sx^2 - 2us + t^2 - 4x^3 + 4ux)x\Theta_5(x)\end{array}}{(2sx^3 + t^2x + 2ux^2 + r^2x^3 - s^2x^2 - u^2 - x^4 + 2rtx^2 - 2sux)^2}, \\ \sum_{k=1}^{n} kx^{k-1}W_{-2k+1} & = & \dfrac{\begin{array}{c}(2sx^3 + t^2x + 2ux^2 + r^2x^3 - s^2x^2 - u^2 - x^4 + 2rtx^2 - 2sux)(\Theta_6(x) + x\Theta_6'(x)) \\ -(3r^2x^2 + 4rtx - 2s^2x + 6sx^2 - 2us + t^2 - 4x^3 + 4ux)x\Theta_6(x)\end{array}}{(2sx^3 + t^2x + 2ux^2 + r^2x^3 - s^2x^2 - u^2 - x^4 + 2rtx^2 - 2sux)^2}, \end{array}$$

i.e.,

$$\begin{aligned}
\sum_{k=1}^{n} kx^k W_{-k} &= x\frac{(rx^3+sx^2+tx+u-x^4)\Theta_4^{'}(x)-(-4x^3+3rx^2+2sx+t)\Theta_4(x)}{(rx^3+sx^2+tx+u-x^4)^2},\\
\sum_{k=1}^{n} kx^k W_{-2k} &= x\frac{\begin{array}{c}(2sx^3+t^2x+2ux^2+r^2x^3-s^2x^2-u^2-x^4+2rtx^2-2sux)(\Theta_5(x)+x\Theta_5^{'}(x))\\ -(3r^2x^2+4rtx-2s^2x+6sx^2-2us+t^2-4x^3+4ux)x\Theta_5(x)\end{array}}{(2sx^3+t^2x+2ux^2+r^2x^3-s^2x^2-u^2-x^4+2rtx^2-2sux)^2},\\
\sum_{k=1}^{n} kx^k W_{-2k+1} &= x\frac{\begin{array}{c}(2sx^3+t^2x+2ux^2+r^2x^3-s^2x^2-u^2-x^4+2rtx^2-2sux)(\Theta_6(x)+x\Theta_6^{'}(x))\\ -(3r^2x^2+4rtx-2s^2x+6sx^2-2us+t^2-4x^3+4ux)x\Theta_6(x)\end{array}}{(2sx^3+t^2x+2ux^2+r^2x^3-s^2x^2-u^2-x^4+2rtx^2-2sux)^2},
\end{aligned}$$

where $\Theta_4^{'}(x)$, $\Theta_5^{'}(x)$ and $\Theta_6^{'}(x)$ denotes the derivatives of $\Theta_4(x)$, $\Theta_5(x)$ and $\Theta_6(x)$ respectively.

5. Specific cases

In this section, for the specific cases of x, we present the closed form solutions (identities) of the sums $\sum_{k=1}^{n} kx^k W_{-k}$, $\sum_{k=1}^{n} kx^k W_{-2k}$ and $\sum_{k=1}^{n} kx^k W_{-2k+1}$ for the specific case of sequence $\{W_n\}$.

5.1. The case $x = 1$

In this subsection we consider the special case $x = 1$.

The case $x = 1$ of Theorem 6 is given in Soykan [34].

We only consider the cases $x = 1, r = 1, s = 1, t = 1, u = 2$ (which is not considered in [34]).

Observe that setting $x = 1, r = 1, s = 1, t = 1, u = 2$ (i.e., for the generalized fourth order Jacobsthal case) in Theorem 6 (a),(b),(c) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule however provides the evaluation of the sum formulas.

Taking $r = 1, s = 1, t = 1, u = 2$ in Theorem 6, we obtain the following theorem.

Theorem 7. *If $r = 1, s = 1, t = 1, u = 2$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} kW_{-k} = \frac{1}{16}(-(4n+2)W_{-n+3} - 4W_{-n+2} + (4n-2)W_{-n+1} + (8n+4)W_{-n} + 2W_3 + 4W_2 + 2W_1 - 4W_0)$.

(b) $\sum_{k=1}^{n} kW_{-2k} = \frac{1}{72}((6n^2-8n-29)W_{-2n+2} - 2(6n^2+4n-19)W_{-2n+1} + (6n^2+28n-11)W_{-2n} - 2(6n^2+4n-19)W_{-2n-1} - 19W_3 + 48W_2 - 19W_1 + 30W_0)$.

(c) $\sum_{k=1}^{n} kW_{-2k+1} = \frac{1}{72}(-(6n^2+16n-9)W_{-2n+2} + 4(3n^2+5n-10)W_{-2n+1} - (6n^2+16n-9)W_{-2n} + 2(6n^2-8n-29)W_{-2n-1} + 29W_3 - 38W_2 + 11W_1 - 38W_0)$.

Proof. **(a)** We use Theorem 6 (a). If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 6 (a) we get (a).

(b) We use Theorem 6 (b). If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 6 (b) then we have

$$\sum_{k=1}^{n} kx^k W_{-2k} = \frac{g_6(x)}{(x-1)^2(x-4)^2(x+1)^4}$$

where $g_6(x) = -x^{n+1}(8x+x^2+6x^3+2x^4-2x^5+x^6+n(-x^2+x+2)(x^4-3x^3-5x^2+3x+4)+8)W_{-2n+2} + x^{n+1}(16x+16x^2+12x^3-4x^5+n(2x+2)(x^4-3x^3-5x^2+3x+4)+8)W_{-2n+1} - x^{n+1}(16x+32x^2-10x^3-12x^4+6x^5+n(-x^3+3x^2+2x-2)(x^4-3x^3-5x^2+3x+4)-8)W_{-2n} + 2x^{n+1}(8x+8x^2+6x^3-2x^5+n(x+1)(x^4-3x^3-5x^2+3x+4)+4)W_{-2n-1} - x(-2x^5+6x^3+8x^2+8x+4)W_3 + x(x^6-4x^5+2x^4+12x^3+9x^2+16x+12)W_2 - x(-2x^5+6x^3+8x^2+8x+4)W_1 - 2x(-2x^5+6x^4+2x^3-20x^2-12x+2)W_0$.

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (b) using

$\sum\limits_{k=1}^{n} kW_{-2k} = \left.\frac{\frac{d^2}{dx^2}(g_6(x))}{\frac{d^2}{dx^2}\left((x-1)^2(x-4)^2(x+1)^4\right)}\right|_{x=1} = \frac{1}{72}((6n^2-8n-29)W_{-2n+2} - 2(6n^2+4n-19)W_{-2n+1} + (6n^2+28n-11)W_{-2n} - 2(6n^2+4n-19)W_{-2n-1} - 19W_3 + 48W_2 - 19W_1 + 30W_0)$.

(c) We use Theorem 6 (c). If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 6 (c) then we have

$$\sum_{k=1}^{n} kx^k W_{-2k+1} = \frac{g_7(x)}{(x-1)^2(x-4)^2(x+1)^4},$$

where $g_7(x) = x^{n+2}(15x+6x^2-2x^3-2x^4-x^5+n(x+1)(x^4-3x^3-5x^2+3x+4)+8)W_{-2n+2} - x^{n+2}(33x-4x^2-10x^3+4x^4+x^5+n(-x^2+2x+3)(x^4-3x^3-5x^2+3x+4)+24)W_{-2n+1} + x^{n+2}(15x+6x^2-2x^3-2x^4-x^5+n(x+1)(x^4-3x^3-5x^2+3x+4)+8)W_{-2n} - 2x^{n+1}(8x+x^2+6x^3+2x^4-2x^5+x^6+n(-x^2+x+2)(x^4-3x^3-5x^2+3x+4)+8)W_{-2n-1} + x(x^6-2x^5+2x^4+6x^3+x^2+8x+8)W_3 - x(-4x^5+12x^3+16x^2+16x+8)W_2 + x(6x^5-12x^4-10x^3+32x^2+16x-8)W_1 - 2x(-2x^5+6x^3+8x^2+8x+4)W_0$.

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (c) using

$\sum\limits_{k=1}^{n} kW_{-2k+1} = \left.\frac{\frac{d^2}{dx^2}(g_7(x))}{\frac{d^2}{dx^2}\left((x-1)^2(x-4)^2(x+1)^4\right)}\right|_{x=1} = \frac{1}{72}(-(6n^2+16n-9)W_{-2n+2} + 4(3n^2+5n-10)W_{-2n+1} - (6n^2+16n-9)W_{-2n} + 2(6n^2-8n-29)W_{-2n-1} + 29W_3 - 38W_2 + 11W_1 - 38W_0)$.

□

Taking $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1$ in the last theorem, we have the following corollary which presents linear sum formula of fourth-order Jacobsthal numbers.

Corollary 22. *For $n \geq 1$, fourth order Jacobsthal numbers have the following property*

(a) $\sum\limits_{k=1}^{n} kJ_{-k} = \frac{1}{16}(-(4n+2)J_{-n+3} - 4J_{-n+2} + (4n-2)J_{-n+1} + (8n+4)J_{-n} + 8)$.

(b) $\sum\limits_{k=1}^{n} kJ_{-2k} = \frac{1}{72}((6n^2-8n-29)J_{-2n+2} - 2(6n^2+4n-19)J_{-2n+1} + (6n^2+28n-11)J_{-2n} - 2(6n^2+4n-19)J_{-2n-1} + 10)$.

(c) $\sum\limits_{k=1}^{n} kJ_{-2k+1} = \frac{1}{72}(-(6n^2+16n-9)J_{-2n+2} + 4(3n^2+5n-10)J_{-2n+1} - (6n^2+16n-9)J_{-2n} + 2(6n^2-8n-29)J_{-2n-1} + 2)$.

From the last theorem, we have the following corollary which gives linear sum formulas of the fourth order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10$).

Corollary 23. *For $n \geq 1$, fourth order Jacobsthal-Lucas numbers have the following property*

(a) $\sum\limits_{k=1}^{n} kj_{-k} = \frac{1}{16}(-(4n+2)j_{-n+3} - 4j_{-n+2} + (4n-2)j_{-n+1} + (8n+4)j_{-n} + 34)$.

(b) $\sum\limits_{k=1}^{n} kj_{-2k} = \frac{1}{72}((6n^2-8n-29)j_{-2n+2} - 2(6n^2+4n-19)j_{-2n+1} + (6n^2+28n-11)j_{-2n} - 2(6n^2+4n-19)j_{-2n-1} + 91)$.

(c) $\sum\limits_{k=1}^{n} kj_{-2k+1} = \frac{1}{72}(-(6n^2+16n-9)j_{-2n+2} + 4(3n^2+5n-10)j_{-2n+1} - (6n^2+16n-9)j_{-2n} + 2(6n^2-8n-29)j_{-2n-1} + 35)$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10$ in the last theorem, we have the following corollary which presents linear sum formula of the modified fourth order Jacobsthal numbers.

Corollary 24. *For $n \geq 1$, modified fourth order Jacobsthal numbers have the following property:*

(a) $\sum\limits_{k=1}^{n} kK_{-k} = \frac{1}{16}(-(4n+2)K_{-n+3} - 4K_{-n+2} + (4n-2)K_{-n+1} + (8n+4)K_{-n} + 22)$.

(b) $\sum\limits_{k=1}^{n} kK_{-2k} = \frac{1}{72}((6n^2-8n-29)K_{-2n+2} - 2(6n^2+4n-19)K_{-2n+1} + (6n^2+28n-11)K_{-2n} - 2(6n^2+4n-19)K_{-2n-1} + 25)$.

(c) $\sum\limits_{k=1}^{n} kK_{-2k+1} = \frac{1}{72}(-(6n^2+16n-9)K_{-2n+2} + 4(3n^2+5n-10)K_{-2n+1} - (6n^2+16n-9)K_{-2n} + 2(6n^2-8n-29)K_{-2n-1} + 73)$.

From the last theorem, we have the following corollary which gives linear sum formula of the fourth-order Jacobsthal Perrin numbers (take $W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8$).

Corollary 25. *For $n \geq 1$, fourth-order Jacobsthal Perrin numbers have the following property:*

(a) $\sum\limits_{k=1}^{n} kQ_{-k} = \frac{1}{16}(-(4n+2)Q_{-n+3} - 4Q_{-n+2} + (4n-2)Q_{-n+1} + (8n+4)Q_{-n} + 12)$.

(b) $\sum\limits_{k=1}^{n} kQ_{-2k} = \frac{1}{72}((6n^2-8n-29)Q_{-2n+2} - 2(6n^2+4n-19)Q_{-2n+1} + (6n^2+28n-11)Q_{-2n} - 2(6n^2+4n-19)Q_{-2n-1} + 34)$.

(c) $\sum\limits_{k=1}^{n} kQ_{-2k+1} = \frac{1}{72}(-(6n^2+16n-9)Q_{-2n+2} + 4(3n^2+5n-10)Q_{-2n+1} - (6n^2+16n-9)Q_{-2n} + 2(6n^2-8n-29)Q_{-2n-1} + 42)$.

Taking $W_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2$ in the last theorem, we have the following corollary which presents linear sum formula of the adjusted fourth-order Jacobsthal numbers.

Corollary 26. *For $n \geq 1$, adjusted fourth-order Jacobsthal numbers have the following property:*

(a) $\sum\limits_{k=1}^{n} kS_{-k} = \frac{1}{16}(-(4n+2)S_{-n+3} - 4S_{-n+2} + (4n-2)S_{-n+1} + (8n+4)S_{-n} + 10)$.

(b) $\sum\limits_{k=1}^{n} kS_{-2k} = \frac{1}{72}((6n^2-8n-29)S_{-2n+2} - 2(6n^2+4n-19)S_{-2n+1} + (6n^2+28n-11)S_{-2n} - 2(6n^2+4n-19)S_{-2n-1} - 9)$.

(c) $\sum\limits_{k=1}^{n} kS_{-2k+1} = \frac{1}{72}(-(6n^2+16n-9)S_{-2n+2} + 4(3n^2+5n-10)S_{-2n+1} - (6n^2+16n-9)S_{-2n} + 2(6n^2-8n-29)S_{-2n-1} + 31)$.

From the last theorem, we have the following corollary which gives linear sum formula of the modified fourth-order Jacobsthal-Lucas numbers (take $W_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7$).

Corollary 27. *For $n \geq 1$, modified fourth-order Jacobsthal-Lucas numbers have the following property:*

(a) $\sum\limits_{k=1}^{n} kR_{-k} = \frac{1}{16}(-(4n+2)R_{-n+3} - 4R_{-n+2} + (4n-2)R_{-n+1} + (8n+4)R_{-n} + 12)$.

(b) $\sum\limits_{k=1}^{n} kR_{-2k} = \frac{1}{72}((6n^2-8n-29)R_{-2n+2} - 2(6n^2+4n-19)R_{-2n+1} + (6n^2+28n-11)R_{-2n} - 2(6n^2+4n-19)R_{-2n-1} + 112)$.

(c) $\sum\limits_{k=1}^{n} kR_{-2k+1} = \frac{1}{72}(-(6n^2+16n-9)R_{-2n+2} + 4(3n^2+5n-10)R_{-2n+1} - (6n^2+16n-9)R_{-2n} + 2(6n^2-8n-29)R_{-2n-1} - 52)$.

5.2. The case $x = -1$

In this subsection we consider the special case $x = -1$.

Taking $x = -1$, $r = s = t = u = 1$ in Theorem 6 (a) and (b) (or (c)), we obtain the following proposition.

Proposition 5. *If $r = s = t = u = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum\limits_{k=1}^{n} k(-1)^k W_{-k} = (-1)^n(-(n-5)W_{-n+3} + (2n-9)W_{-n+2} - (n-2)W_{-n+1} + (2n-6)W_{-n}) - 5W_3 + 9W_2 - 2W_1 + 6W_0$.

(b) $\sum\limits_{k=1}^{n} k(-1)^k W_{-2k} = (-1)^n(-(n-2)W_{-2n+2} + (n-3)W_{-2n+1} + (2n-2)W_{-2n} + W_{-2n-1}) - W_3 - W_2 + 4W_1 + 3W_0$.

(c) $\sum\limits_{k=1}^{n} k(-1)^k W_{-2k+1} = (-1)^n(-W_{-2n+2} + nW_{-2n+1} - (n-3)W_{-2n} - (n-2)W_{-2n-1}) - 2W_3 + 3W_2 + 2W_1 - W_0$.

From the above proposition, we have the following corollary which gives linear sum formulas of Tetranacci numbers (take $W_n = M_n$ with $M_0 = 0, M_1 = 1, M_2 = 1, M_3 = 2$).

Corollary 28. *For $n \geq 1$, Tetranacci numbers have the following properties.*

(a) $\sum_{k=1}^{n} k(-1)^k M_{-k} = (-1)^n \left(-(n-5)M_{-n+3} + (2n-9)M_{-n+2} - (n-2)M_{-n+1} + (2n-6)M_{-n}\right) - 3.$

(b) $\sum_{k=1}^{n} k(-1)^k M_{-2k} = (-1)^n \left(-(n-2)M_{-2n+2} + (n-3)M_{-2n+1} + (2n-2)M_{-2n} + M_{-2n-1}\right) + 1.$

(c) $\sum_{k=1}^{n} k(-1)^k M_{-2k+1} = (-1)^n \left(-M_{-2n+2} + nM_{-2n+1} - (n-3)M_{-2n} - (n-2)M_{-2n-1}\right) + 1.$

Taking $W_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7$ in the above proposition, we have the following corollary which presents linear sum formulas of Tetranacci-Lucas numbers.

Corollary 29. *For $n \geq 1$, Tetranacci-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} k(-1)^k R_{-k} = (-1)^n \left(-(n-5)R_{-n+3} + (2n-9)R_{-n+2} - (n-2)R_{-n+1} + (2n-6)R_{-n}\right) + 14.$

(b) $\sum_{k=1}^{n} k(-1)^k R_{-2k} = (-1)^n \left(-(n-2)R_{-2n+2} + (n-3)R_{-2n+1} + (2n-2)R_{-2n} + R_{-2n-1}\right) + 6.$

(c) $\sum_{k=1}^{n} k(-1)^k R_{-2k+1} = (-1)^n \left(-R_{-2n+2} + nR_{-2n+1} - (n-3)R_{-2n} - (n-2)R_{-2n-1}\right) - 7.$

Taking $x = -1, r = 2, s = t = u = 1$ in Theorem 6 (a) and (b) (or (c)), we obtain the following proposition.

Proposition 6. *If $r = 2, s = t = u = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} k(-1)^k W_{-k} = \frac{1}{4}\left((-1)^n \left(-(2n-7)W_{-n+3} + (6n-19)W_{-n+2} - (4n-6)W_{-n+1} + (6n-9)W_{-n}\right) - 7W_3 + 19W_2 - 6W_1 + 9W_0\right).$

(b) $\sum_{k=1}^{n} k(-1)^k W_{-2k} = \frac{1}{4}\left((-1)^n \left(-(2n-3)W_{-2n+2} + (2n-3)W_{-2n+1} + (8n-10)W_{-2n} + (2n-5)W_{-2n-1}\right) + 5W_3 - 13W_2 - 2W_1 + 5W_0\right).$

(c) $\sum_{k=1}^{n} k(-1)^k W_{-2k+1} = \frac{1}{4}\left((-1)^n \left(-(2n-3)W_{-2n+2} + (6n-7)W_{-2n+1} - 2W_{-2n} - (2n-3)W_{-2n-1}\right) - 3W_3 + 3W_2 + 10W_1 + 5W_0\right).$

From the last proposition, we have the following corollary which gives linear sum formulas of the fourth-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5$).

Corollary 30. *For $n \geq 1$, fourth-order Pell numbers have the following properties:*

(a) $\sum_{k=1}^{n} k(-1)^k P_{-k} = \frac{1}{4}\left((-1)^n \left(-(2n-7)P_{-n+3} + (6n-19)P_{-n+2} - (4n-6)P_{-n+1} + (6n-9)P_{-n}\right) - 3\right).$

(b) $\sum_{k=1}^{n} k(-1)^k P_{-2k} = \frac{1}{4}\left((-1)^n \left(-(2n-3)P_{-2n+2} + (2n-3)P_{-2n+1} + (8n-10)P_{-2n} + (2n-5)P_{-2n-1}\right) - 3\right).$

(c) $\sum_{k=1}^{n} k(-1)^k P_{-2k+1} = \frac{1}{4}\left((-1)^n \left(-(2n-3)P_{-2n+2} + (6n-7)P_{-2n+1} - 2P_{-2n} - (2n-3)P_{-2n-1}\right) + 1\right).$

Taking $W_n = Q_n$ with $Q_0 = 4, Q_1 = 2, Q_2 = 6, Q_3 = 17$ in the last proposition, we have the following corollary which presents linear sum formulas of the fourth-order Pell-Lucas numbers.

Corollary 31. *For $n \geq 1$, fourth-order Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} k(-1)^k Q_{-k} = \frac{1}{4}\left((-1)^n \left(-(2n-7)Q_{-n+3} + (6n-19)Q_{-n+2} - (4n-6)Q_{-n+1} + (6n-9)Q_{-n}\right) + 19\right).$

(b) $\sum_{k=1}^{n} k(-1)^k Q_{-2k} = \frac{1}{4}\left((-1)^n \left(-(2n-3)Q_{-2n+2} + (2n-3)Q_{-2n+1} + (8n-10)Q_{-2n} + (2n-5)Q_{-2n-1}\right) + 23\right).$

(c) $\sum_{k=1}^{n} k(-1)^k Q_{-2k+1} = \frac{1}{4}\left((-1)^n \left(-(2n-3)Q_{-2n+2} + (6n-7)Q_{-2n+1} - 2Q_{-2n} - (2n-3)Q_{-2n-1}\right) + 7\right).$

Observe that setting $x=-1, r=1, s=1, t=1, u=2$ (i.e. for the generalized fourth order Jacobsthal case) in Theorem 6 (a),(b),(c) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule however provides the evaluation of the sum formulas.

Taking $r=1, s=1, t=1, u=2$ in Theorem 6, we obtain the following theorem.

Theorem 8. *If $r=1, s=1, t=1, u=2$ then for $n\geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} k(-1)^k W_{-k} = \frac{1}{36}(-(-1)^n(3n^2-5n-39)W_{-n+3}+2(-1)^n(3n+10)(n-4)W_{-n+2}-(-1)^n(3n^2+13n-39)W_{-n+1}+2(-1)^n(3n^2+7n-31)W_{-n}-39W_3+80W_2-39W_1+62W_0)$.

(b) $\sum_{k=1}^{n} k(-1)^k W_{-2k} = \frac{1}{100}((-1)^n(-(15n^2+4n-69)W_{-2n+2}+2(5n^2-2n-24)W_{-2n+1}+(35n^2+46n-140)W_{-2n}+2(5n^2-2n-24)W_{-2n-1})+24W_3-93W_2+24W_1+116W_0)$.

(c) $\sum_{k=1}^{n} k(-1)^k W_{-2k+1} = \frac{1}{100}((-1)^n(-(n+3)(5n-7)W_{-2n+2}+(20n^2+42n-71)W_{-2n+1}-(n+3)(5n-7)W_{-2n}-2(15n^2+4n-69)W_{-2n-1})-69W_3+48W_2+140W_1+48W_0)$.

Proof. **(a)** We use Theorem 6 (a). If we set $r=1, s=1, t=1, u=2$ in Theorem 6 (a) then we have

$$\sum_{k=1}^{n} x^k W_{-k} = \frac{g_8(x)}{(x-2)^2(x+1)^2(x^2+1)^2}$$

where $g_8(x) = -x^{n+1}(n(-x^4+x^3+x^2+x+2)-x^2-2x^3+3x^4+2)W_{-n+3}-x^{n+1}(4x+2x^2+2x^3-4x^4+2x^5+n(x-1)(-x^4+x^3+x^2+x+2)-2)W_{-n+2}+x^{n+1}(4x-6x^2-4x^3+x^4+2x^5-x^6+n(-x^2+x+1)(-x^4+x^3+x^2+x+2)+2)W_{-n+1}+x^{n+1}(4x+6x^2-8x^3+n(-x^3+x^2+x+1)(-x^4+x^3+x^2+x+2)+2)W_{-n}-x(-3x^4+2x^3+x^2-2)W_3+x(2x^5-4x^4+2x^3+2x^2+4x-2)W_2-x(-x^6+2x^5+x^4-4x^3-6x^2+4x+2)W_1-2x(-4x^3+3x^2+2x+1)W_0$. For $x=-1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (a) using

$$\sum_{k=1}^{n} k(-1)^k W_{-k} = \left.\frac{\frac{d^2}{dx^2}(g_8(x))}{\frac{d^2}{dx^2}((x-2)^2(x+1)^2(x^2+1)^2)}\right|_{x=-1} = \frac{1}{36}(-(-1)^n(3n^2-5n-39)W_{-n+3}+2(-1)^n(3n+10)(n-4)W_{-n+2}-(-1)^n(3n^2+13n-39)W_{-n+1}+2(-1)^n(3n^2+7n-31)W_{-n}-39W_3+80W_2-39W_1+62W_0).$$

(b) We use Theorem 6 (b). If we set $r=1, s=1, t=1, u=2$ in Theorem 6 (b) then we have

$$\sum_{k=1}^{n} x^k W_{-2k} = \frac{g_9(x)}{(x-1)^2(x-4)^2(x+1)^4}$$

where $g_9(x) = -x^{n+1}(8x+x^2+6x^3+2x^4-2x^5+x^6+n(-x^2+x+2)(x^4-3x^3-5x^2+3x+4)+8)W_{-2n+2}+x^{n+1}(16x+16x^2+12x^3-4x^5+n(2x+2)(x^4-3x^3-5x^2+3x+4)+8)W_{-2n+1}-x^{n+1}(16x+32x^2-10x^3-12x^4+6x^5+n(-x^3+3x^2+2x-2)(x^4-3x^3-5x^2+3x+4)-8)W_{-2n}+2x^{n+1}(8x+8x^2+6x^3-2x^5+n(x+1)(x^4-3x^3-5x^2+3x+4)+4)W_{-2n-1}-x(-2x^5+6x^3+8x^2+8x+4)W_3+x(x^6-4x^5+2x^4+12x^3+9x^2+16x+12)W_2-x(-2x^5+6x^3+8x^2+8x+4)W_1-2x(-2x^5+6x^4+2x^3-20x^2-12x+2)W_0$. For $x=-1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (b) using

$$\sum_{k=1}^{n} k(-1)^k W_{-2k} = \left.\frac{\frac{d^4}{dx^4}(g_9(x))}{\frac{d^4}{dx^4}((x-1)^2(x-4)^2(x+1)^4)}\right|_{x=-1} = \frac{1}{100}((-1)^n(-(15n^2+4n-69)W_{-2n+2}+2(5n^2-2n-24)W_{-2n+1}+(35n^2+46n-140)W_{-2n}+2(5n^2-2n-24)W_{-2n-1})+24W_3-93W_2+24W_1+116W_0).$$

(c) We use Theorem 6 (c). If we set $r=1, s=1, t=1, u=2$ in Theorem 6 (c) then we have

$$\sum_{k=1}^{n} x^k W_{-2k+1} = \frac{g_{10}(x)}{(x-1)^2(x-4)^2(x+1)^4}$$

where $g_{10}(x) = x^{n+2}(15x+6x^2-2x^3-2x^4-x^5+n(x+1)(x^4-3x^3-5x^2+3x+4)+8)W_{-2n+2}-x^{n+2}(33x-4x^2-10x^3+4x^4+x^5+n(-x^2+2x+3)(x^4-3x^3-5x^2+3x+4)+24)W_{-2n+1}+x^{n+2}(15x+6x^2-2x^3-2x^4-x^5+n(x+1)(x^4-3x^3-5x^2+3x+4)+8)W_{-2n}-2x^{n+1}(8x+x^2+6x^3+2x^4-2x^5+x^6+n(-x^2+x+2)(x^4-$

$3x^3-5x^2+3x+4)+8)W_{-2n-1}+x(x^6-2x^5+2x^4+6x^3+x^2+8x+8)W_3-x(-4x^5+12x^3+16x^2+16x+8)W_2+x(6x^5-12x^4-10x^3+32x^2+16x-8)W_1-2x(-2x^5+6x^3+8x^2+8x+4)W_0$. For $x=-1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (c) using

$\sum\limits_{k=1}^{n}k(-1)^kW_{-2k+1} = \left.\frac{\frac{d^4}{dx^4}(g_{10}(x))}{\frac{d^4}{dx^4}((x-1)^2(x-4)^2(x+1)^4)}\right|_{x=-1} = \frac{1}{100}((-1)^n(-(n+3)(5n-7)W_{-2n+2}+(20n^2+42n-71)W_{-2n+1}-(n+3)(5n-7)W_{-2n}-2(15n^2+4n-69)W_{-2n-1})-69W_3+48W_2+140W_1+48W_0)$.

□

Taking $W_n=J_n$ with $J_0=0, J_1=1, J_2=1, J_3=1$ in the last proposition, we have the following corollary which presents linear sum formula of the fourth-order Jacobsthal numbers.

Corollary 32. *For $n\geq 1$, fourth order Jacobsthal numbers have the following property*

(a) $\sum\limits_{k=1}^{n}k(-1)^kJ_{-k}=\frac{1}{36}(-(-1)^n(3n^2-5n-39)J_{-n+3}+2(-1)^n(3n+10)(n-4)J_{-n+2}-(-1)^n(3n^2+13n-39)J_{-n+1}+2(-1)^n(3n^2+7n-31)J_{-n}+2)$.

(b) $\sum\limits_{k=1}^{n}k(-1)^kJ_{-2k}=\frac{1}{100}((-1)^n(-(15n^2+4n-69)J_{-2n+2}+2\left(5n^2-2n-24\right)J_{-2n+1}+(35n^2+46n-140)J_{-2n}+2\left(5n^2-2n-24\right)J_{-2n-1})-45)$.

(c) $\sum\limits_{k=1}^{n}k(-1)^kJ_{-2k+1}=\frac{1}{100}((-1)^n(-(n+3)(5n-7)J_{-2n+2}+(20n^2+42n-71)J_{-2n+1}-(n+3)(5n-7)J_{-2n}-2(15n^2+4n-69)J_{-2n-1})+119)$.

From the last proposition, we have the following corollary which gives linear sum formulas of the fourth order Jacobsthal-Lucas numbers (take $W_n=j_n$ with $j_0=2, j_1=1, j_2=5, j_3=10$).

Corollary 33. *For $n\geq 1$, fourth order Jacobsthal-Lucas numbers have the following property*

(a) $\sum\limits_{k=1}^{n}k(-1)^kj_{-k}=\frac{1}{36}(-(-1)^n(3n^2-5n-39)j_{-n+3}+2(-1)^n(3n+10)(n-4)j_{-n+2}-(-1)^n(3n^2+13n-39)j_{-n+1}+2(-1)^n(3n^2+7n-31)j_{-n}+95)$.

(b) $\sum\limits_{k=1}^{n}k(-1)^kj_{-2k}=\frac{1}{100}((-1)^n(-(15n^2+4n-69)j_{-2n+2}+2\left(5n^2-2n-24\right)j_{-2n+1}+(35n^2+46n-140)j_{-2n}+2\left(5n^2-2n-24\right)j_{-2n-1})+31)$.

(c) $\sum\limits_{k=1}^{n}k(-1)^kj_{-2k+1}=\frac{1}{100}((-1)^n(-(n+3)(5n-7)j_{-2n+2}+(20n^2+42n-71)j_{-2n+1}-(n+3)(5n-7)j_{-2n}-2(15n^2+4n-69)j_{-2n-1})-214)$.

Taking $W_n=K_n$ with $K_0=3, K_1=1, K_2=3, K_3=10$ in the last proposition, we have the following corollary which presents linear sum formula of the modified fourth order Jacobsthal numbers.

Corollary 34. *For $n\geq 1$, modified fourth order Jacobsthal numbers have the following property:*

(a) $\sum\limits_{k=1}^{n}k(-1)^kK_{-k}=\frac{1}{36}(-(-1)^n(3n^2-5n-39)K_{-n+3}+2(-1)^n(3n+10)(n-4)K_{-n+2}-(-1)^n(3n^2+13n-39)K_{-n+1}+2(-1)^n(3n^2+7n-31)K_{-n}-3)$.

(b) $\sum\limits_{k=1}^{n}k(-1)^kK_{-2k}=\frac{1}{100}((-1)^n(-(15n^2+4n-69)K_{-2n+2}+2\left(5n^2-2n-24\right)K_{-2n+1}+(35n^2+46n-140)K_{-2n}+2\left(5n^2-2n-24\right)K_{-2n-1})+333)$.

(c) $\sum\limits_{k=1}^{n}k(-1)^kK_{-2k+1}=\frac{1}{100}((-1)^n(-(n+3)(5n-7)K_{-2n+2}+(20n^2+42n-71)K_{-2n+1}-(n+3)(5n-7)K_{-2n}-2(15n^2+4n-69)K_{-2n-1})-262)$.

From the last proposition, we have the following corollary which gives linear sum formula of the fourth-order Jacobsthal Perrin numbers (take $W_n=Q_n$ with $Q_0=3, Q_1=0, Q_2=2, Q_3=8$).

Corollary 35. *For $n \geq 1$, fourth-order Jacobsthal Perrin numbers have the following property:*

(a) $\sum_{k=1}^{n} k(-1)^k Q_{-k} = \frac{1}{36}(-(-1)^n (3n^2-5n-39)Q_{-n+3} + 2(-1)^n (3n+10)(n-4)Q_{-n+2} - (-1)^n (3n^2+13n-39)Q_{-n+1} + 2(-1)^n (3n^2+7n-31)Q_{-n} + 34)$.

(b) $\sum_{k=1}^{n} k(-1)^k Q_{-2k} = \frac{1}{100}((-1)^n (-(15n^2+4n-69)Q_{-2n+2} + 2\left(5n^2-2n-24\right) Q_{-2n+1} + (35n^2+46n-140)Q_{-2n} + 2\left(5n^2-2n-24\right) Q_{-2n-1}) + 354)$.

(c) $\sum_{k=1}^{n} k(-1)^k Q_{-2k+1} = \frac{1}{100}((-1)^n (-(n+3)(5n-7)Q_{-2n+2} + (20n^2+42n-71)Q_{-2n+1} - (n+3)(5n-7)Q_{-2n} - 2(15n^2+4n-69)Q_{-2n-1}) - 312)$.

Taking $W_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2$ in the last proposition, we have the following corollary which presents linear sum formula of adjusted fourth-order Jacobsthal numbers.

Corollary 36. *For $n \geq 1$, adjusted fourth-order Jacobsthal numbers have the following property:*

(a) $\sum_{k=1}^{n} k(-1)^k S_{-k} = \frac{1}{36}(-(-1)^n (3n^2-5n-39)S_{-n+3} + 2(-1)^n (3n+10)(n-4)S_{-n+2} - (-1)^n (3n^2+13n-39)S_{-n+1} + 2(-1)^n (3n^2+7n-31)S_{-n} - 37)$.

(b) $\sum_{k=1}^{n} k(-1)^k S_{-2k} = \frac{1}{100}((-1)^n (-(15n^2+4n-69)S_{-2n+2} + 2\left(5n^2-2n-24\right) S_{-2n+1} + (35n^2+46n-140)S_{-2n} + 2\left(5n^2-2n-24\right) S_{-2n-1}) - 21)$.

(c) $\sum_{k=1}^{n} k(-1)^k S_{-2k+1} = \frac{1}{100}((-1)^n (-(n+3)(5n-7)S_{-2n+2} + (20n^2+42n-71)S_{-2n+1} - (n+3)(5n-7)S_{-2n} - 2(15n^2+4n-69)S_{-2n-1}) + 50)$.

From the last proposition, we have the following corollary which gives linear sum formula of the modified fourth-order Jacobsthal-Lucas numbers (take $W_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7$).

Corollary 37. *For $n \geq 1$, modified fourth-order Jacobsthal-Lucas numbers have the following property:*

(a) $\sum_{k=1}^{n} k(-1)^k R_{-k} = \frac{1}{36}(-(-1)^n (3n^2-5n-39)R_{-n+3} + 2(-1)^n (3n+10)(n-4)R_{-n+2} - (-1)^n (3n^2+13n-39)R_{-n+1} + 2(-1)^n (3n^2+7n-31)R_{-n} + 176)$.

(b) $\sum_{k=1}^{n} k(-1)^k R_{-2k} = \frac{1}{100}((-1)^n (-(15n^2+4n-69)R_{-2n+2} + 2\left(5n^2-2n-24\right) R_{-2n+1} + (35n^2+46n-140)R_{-2n} + 2\left(5n^2-2n-24\right) R_{-2n-1}) + 377)$.

(c) $\sum_{k=1}^{n} k(-1)^k R_{-2k+1} = \frac{1}{100}((-1)^n (-(n+3)(5n-7)R_{-2n+2} + (20n^2+42n-71)R_{-2n+1} - (n+3)(5n-7)R_{-2n} - 2(15n^2+4n-69)R_{-2n-1}) - 7)$.

Taking $x = -1, r = 2, s = 3, t = 5, u = 7$ in Theorem 6 (a), (b) and (c), we obtain the following proposition.

Proposition 7. *If $r = 2, s = 3, t = 5, u = 7$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} k(-1)^k W_{-k} = \frac{1}{4}((-1)^n ((2n+11)W_{-n+3} - (6n+35)W_{-n+2} + 8W_{-n+1} - (10n+63)W_{-n}) - 11W_3 + 35W_2 - 8W_1 + 63W_0)$.

(b) $\sum_{k=1}^{n} k(-1)^k W_{-2k} = \frac{1}{36}((-1)^n ((6n+7)W_{-2n+2} - (6n-5)W_{-2n+1} - 36(n+2)W_{-2n} - 7(6n+13)W_{-2n-1}) + 13W_3 - 33W_2 - 44W_1 + 7W_0)$.

(c) $\sum_{k=1}^{n} k(-1)^k W_{-2k+1} = \frac{1}{36}((-1)^n ((6n+19)W_{-2n+2} - 3(6n+17)W_{-2n+1} - 4(3n+14)W_{-2n} + 7(6n+7)W_{-2n-1}) - 7W_3 - 5W_2 + 72W_1 + 91W_0)$.

From the last proposition, we have the following corollary which gives linear sum formulas of 4-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 1, G_3 = 2$).

Corollary 38. *For $n \geq 1$, 4-primes numbers have the following properties:*

(a) $\sum_{k=1}^{n} k(-1)^k G_{-k} = \frac{1}{4}((-1)^n((2n+11)G_{-n+3} - (6n+35)G_{-n+2} + 8G_{-n+1} - (10n+63)G_{-n}) + 13)$.

(b) $\sum_{k=1}^{n} k(-1)^k G_{-2k} = \frac{1}{36}((-1)^n((6n+7)G_{-2n+2} - (6n-5)G_{-2n+1} - 36(n+2)G_{-2n} - 7(6n+13)G_{-2n-1}) - 7)$.

(c) $\sum_{k=1}^{n} k(-1)^k G_{-2k+1} = \frac{1}{36}((-1)^n((6n+19)G_{-2n+2} - 3(6n+17)G_{-2n+1} - 4(3n+14)G_{-2n} + 7(6n+7)G_{-2n-1}) - 19)$.

Taking $G_n = H_n$ with $H_0 = 4, H_1 = 2, H_2 = 10, H_3 = 41$ in the last proposition, we have the following corollary which presents linear sum formulas of Lucas 4-primes numbers.

Corollary 39. *For $n \geq 1$, Lucas 4-primes numbers have the following properties:*

(a) $\sum_{k=1}^{n} k(-1)^k H_{-k} = \frac{1}{4}((-1)^n((2n+11)H_{-n+3} - (6n+35)H_{-n+2} + 8H_{-n+1} - (10n+63)H_{-n}) + 135)$.

(b) $\sum_{k=1}^{n} k(-1)^k H_{-2k} = \frac{1}{36}((-1)^n((6n+7)H_{-2n+2} - (6n-5)H_{-2n+1} - 36(n+2)H_{-2n} - 7(6n+13)H_{-2n-1}) + 143)$.

(c) $\sum_{k=1}^{n} k(-1)^k H_{-2k+1} = \frac{1}{36}((-1)^n((6n+19)H_{-2n+2} - 3(6n+17)H_{-2n+1} - 4(3n+14)H_{-2n} + 7(6n+7)H_{-2n-1}) + 171)$.

From the last proposition, we have the following corollary which gives linear sum formulas of modified 4-primes numbers (take $H_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 1, E_3 = 1$).

Corollary 40. *For $n \geq 1$, modified 4-primes numbers have the following properties:*

(a) $\sum_{k=1}^{n} k(-1)^k E_{-k} = \frac{1}{4}((-1)^n((2n+11)E_{-n+3} - (6n+35)E_{-n+2} + 8E_{-n+1} - (10n+63)E_{-n}) + 24)$.

(b) $\sum_{k=1}^{n} k(-1)^k E_{-2k} = \frac{1}{36}((-1)^n((6n+7)E_{-2n+2} - (6n-5)E_{-2n+1} - 36(n+2)E_{-2n} - 7(6n+13)E_{-2n-1}) - 20)$.

(c) $\sum_{k=1}^{n} k(-1)^k E_{-2k+1} = \frac{1}{36}((-1)^n((6n+19)E_{-2n+2} - 3(6n+17)E_{-2n+1} - 4(3n+14)E_{-2n} + 7(6n+7)E_{-2n-1}) - 12)$.

5.3. The case $x = i$

In this subsection, we consider the special case $x = i$. Taking $r = s = t = u = 1$ in Theorem 6, we obtain the following proposition.

Proposition 8. *If $r = s = t = u = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} ki^k W_{-k} = i^n(i(n-(5+2i))W_{-n+3} + (-1-i)(n-(\frac{9}{2}+\frac{5}{2}i))W_{-n+2} + (1-2i)(n-(4+2i))W_{-n+1} + 2(n-(3+i))W_{-n}) - (2-5i)W_3 - (2+7i)W_2 + (8-6i)W_1 + (6+2i)W_0$.

(b) $\sum_{k=1}^{n} ki^k W_{-2k} = \frac{1}{9+40i}(i^n((13-6i)(n-(\frac{8}{41}+\frac{10}{41}i))W_{-2n+2} + (-14-3i)(n-(\frac{81}{205}-\frac{32}{205}i))W_{-2n+1} + (-6+28i)(n+(\frac{83}{205}-\frac{91}{205}i))W_{-2n} + (-9+i)(n-(\frac{57}{82}-\frac{21}{82}i))W_{-2n-1}) - (6-3i)W_3 + (10-i)W_2 - 2iW_1 - (4+17i)W_0)$.

(c) $\sum_{k=1}^{n} ki^k W_{-2k+1} = \frac{1}{9+40i}(i^n((-1-9i)(n+(\frac{25}{82}+\frac{21}{82}i))W_{-2n+2} + (7+22i)W_{-2n+1}(n+(\frac{306}{533}-\frac{48}{533}i)) + (4-5i)(n+(\frac{33}{41}-\frac{10}{41}i))W_{-2n} + (13-6i)(n-(\frac{8}{41}+\frac{10}{41}i))W_{-2n-1}) + (4+2i)W_3 - (6-i)W_2 - (10+14i)W_1 - (6-3i)W_0)$.

From the above proposition, we have the following corollary which gives linear sum formulas of Tetranacci numbers (take $W_n = M_n$ with $M_0 = 0, M_1 = 1, M_2 = 1, M_3 = 2$).

Corollary 41. *For $n \geq 1$, Tetranacci numbers have the following properties.*

(a) $\sum_{k=1}^{n} ki^k M_{-k} = i^n(i(n-(5+2i))M_{-n+3} + (-1-i)(n-(\frac{9}{2}+\frac{5}{2}i))M_{-n+2} + (1-2i)(n-(4+2i))M_{-n+1} + 2(n-(3+i))M_{-n}) + (2-3i)$.

(b) $\sum_{k=1}^{n} ki^k M_{-2k} = \frac{1}{9+40i}(i^n((13-6i)(n-(\frac{8}{41}+\frac{10}{41}i))M_{-2n+2} + (-14-3i)(n-(\frac{81}{205}-\frac{32}{205}i))M_{-2n+1} + (-6+28i)(n+(\frac{83}{205}-\frac{91}{205}i))M_{-2n} + (-9+i)(n-(\frac{57}{82}-\frac{21}{82}i))M_{-2n-1}) + (-2+3i))$.

(c) $\sum\limits_{k=1}^{n} ki^{k}M_{-2k+1} = \frac{1}{9+40i}(i^{n}((-1-9i)(n+(\frac{25}{82}+\frac{21}{82}i))M_{-2n+2}+(7+22i)M_{-2n+1}(n+(\frac{306}{533}-\frac{48}{533}i))+(4-5i)(n+(\frac{33}{41}-\frac{10}{41}i))M_{-2n}+(13-6i)(n-(\frac{8}{41}+\frac{10}{41}i))M_{-2n-1})+(-8-9i))$.

Taking $W_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7$ in the above proposition, we have the following corollary which presents linear sum formulas of Tetranacci-Lucas numbers.

Corollary 42. *For $n \geq 1$, Tetranacci-Lucas numbers have the following properties.*

(a) $\sum\limits_{k=1}^{n} ki^{k}R_{-k} = i^{n}(i(n-(5+2i))R_{-n+3}+(-1-i)(n-(\frac{9}{2}+\frac{5}{2}i))R_{-n+2}+(1-2i)(n-(4+2i))R_{-n+1}+2(n-(3+i))R_{-n})+(12+16i)$.

(b) $\sum\limits_{k=1}^{n} ki^{k}R_{-2k} = \frac{1}{9+40i}(i^{n}((13-6i)(n-(\frac{8}{41}+\frac{10}{41}i))R_{-2n+2}+(-14-3i)(n-(\frac{81}{205}-\frac{32}{205}i))R_{-2n+1}+(-6+28i)(n+(\frac{83}{205}-\frac{91}{205}i))R_{-2n}+(-9+i)(n-(\frac{57}{82}-\frac{21}{82}i))R_{-2n-1})+(-28-52i))$.

(c) $\sum\limits_{k=1}^{n} ki^{k}R_{-2k+1} = \frac{1}{9+40i}(i^{n}((-1-9i)(n+(\frac{25}{82}+\frac{21}{82}i))R_{-2n+2}+(7+22i)R_{-2n+1}(n+(\frac{306}{533}-\frac{48}{533}i))+(4-5i)(n+(\frac{33}{41}-\frac{10}{41}i))R_{-2n}+(13-6i)(n-(\frac{8}{41}+\frac{10}{41}i))R_{-2n-1})+(-24+15i))$.

Corresponding sums of the other fourth order generalized Tetranacci numbers can be calculated similarly.

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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