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Results of semigroup of linear operators generating a nonlinear Schrödinger equation

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Abstract: In this paper, we present results of ω-order preserving partial contraction mapping generating a nonlinear Schrödinger equation. We used the theory of semigroup to generate a nonlinear Schrödinger equation by considering a simple application of Lipschitz perturbation of linear evolution equations. We considered the space $L^2(\mathbb{R}^2)$ and of linear operator A_0 by $D(A_0) = H^2(\mathbb{R}^2)$ and $A_0u = -i\Delta u$ for $u \in D(A_0)$ for the initial value problem, we hereby established that A_0 is the infinitesimal generator of a C_0-semigroup of unitary operators $T(t)$, $-\infty < t < \infty$ on $L^2(\mathbb{R}^2)$.

Keywords: ω-OCP_n; Evolution equation; C_0-semigroup; Schrödinger equation.

MSC: 06F15; 06F05; 20M05.

1. Introduction

Consider the initial value problem for the following nonlinear Schrödinger equation in $\mathbb{R}^2$

$$\begin{cases} \frac{1}{i}\frac{\partial u}{\partial t} - \Delta u + k|u|^2 u = 0, & \text{in } (0,\infty)\times\mathbb{R}^2, \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^2, \end{cases} \tag{1}$$

where u is a complex valued function and k a real constant. The space in which this problem is considered is $L^2(\mathbb{R}^2)$. By defining the linear operator A_0 by $D(A_0) = H^2(\mathbb{R}^2)$ and $A_0u = -i\Delta u$ for $u \in D(A_0)$ and $A \in \omega - OCP_n$, the initial value problem (1) can be rewritten as

$$\begin{cases} \frac{du}{dt} + A_0u + F(u) = 0, & \text{for} \quad t > 0, \\ u(0) = u_0, \end{cases} \tag{2}$$

where $F(u) = ik|u|^2u$.

It follows that the operator A_0 is the infinitesimal generator of a C_0-semigroup of unitary operators $T(t)$, $-\infty < t < \infty$, on $L^2(\mathbb{R}^2)$. A simple application of the Fourier transform gives the following explicit formula for $T(t)$;

$$(T(t)u)(x) = \frac{1}{4\pi it}\int_{\mathbb{R}^2} \exp\left\{i\frac{|x-y|^2}{4t}\right\} u(y)dy\,. \tag{3}$$

Suppose X is a Banach space, H is Hilbert space, $X_n \subseteq X$ is a finite set, $\omega - OCP_n$ the ω-order preserving partial contraction mapping, M_m be a matrix, $L(X)$ be a bounded linear operator on X, P_n a partial transformation semigroup, $\rho(A)$ a resolvent set, $\sigma(A)$ a spectrum of A and A is a generator of C_0-semigroup. This paper consists of results of ω-order preserving partial contraction mapping generating a nonlinear Schrödinger equation.

Akinyele *et al.*, [1], obtained a continuous time Markov semigroup of linear operators and also in [2], Akinyele *et al.*, established results of ω-order reversing partial contraction mapping generating a differential operator. Balakrishnan [3], presented an operator calculus for infinitesimal generators of the semigroup. Banach [4], established and introduced the concept of Banach spaces. Brezis and Gallouet [5] generated a nonlinear Schrödinger evolution equation. Chill and Tomilov [6], introduced some resolvent approaches to stability operator semigroup. Davies [7] deduced linear operators and their spectra. Engel and Nagel [8] obtained a one-parameter semigroup for linear evolution equations. Omosowon *et al.*, [9], generated some analytic results of the semigroup of the linear operator with dynamic boundary conditions, and also in [10], Omosowon *et al.*, introduced dual properties of ω-order reversing partial contraction mapping in semigroup of linear operator. Omosowon *et al.*, [11], established a regular weak*-continuous semigroup of linear operators, and also in [12], Omosowon *et al.*, generated quasilinear equations of evolution on semigroup of a linear operator. Pazy [13] presented the asymptotic behaviour of the solution of an abstract evolution and some applications and also, in [14], obtained a class of semi-linear equations of evolution. Rauf and Akinyele [15] obtained ω-order preserving partial contraction mapping and obtained its properties, also in [16], Rauf *et al.*, introduced some results of stability and spectra properties on semigroup of a linear operator. Vrabie [17], proved some results of C_0-semigroup and its applications. Yosida [18] deduced some results on differentiability and representation of one-parameter semigroup of linear operators.

2. Preliminaries

Definition 1. (C_0-Semigroup) [17] A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.

Definition 2. (ω-OCP_n) [15] A transformation $\alpha \in P_n$ is called ω-order preserving partial contraction mapping if $\forall x, y \in \text{Dom}\alpha : x \leq y \implies \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

Definition 3. (Evolution Equation) [13] An evolution equation is an equation that can be interpreted as the differential law of the development (evolution) in time of a system. The class of evolution equations includes, first of all, ordinary differential equations and systems of the form

$$u = f(t,u), u = f(t,u,u),$$

etc., in the case where $u(t)$ can be regarded naturally as the solution of the Cauchy problem; these equations describe the evolution of systems with finitely many degrees of freedom.

Definition 4. (Mild Solution) [14] A continuous solution u of the integral equation.

$$u(t) = T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)f(s,u(s))ds \tag{4}$$

will be called a mild solution of the initial value problem

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = f(t,u(t)), \ t > t_0, \\ u(t_0) = u_0, \end{cases} \tag{5}$$

if the solution is a Lipschitz continuous function.

Definition 5. (Schrödinger Equation) [19] The Schrödinger equation is a linear partial differential equation that governs the wave function of a quantum-mechanical system. It is a key result in quantum mechanics, and its discovery was a significant landmark in the development of the subject.

Example 1. 2×2 matrix $[M_m(\mathbb{R}^n)]$: Suppose

$$A = \begin{pmatrix} 2 & 0 \\ \Delta & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & I \\ e^{\Delta t} & e^{2t} \end{pmatrix}.$$

Example 2. 3×3 matrix $[M_m(\mathbb{C})]$: We have for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on X. Suppose we have

$$A = \begin{pmatrix} 2 & 2 & I \\ 2 & 2 & 2 \\ \Delta & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA_\lambda}$, then

$$e^{tA_\lambda} = \begin{pmatrix} e^{2t\lambda} & e^{2t\lambda} & I \\ e^{2t\lambda} & e^{2t\lambda} & e^{2t\lambda} \\ e^{\Delta t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

Example 3. Let $X = C_{ub}(\mathbb{N} \cup \{0\})$ be the space of all bounded and uniformly continuous function from $\mathbb{N} \cup \{0\}$ to $\mathbb{R}$, endowed with the sup-norm $\|\cdot\|_\infty$ and let $\{T(t); t \in \mathbb{R}_+\} \subseteq L(X)$ be defined by

$$[T(t)f](s) = f(t+s).$$

For each $f \in X$ and each $t, s \in \mathbb{R}_+$, one may easily verify that $\{T(t); t \in \mathbb{R}_+\}$ satisfies Examples 1 and 2 above.

3. Main results

This section present results of semigroup of linear operator by using ω-OCP_n to generates a nonlinear Schrödinger equation:

Theorem 1. *Suppose $A : D(A) \subseteq L^2(\mathbb{R}^2)$ is the infinitesimal generator of a semigroup $\{T(t),\ t \geq 0\}$ given by (3) where $A \in \omega - OCP_n$. If $2 \leq p \leq \infty$ and $\frac{1}{q} + \frac{1}{p} = 1$, then $T(t)$ can be extended in a unique way to an operator from $L^q(\mathbb{R}^2)$ into $L^p(\mathbb{R}^2)$ and*

$$\|T(t)u\|_{0,p} \leq (4\pi t)^{-(\frac{2}{q}-1)} \|u\|_{0,q}. \tag{6}$$

Proof. Since $T(t)$ is a unitary operator on $L^2(\mathbb{R}^2)$ we have

$$\|T(t)u\|_{0,2} = \|u\|_{0,2} \quad for\ u \in L^2(\mathbb{R}^2).$$

On the other hand it is clear from (3) that $T(t) : L^1(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)$ and that for $t > 0$, we have

$$\|T(t)u\|_{0,\infty} \leq (4\pi t)^{-1} \|u\|_{0,1}.$$

The Riesz convexity theorem implies in this situation that $T(t)$ can be extended uniquely to an operator from $L^q(\mathbb{R}^2)$ into $L^p(\mathbb{R}^2)$ and that (6) holds. In order to prove the existence of a local solution of the initial value problem (2) for every $u \in H^2(\mathbb{R}^2)$ and $A \in \omega - OCP_n$. We note that the graph norm of the operator A_0 in $L^2(\mathbb{R}^2)$, that is the norm $\|u\| = \|u\|_{0,2} + \|A_0 u\|$, for $u \in D(A_0)$ and $A \in \omega - OCP_n$ is equivalent to the norm $\|\cdot\|_{2,2}$ in $H^2(\mathbb{R}^2)$. Therefore $D(A_0)$ equipped with the graph norm is the space $H^2(\mathbb{R}^2)$. Hence the proof in competed. □

Theorem 2. *Assume $A : D(A) \subseteq H^2(\mathbb{R}^2) \to H^2(\mathbb{R}^2)$ is the infinitesimal generator of a C_0-semigroup $\{T(t);\ t \geq 0\}$. The nonlinear mapping $Fu = ik|u|^2 u$ maps $H^2(\mathbb{R}^2)$ into itself and satisfies for $u, v \in H^2(\mathbb{R}^2)$ and $A \in \omega - OCP_n$, we have*

$$\|F(u)\|_{2,2} \leq C\|u\|_{0,\infty}^2 \|u\|_{2,2}, \tag{7}$$

$$\|F(u) - F(v)\|_{2,2} \leq C(\|u\|_{2,2}^2 + \|v\|_{2,2}^2)\|u - v\|_{2,2}. \tag{8}$$

Proof. From Sobolev's theorem in $\mathbb{R}^2$, it follows that $H^2(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$ and that there is a constant C such that

$$\|u\|_{0,\infty} \leq C\|u\|_{2,2} \quad for\ u \in H^2(\mathbb{R}^2)\,. \tag{9}$$

Denoting by D any first order differential operator we have for every $u \in H^2(\mathbb{R}^2)$

$$|D^2(|u|^2u)| \leq C(|u|^2|D^2u| + |u||Du|^2)\,,$$

and therefore

$$\||u|^2u\|_{2,2} \leq C(\|u\|^2_{0,\infty}\|u\|_{2,2} + \|u\|_{0,\infty}\|u\|^2_{1,4}). \tag{10}$$

From Gagliardo-Nirenberg inequalities we have

$$\|u\|_{1,4} \leq C\|u\|^{\frac{1}{2}}_{0,\infty}\|u\|^{\frac{1}{2}}_{2,2}. \tag{11}$$

Combining (10) and (11), we obtain (7). The inequality (8) is proved similarly using Leibnitz formula for the derivatives of of products and estimates (9) and (11), and this achieved the proof. □

Theorem 3. *Suppose $A : D(A) \subseteq H^2(\mathbb{R}^2) \to H^2(\mathbb{R}^2)$ is the infinitesimal generator of a C_0-semigroup $\{T(t);\ t \geq 0\}$. Let $u_0 \in H^2\mathbb{R}^2$, $A \in \omega - OCP_n$ and u be the solution of initial value problem (2) on $[0, T)$. If $K \geq 0$, then $\|u(t)\|_{2,2}$ is bounded on $[0, T)$.*

Proof. We will first show that $\|u(t)\|_{1,2}$ is bounded on $[0, T)$. To this end we multiply the equation

$$\frac{1}{i}\frac{\partial u}{\partial t} - \Delta u + K|u|^2u = 0\,, \tag{12}$$

by $\overline{u}$ and integrate over $\mathbb{R}^2$. Then taking the imaginary part of the result gives $\frac{d}{dt}\|u\|^2_{0,2} = 0$ and therefore,

$$\|u(t)\|_{0,2} = \|u_0\|_{0,2} \quad for\ 0 \leq t \leq T. \tag{13}$$

Next we multiply (12) by $\partial\overline{u}/\partial t$, integrate over $\mathbb{R}^2$ and consider the real part of the result. This lead to

$$\frac{1}{2}\int_{\mathbb{R}^2}|\nabla u(t,x)|^2dx + \frac{K}{4}\int_{\mathbb{R}^2}|u(t,x)|^4dx = \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u_0(x)|^2dx + \frac{K}{4}\int_{\mathbb{R}^2}|u_0(x)|^4dx. \tag{14}$$

Therefore, since $K \geq 0$, then $\|u\|_{1,2}$ is bounded on $[0, T)$. To prove that $\|u(t)\|_{2,2}$ is bounded on $[0, T)$, we first note that from Sobolev's theorem it follows that $H^1(\mathbb{R}^2) \subset L^p(\mathbb{R}^2)$ for $p > 2$ and that

$$\|v\|_{0,p} \leq C\|v\|_{1,2} \quad for\ v \in H^1(\mathbb{R}^2). \tag{15}$$

Therefore if u is the solution of (2) on $[0, T)$ it follows from the boundedness of $\|u(t)\|_{1,2}$ on $[0, T)$ and (15) that

$$\|u(t)\|_{0,p} \leq C \quad for\ p > 2,\ 0 \leq t < T. \tag{16}$$

Since u is the solution of (2), it is also the solution of the integral equation

$$u(t) = T(t)u_0 - \int_0^t T(t-s)F(u(s))ds. \tag{17}$$

Denoting by D any first order derivative, we have

$$Du(t) = T(t)Du_0 - \int_0^t T(t-s)DF(u(s))ds. \tag{18}$$

We fix now $p > 2$ and let $q = p/(p-1)$ and $r = 4p/(p-2)$. Then denoting by C a generic constant and using Theorem 1, (18) and Hölder's inequality, we find

$$\begin{aligned}\|Du(t)\|_{0,p} &\leq \|T(t)Du_0\|_{0,p} + C\int_0^t (t-s)^{1-\frac{2}{q}} \||u(s)|^2|Du(s)|\|_{0,q} ds\\ &\leq C\|u_0\|_{2,2} + C\int_0^t (t-s)^{1-\frac{2}{q}} \|u(s)\|_{0,r}\|Du(s)\|_{0,2} ds\\ &\leq C\|u_0\|_{2,2} + C\int_0^t (t-s)^{1-\frac{2}{q}} ds \leq C(t),\end{aligned}$$

where in the last inequality we used the fact that $r > 2$ and therefore $\|u(s)\|_{0,r} \leq C$ by (16) and that

$$\|Du(s)\|_{0,2} \leq C\|u(s)\|_{1,2} \leq C.$$

Therefore, $\|u(t)\|_{1,p} \leq C$ and since by Sobolev's theorem, $W^{1,p}(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$ for $p > 2$, it follows that

$$\|u(t)\|_{0,\infty} \leq C \quad for\ 0 \leq t < T.$$

Finally, since $T(t)$ is an isometry on $L^2(\mathbb{R}^2)$ it follows from (17) that

$$\begin{aligned}\|u(t)\|_{2,2} &\leq \|T(t)u_0\|_{2,2} + \int_0^t \|T(t-s)F(u(s))\|_{2,2} ds\\ &\leq \|u_0\|_{2,2} + C\int_0^t \|u(s)\|_{0,\infty}^2 \|u(s)\|_{2,2} ds,\end{aligned}$$

which by Gronwall's inequality implies the boundedness of $\|u(t)\|_{2,2}$ on $[0, t)$ as desired. Hence the proof is completed. □

4. Conclusion

In this paper, it has been established that ω-order preserving partial contraction mapping generates some results of a nonlinear Schrödinger equation.

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