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Article

Stabilities of non-standard Euler-Maruyama scheme's for Vasicek and geometric brownian motion models

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Abstract: Stochastic differential equations (SDEs) are a powerful tool for modeling certain random trajectories of diffusion phenomena in the physical, ecological, economic, and management sciences. However, except in some cases, it is generally impossible to find an explicit solution to these equations. In this case, the numerical approach is the only favorable possibility to find an approximative solution. In this paper, we present the mean and mean-square stability of the Non-standard Euler-Maruyama numerical scheme using the Vasicek and geometric Brownian motion models.

Keywords: Brownian motion; Stochastic differential equations; Stabilities; Non-standard Euler-Maruyama scheme; Vasicek and geometric brownian motion.

MSC: 65C30.

1. Introduction

In order to construct continuous and strongly Markovian processes whose generators are second-order differential operators called diffusions [1,2], Ito developed stochastic differential equations, which are considered random perturbations added to ordinary differential equations or integral equations in which integrals are involved with respect to a Brownian motion. However, in general, finding explicit solutions for stochastic differential equations (SDEs), except in cases where diffusion and drift coefficients are linear, seems difficult or impossible [3].

This is why the numerical approach is relevant because there are numerical methods to predict in advance the qualitative behavior of solutions of stochastic differential equations such as stability. In this paper, we apply the approach described in [1,4] to analyze the stability of the Vasicek and geometric Brownian motion models using the non-standard Euler-Maruyama scheme. At first glance, we present some basic concepts on the non-standard finite difference scheme of ordinary differential equations, the classical Euler-Maruyama scheme, and the non-standard scheme of SDEs.

2. Preliminaries notions

In this section, we present some important tools in connection with stochastic differential equations, stabilities, and numerical schemes, such as the Non-standard finite difference scheme, the Euler-Maruyama scheme, and the Non-standard Euler-Maruyama scheme.

2.1. Stochastic differential equation and stabilities

This section presents some definitions in connection with stochastic differential equations and the stabilities of solutions of SDEs.

Definition 1 (Stochastic differential equation (SDE), [5])**.** Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ be a filtered probability space, $(B_t)_{t\geq 0}$ a standard Brownian motion on $\mathbb{R}^d$ defines in a filtered probability space. A stochastic differential equation (SDE) on $\mathbb{R}^d$ is an equation of the form:

$$\begin{cases} dX_t & = b\left(t, X_t\right)\, dt + \sigma\left(t, X_t\right)\, dB_t, \\ X\left(o\right) & = X_o. \end{cases} \tag{1}$$

with the drift coefficient:

$$b\left(t, X_t\right) \in [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n\,,$$

and the diffusion:

$$\sigma\left(t, X_t\right) \in [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times d}\,,$$

when X_o is random variable independent of $(B_t)_{t\geq 0}$.

Remark 1.
1. The white noise $\sigma\left(t, X_t\right)$ can be additive, it does not influence the state of the system.
2. The white noise $\sigma\left(t, X_t\right)$ can be multiplicative, it's influence the state of the system.

Theorem 1 (Existence and uniqueness [6])**.** *We assume that there is a positive constant K such that $\forall \quad t \geq 0$, $X, Y \in \mathbb{R}^d$*

1. ***Lipschitz condition:***
$$|b\left(t, X\right) - b\left(t, Y\right)| + |\sigma\left(t, X\right) - \sigma\left(t, Y\right)| \leq K|X - Y|.$$
2. ***Linear growth condition:***
$$|b\left(t, X\right)| \leq K\left(1 + |X|\right),$$
$$|\sigma\left(t, X\right)| \leq K\left(1 + |X|\right).$$

So the SDE (1) admits, for any initial condition X_o of square integrable $\left(E\left[|X_o|^2\right] < \infty\right)$ the strong solution $(X_t)_{t\in[0,T]}$, unique, almost surely continuous and satisfying the following condition:

$$E\left(\underset{0\leq t\leq T}{Sup}\,|X_t^2|\right) < \infty\,.$$

According to [6] there exists one and only one solution for the Eq. (1), verifying the Lipschitz conditions.

Definition 2 (Asymptotic stability in probability in large sense [7])**.** The solution is said to be asymptotically and stochastically stable in the large sense if

$$\forall \quad X_o \in \mathcal{L}^2_{\mathcal{F}_t}\left([-T, 0], \mathbb{R}^n\right),$$

then

$$\mathbb{P}\left\{\lim_{t\longrightarrow\infty} X\left(t\right) = 0\right\} = 1.$$

Definition 3 (Stability of p^{th} moment [1])**.**
1. Let $p \geq 2$ we say that a solution of (1) is stable in p^{th} moment if $\forall\epsilon > 0$ it exists $\delta > 0$ such as
$$E\left[\underset{t>0}{\text{Sup}}|X\left(t\right)|^p\right] < \epsilon \text{ avec } |X_o| < \delta.$$
2. Let $p \geq 2$, we say that a solution of (1) is stable asymptoticaly in p^{th} moment if it is stable from p^{eme} moment
$$\forall \quad X_o \in \mathcal{L}^2_{\mathcal{F}_{t_o}}\left([-T, 0], \mathbb{R}^n\right),$$

then we have

$$\lim_{T\longrightarrow\infty} E\left[\operatorname{Sup}_{t>T}|X\left(t\right)|^{p}\right] = 0\,.$$

2.2. Numerical schemes of SDEs

In this section, we present two numerical schemes adapted to SDEs, firtly the Euler-Maruyama scheme and secondly the non-standard Euler-Maruyama scheme.

Definition 4 (Non-standard finite difference schemes [8,9])**.** Let consider an ordinary differential equation whose general form is given by:

$$\frac{dX}{dt} = f\left(X(t)\right), \tag{2}$$

with $X \in \mathbb{R}^n, f \in C^2(\mathbb{R}^m, \mathbb{R}^m)$.

Let $I = [0, T]$ with $T \in \mathbb{R}^+$ and $\Delta t \in \mathbb{R}\ \forall \Delta t > 0$ for $k \in \mathbb{N}$, we note t_k the discrete time defined as $t_k = k\Delta t$ or $\Delta t = \frac{t_k}{k}$.

A general numerical scheme of the non-standard type with one step Δt which approach the general solution of a system of the form (2) has the form:

$$X_{k+1} = \phi(\Delta t)(X_k) = \phi(\Delta t, X_k)\,,$$

when $\phi(\Delta t)$ is $C^2(\mathbb{R}^m, \mathbb{R}^m)$, X_k initial condition and (2) can be written as:

$$\frac{dY}{dX} \simeq \frac{X_{k+1} - X_k}{\phi(\Delta t)} = f(X_k, X_{k+1}, \Delta t)\,,$$

when

$$X_{k+1} = X_k + \phi(\Delta t) f(X_k, X_{k+1}, \Delta t)\,, \tag{3}$$

with

$$\phi(\Delta t) = \Delta t + 0(\Delta t^2) \text{ and } \Delta t \longrightarrow 0\,.$$

Among the forms of the function $\phi(\Delta t)$ the most used form of the function ϕ is of the form [10]:

$$\phi(\Delta t) = \frac{1 - e^{\lambda \Delta t}}{\lambda} \quad \forall \quad \lambda \in \mathbb{R}\,.$$

Mickens in [11–13] states five rules to be respected in order to build a good non-standard finite difference scheme. An advantage of using this scheme is that issues related to consistency, stability and convergence do not appear. Let's state the convergence of this scheme.

Definition 5 (Convergence of the Non-standard scheme [14])**.** A numerical scheme converges if the numerical solution X_k satisfies

$$\sup_{0\leq t_k\leq T} \|X_k - x(t_x)\| \longrightarrow 0, \Delta t \longrightarrow 0 \text{ et } x_0 \longrightarrow x(t_0)\,.$$

It is of order p if

$$\sup_{0\leq t_k\leq T} \|X_x - x(t_x)\|_\infty = 0, (\Delta t)^p + 0 + (\|X_x - x(\Delta t)\|)\,,$$

when $\Delta t \longrightarrow 0$ and $x_0 \longrightarrow x(t_0)$.

Let us consider another improving definition of Mickens on the non-standard finite difference scheme according to Lubuma and Anguelov:

Definition 6 (DFNS according to Lubuma and Anguelov [15,16])**.** A general one-step numerical scheme that approximates the solution of (2) is said to be a non-standard finite difference scheme if at least one of the following conditions is satisfied:

1. $\frac{d}{dx}X(t_k) \approx \frac{X_{k+1}-X_k}{\phi(\Delta t)}$ with $\phi(\Delta t) = \Delta t + 0(\Delta t)^2$ is a positive function.
2. The nonlocal approximation of $f(t_k, X(t_k))$ is $\phi_{\Delta t}(f, X_k) = \overline{\phi}_{\Delta t}(f, X_k, X_{k+1})$.

Definition 7 (Euler-Maruyama scheme [5,17])**.** Let $\{X_t\}$ be the diffusion solution of the SDE (1). Let us consider the interval $[0, T]$ and a regular subdivision:

$$t_0 = 0 < t_1 < t_2 < t_0 < \cdots < t_k = T \quad \text{with step} \quad \Delta t = \frac{T}{N} = \frac{T}{k}.$$

The Euler-Maruyama scheme of (1) is defined as:

$$\begin{cases} X_{k+1}^{EM} = X_k + b(t_k, X_k)(t_{k+1} - t_k) + \sigma(t_k, X_k)(B_{k+1} - B_k), \\ X(0) = X_0. \end{cases} \tag{4}$$

Let us now state the definition of the non-standard Euler-Maruyama scheme based on the definitions of the non-standard finite difference scheme and the Euler-Maruyama scheme discussed earlier.

Definition 8 (Non-standard Euler-Maruyama scheme [14,18])**.** Considering the non-standard scheme definition rules, we define the Non-standard Euler-Maruyama scheme (EMNS) applied to (1) and (4) which is given by:

$$X_{k+1}^{EMNS} = X_k + b(X_k)\phi(\Delta t) + \sigma(X_k)\Delta B_k\,, \tag{5}$$

with $\phi(\Delta t) = \Delta t + C(\Delta t)$, a positive function of Δt and $\Delta B_k = B_{k+1} - B_k$.

In the following subsection, we present some elements of the construction of this scheme, including convergence.

2.2.1. Non-standard Euler-Maruyama scheme convergence

Consider the following three assumptions of convergence for non-standard Euler-Maruyama scheme:

Theorem 2 (Scheme convergence assumptions [19])**.** H_1 *For the initial condition, we assume that it is chosen independently of the Brownian motion B_t of the Eq.* (5)*.*

H_2 *The local Liptschitz condition i.e.* $\forall\ R > 0\ \exists L_R$ *depends on R such us*

$$|b(x) - b(y)| \vee |\sigma(x) - \sigma(y)| \leq L_R\,|x - y| \quad x, y \in \mathbb{R}^n$$

with $|x| \vee |y| \leq R$.

H_3 *Linear growth condition* $\exists\ k > 0$ *such that*

$$|b(x)| \vee |\sigma(x)| \leq k(1 + |x|) \quad \forall x \in \mathbb{R}^n\,.$$

Let us consider another result useful concerning the convergence of the non-standard Euler-Maruyama scheme.

Theorem 3 (Strong convergence of the NSEMS [18])**.** *Under assumptions H_1, H_2 and H_3, for any $p \geq 2$ there exists a constant c_1 depending only on δt and p such that the exact solution of the approximation given by the non-standard Euler-Maruyama scheme of* (5) *have the following property:*

$$E\left[\sup_{0\leq t\leq T} |Y_t|^p\right] \vee E\left[\sup_{0\leq t\leq T} \left|X^{EMNS}(t)\right|^p\right] \leq c_1(\Delta t, p)\,,$$

the solution obtained with the non-standard Euler-Maruyama scheme scheme of (5) *is strongly convergent.*

The remaining paper is organized as follows. In §3, non-standard Euler-Maruyama asymptotic stability in mean and mean-square for Vasicek and Geometric Brownian motion is carried out and classical proof will be given for each case. Finally, in §4, numerical investigations and residual calculations are provided and discussed.

3. Non-standard Euler-Maruyama stabilities

In this section we will consider two models of SDEs; the first SDE model, of Vasicek, involves white noise of an additive nature, while the second model, the geometric Brownian motion, is of multiplicative type. Let consider the first, the Vasicek model.

3.1. Vasicek model

Let's consider the following SDE representing the Vasicek model [1]:

$$\begin{cases} dX_t = (\theta_1 - \theta_2 X_t)dt + \theta_3 dB_t, \\ X(0) = X_0; \quad \text{with} \quad \theta_1, \theta_2 \in \mathbb{R}^* \quad \text{et} \quad \theta_3 \in \mathbb{R}^*_+. \end{cases} \tag{6}$$

The analytical solution of (6) model is:

$$X_t = \frac{\theta_1}{\theta_2} + \left(X_0 - \frac{\theta_1}{\theta_2} \right) e^{-\theta_2 t} + \theta_3 \int_0^{+\infty} e^{-\theta_2 (t-u)} dB_u \,. \tag{7}$$

Considering the solution of (7), the mean and the mean-square give respectively:

$$E[|X_t|] = \frac{\theta_1}{\theta_2},$$

and

$$E(|X_t|^2) = \frac{\theta_3^2}{2\theta_2}.$$

Which means that the stochastic process

$$X_t \simeq \mathcal{N}\left(\frac{\theta_1}{\theta_2}, \frac{\theta_3^2}{2\theta_2} \right).$$

By using some properties of Brownian motion, the solution of the model (7) can be written as follows:

$$X_t = \frac{\theta_1}{\theta_2} + \frac{\theta_3 e^{-2\theta_2 t}}{\sqrt{2\theta_2}} B(e^{2\theta_2 t}) \,. \tag{8}$$

Now, we present some numerical stabilities conditions for the system (6) of non-standard scheme and the proofs of these based on the approach described in [4] and used in [1]. The Euler-Maruyama scheme associated with (6) is:

$$X_{k+1}^{EM} = X_k + (\theta_1 - \theta_2 X_k)\Delta t_k + \theta_3 \Delta B_k.$$

After calculation, we get

$$X_{k+1}^{EM} = \theta_1 \Delta t + (1 - \Delta t \theta_2) X_k + \theta_3 \sqrt{\Delta t} Z_k. \tag{9}$$

Remark 2. We assume that the variable follows the normal distribution of expectation 0 and mean-square 1. i.e.,

$$Z_k \simeq \mathcal{N}(0,1)\,.$$

The non-standard Euler-Maruyama scheme associated to (9) is

$$X_{k+1}^{EMNS} = X_k + (\theta_1 - \theta_2 X_k)\phi(\Delta t) + \theta_3 \Delta B_k = X_k + \theta_1 \phi(\Delta t) - \theta_2 \phi(\Delta t) X_k + \theta_3 \Delta B_k \,.$$

After calculation, we obtain

$$X_{k+1}^{EMNS} = \theta_1 \phi(\Delta t) + (1 - \theta_2 \phi(\Delta t)) X_k + \theta_3 \sqrt{\Delta t} Z_k \,. \tag{10}$$

Remark 3. The positive function, $\phi(\Delta t)$ is calculated using the procedure described by Mickens in [20]. In the framework of this stochastic differential equation of Vasicek, we consider only the part containing the drift b for the computed, i.e., $b(t, X_t) = (\theta_1 - \theta_2 X_t)dt$, and the part containing the diffusion σ, i.e., $\sigma(t, X_t) = \theta_3 dB_t$ remains unchanged.

After determination of the function $\phi(\Delta t)$, while respecting the rules described in [20], the function $\phi(\Delta t)$ gives:

$$\phi(\Delta t) = \frac{1 - e^{-\theta_2 \Delta t}}{\theta_2} . \tag{11}$$

Carrying (11) into (10), we have

$$\begin{aligned} X_{k+1}^{EMNS} &= X_k + (\theta_1 - \theta_2 X_k) \left(\frac{1 - e^{-\theta_2 \Delta t}}{\theta_2} \right) \quad + \quad \theta_3 \Delta B_k \\ &= \frac{\theta_1}{\theta_2} \left(1 - e^{-\theta_2 \Delta t} \right) + \left(1 - \theta_2 \left(\frac{1 - e^{-\theta_2 \Delta t}}{\theta_2} \right) \right) X_k + \theta_3 \Delta B_k \\ &= \frac{\theta_1}{\theta_2} (1 - e^{-\theta_2 \Delta t}) + e^{-\theta_2 \Delta t} X_k + \theta_3 \Delta B_k . \end{aligned}$$

The non-standard Euler-Maruyama scheme of the Vasicek model after insertion of the function $\phi(\Delta t)$ gives:

$$X_{k+1}^{EMNS} = \frac{\theta_1}{\theta_2} (1 - e^{-\theta_2 \Delta t}) + e^{-\theta_2 \Delta t} X_k + \theta_3 \Delta B_k .$$

We obtain after manipulations that

$$X_{k+1}^{EMNS} = \frac{\theta_1}{\theta_2} (1 - e^{-\theta_2 \Delta t}) + e^{-\theta_2 \Delta t} X_k + \theta_3 \sqrt{\Delta t} Z_k . \tag{12}$$

Let us analyse the numerical mean and mean-square stabilities of the Non-standard Euler-Maruyama scheme of the Vasicek stochastic differential equation model.

3.1.1. Mean stability of non-standard Euler-Maruyama scheme

Theorem 4 (Mean stability of non-standard Euler-Maruyama scheme)**.** *The non-standard Euler-Maruyama scheme of the model (6) is asymptotically stable in mean if*

$$E\left(X_{k+1}^{EMNS}\right) = \left|e^{-\theta_2 \Delta t}\right|^{k+1} E\left(X_0\right) + \left|\frac{3\theta_1}{2\theta_2}(1 - \frac{1}{3} e^{-\theta_2 \Delta t})\right| \left(1 + \sum_{i=1}^{k+1} \left|e^{-\theta_2 \Delta t}\right|^i\right) ,$$

with $\left|e^{-\theta_2 \Delta t}\right| < 1$ *and* $\lim_{\Delta t \to 0} \left(\lim_{x \to \infty} E\left(X_{k+1}^{EMNS}\right)\right) = \frac{\theta_1}{\theta_2}$.

Proof. Let us begin by evaluating the mean of (12), we have

$$\begin{aligned} E\left(X_{k+1}^{EMNS}\right) &= E\left(\frac{\theta_1}{\theta_2}\left(3 - e^{-\theta_2 \Delta t}\right) + e^{-\theta_2 \Delta} X_k + \theta_3 \Delta B_k\right) \\ &= E\left(\frac{\theta_1}{\theta_2}\left(1 - e^{-\theta_2 \Delta t}\right)\right) + E\left(e^{-\theta_2 \Delta t} X_k\right) + E\left(\theta_3 \Delta B_k\right) \\ &= E\left(\frac{\theta_1}{\theta_2}\left(1 - e^{-\theta_2 \Delta t}\right)\right) + \left|e^{-\theta_2 \Delta t}\right| E\left(X_k\right) + \theta_3 \sqrt{\Delta t} E\left(Z_k\right) \\ &= E\left(\frac{\theta_1}{\theta_2}\left(1 - e^{-\theta_2 \Delta t}\right)\right) + \left|e^{-\theta_2 \Delta t}\right| E\left(X_k\right) + 0 \text{ car } E\left(Z_k\right) = 0 \\ &= \left|\frac{\theta_1}{\theta_2}\left(1 - e^{-\theta_2 \Delta t}\right)\right| + \left|e^{-\theta_2 \Delta t}\right| E(X_k) \\ &= \left|\frac{3\theta_1}{2\theta_2}\left(1 - \frac{1}{3} e^{-\theta_2 \Delta t}\right)\right| + \left|e^{-\theta_2 \Delta t}\right| \left\{\left|\frac{3\theta_1}{2\theta_2}(1 - \frac{1}{3} e^{-\theta_2 \Delta t})\right| + \left|e^{-\theta_2 \Delta t}\right| E\left(X_{k-1}\right)\right\} \end{aligned}$$

$$= \left|\frac{3\theta_1}{\theta_2}\left(1-\frac{1}{3}e^{-\theta_2\Delta t}\right)\right|\left(1+\left|e^{-\theta_2\Delta t}\right|+\left|e^{-\theta_2\Delta t}\right|^2+\left|e^{-\theta_2\Delta t}\right|^3+\cdots+\left|e^{-\theta_2\Delta t}\right|^{k+1}\right)E\left(X_0\right)$$
$$= \left|\frac{3\theta_1}{2\theta_2}\left(1-\frac{1}{3}e^{-\theta_2\Delta t}\right)\right|\left(1+\sum_{i=1}^{k+1}\left|e^{-\theta_2\Delta t}\right|^i\right)+\left|e^{-\theta_2\Delta t}\right|^{k+1}E\left(X_0\right).$$

Continuing with the iterations, we get

$$E\left(X_{k+1}^{EMNS}\right) = \left|e^{-\theta_2\Delta t}\right|^{k+1}E\left(X_0\right)+\left|\frac{3\theta_1}{2\theta_2}(1-\frac{1}{3}e^{-\theta_2\Delta t})\right|\left(1+\sum_{i=1}^{k+1}\left|e^{-\theta_2\Delta t}\right|^i\right),$$

Let's assume that

$$I_1 = \left|e^{-\theta_2\Delta t}\right|^{k+1}E(X_0),$$

which represents a geometric sequence, it converges, for $\left|e^{-\theta_2\Delta t}\right| < 1$ and as $\Delta t \to 0$ with $\theta_2 > 0$ we have by passing to the limit that

$$\lim_{\Delta t\to 0}\left(\lim_{k\to\infty}\left|e^{-\theta_2\Delta t}\right|^{k+1}E\left(X_0\right)\right) = 0, \tag{13}$$

and just as in the case of $I_2 = \left|\frac{3\theta_1}{2\theta_2}(1-\frac{1}{3}e^{-\theta_2\Delta t})\right|\left(1+\sum_{i=1}^{k+1}|e^{-\theta_2\Delta t}|^i\right)$ as $\Delta t \to 0$ and $\theta_2 > 0$ with $\left|e^{-\theta_2\Delta t}\right| < 1$, then we have that

$$\lim_{\Delta t\to 0}\left(\lim_{k\to\infty}\left|\frac{3\theta_1}{2\theta_2}(1-\frac{1}{3}e^{-\theta_2\Delta t})\right|\left(1+\sum_{i=1}^{k+1}|e^{-\theta_2\Delta t}|^i\right)\right) = \frac{\theta_1}{\theta_2}. \tag{14}$$

From (13) and (14), we have $\forall\ \theta_2 > 0, |e^{-\theta_2\Delta t}| < 1$,

$$\lim_{\Delta t\to 0}\left(\lim_{k\to\infty}E\left(X_{k+1}\right)\right) = \frac{\theta_1}{\theta_2}.$$

□

3.1.2. Mean-square stability of non-standard Euler-Maruyama scheme

Theorem 5 (Mean-square stability of non-standard Euler-Maruyama scheme)**.** *The Non-standard Euler-Maruyama scheme of the model* (6) *is mean square asymptotically stable if*

$$E\left(X_{k+1}^{EMNS}\right) = \left|e^{-\theta_2\Delta t}\right|^{2(k+1)}E\left(|X_0|^2\right)+\left(\left|\frac{\theta_1}{\theta_2}\left(1-e^{-\theta_2\Delta t}\right)\right|^2+\frac{\left|\theta_3\sqrt{\Delta t}\right|^2}{|2\Delta t\theta_2|}\right)\sum_{i=1}^{k+1}\left|1-e^{-\theta_2\Delta t}\right|^{2i},$$

with $\left|e^{-\theta_2\Delta t}\right| < 1,\ \theta_2 > 0$*, then* $\lim_{\Delta t\to 0}\left(\lim_{x\to\infty}E\left(X_{k+1}^{EMNS}\right)\right) = \frac{\theta_3{}^2}{2\theta_2}$.

Proof. First, let's calculate the root mean square of the EMNS scheme of the model (6), we have

$$E\left(\left|X_{k+1}^{EMNS}\right|^2\right) = E\left(\left|\frac{\theta_1}{\theta_2}\left(1-e^{-\theta_2\Delta t}\right)+\left(e^{-\theta_2\Delta t}\right)X_k+\theta_3\sqrt{\Delta t}Z_k\right|^2\right)$$
$$= E\left(\left|\frac{\theta_1}{\theta_2}\left(1-e^{-\theta_2\Delta t}\right)\right)\right|^2+E\left(\left|e^{-\theta_2\Delta t}X_k\right)\right|^2+E\left(\left|\theta_3\sqrt{\Delta t}Z_k\right|^2\right)$$
$$= \left|\frac{\theta_1}{\theta_2}\left(1-e^{-\theta_2\Delta t}\right)\right|^2+\left|\theta_3\sqrt{\Delta t}\right|^2+\left|e^{-\theta_2\Delta t}\right|^2E\left(|X_k|^2\right)$$
$$= \left|\frac{\theta_1}{\theta_2}\left(1-e^{-\theta_2\Delta t}\right)\right|^2+\frac{\left|\theta_3\sqrt{\Delta t}\right|^2}{|\Delta t\theta_2|}+\left|e^{-\theta_2\Delta t}\right|^2\left\{\left|\frac{\theta_1}{\theta_2}\left(1-e^{-\theta_2\Delta t}\right)\right|^2+\frac{\left|\theta_3\sqrt{\Delta t}\right|^2}{|\Delta t\theta_2|}\right.$$

$$
+ \left|1-e^{-\theta_2\Delta t}\right|^2 E\left(|X_{k-1}|^2\right)\Bigg\}
$$

$$
\vdots
$$

$$
= \left|e^{-\theta_2\Delta t}\right|^{2(k+1)} E\left(|X_0|^2\right) + \left(\left|\frac{\theta_1}{\theta_2}\left(1-e^{-\theta_2\Delta t}\right)\right|^2 + \frac{\left|\theta_3\sqrt{\Delta t}\right|^2}{|2\Delta t\theta_2|}\right)\sum_{i=1}^{k+1}\left|1-e^{-\theta_2\Delta t}\right|^{2i}
$$

By recurrence, we obtain that

$$
E(|X_{k+1}^{EMNS}|^2) = \left|e^{-\theta_2\Delta t}\right|^{2(k+1)} E\left(|X_0|^2\right) + \left(\left|\frac{\theta_1}{\theta_2}\left(1-e^{-\theta_2\Delta t}\right)\right|^2 + \frac{\left|\theta_3\sqrt{\Delta t}\right|^2}{|2\Delta t\theta_2|}\right)\sum_{i=1}^{k+1}\left|1-e^{-\theta_2\Delta t}\right|^{2i}.
$$

By going to the limit for $\Delta t \to 0$ and $k \to +\infty$, with

$$
\left|1-e^{-\theta_2\Delta t}\right| < 1 \quad \text{and} \quad \theta_2 > 0,
$$

we obtain that

$$
\lim_{\Delta t\to 0}\left(\lim_{k\to\infty} E\left(X_{k+1}^{EMNS}\right)\right) = \frac{{\theta_3}^2}{2\theta_2}.
$$

□

Theorem 6. *Under the assumptions, the non-standard Euler-Maruyama scheme in mean and mean-square is asymptotically stable if we have respectively* $\left|e^{-\theta_2\Delta t}\right| < 1$ *and* $\left|1-e^{-\theta_2\Delta t}\right| < 1$ *with* $\theta_2 > 0$.

Let's consider the second model, the Geometric Brownian motion.

3.2. Geometric Brownian motion

Consider the following stochastic differential equation [1], describing geometric Brownian motion

$$
\begin{cases} dX_t = \theta_1 X_t dt + \theta_2 X_t dB_t, \\ X(0) = X_0, \end{cases} \qquad \theta_1, \theta_2 \in \mathbb{R}. \tag{15}
$$

The explicit solution of this model is

$$
X_t = X_0 e^{\left\{(\theta_1 - \frac{1}{2}\theta_2^2)t + \theta_2 B_t\right\}}. \tag{16}
$$

The the right hand variable, follows a normal distribution, it can also be written in the form:

$$
X_t = X_s e^{\left\{\left(\theta_1 - \frac{1}{2}\theta_2^2\right)t + \theta_2(B_t - B_s)\right\}}.
$$

The mean and the mean-square of (16) give respectively:

$$
E(X_t) = X_0 e^{\theta_1 t}, \quad E(X_t^2) = X_0^2 e^{\left(2\theta_1 + \theta_2^2\right)t}. \tag{17}
$$

Remark 4. ([1]) It should be noted that

1. For mean, if $t \to \infty$ and $\theta_1 < 0$, we have

$$
\lim_{t\to\infty} E(X_t) = \lim_{k\to\infty} X_0 e^{\theta_1 t} = 0.
$$

2. For mean-square, if $\left(2\theta_1 + \theta_2^2\right) < 0$ and $t \to \infty$ i.e.,

$$
\lim_{t\to\infty} E(X_t^2) = 0 \text{ with } \left(2\theta_1 + \theta_2^2\right) < 0.
$$

The Euler-Maruyama scheme of the geometric Brownian motion model [1] associated with (15) is

$$X_{k+1}^{EM} = X_k(1+\theta_1\Delta t) + \theta_2 X_k \Delta B_k\,, \tag{18}$$

with $\Delta B_k = \sqrt{\Delta t} Z_k$, and $Z_k \simeq \mathcal{N}(0,1)$.

The non-standard Euler-Maruyama scheme associated to (18) is

$$X_{k+1}^{EMNS} = X_k + \theta_1\phi(\Delta t) X_k + \theta_2 X_k \Delta B_k\,.$$

After some calculations, we get

$$X_{k+1}^{EMNS} = X_k\left(1+\theta_1\phi(\Delta t)\right) + \theta_2 X_k\sqrt{\Delta t} Z_k\,. \tag{19}$$

Remark 5. Given, the nature of the function $\phi(\Delta t)$, a positive function. In the framework of the stochastic differential equation of geometrical Brownian motion that we treat, to calculate the function $\phi(\Delta t)$, we consider only the part containing the drift $b(t, X_t) = \theta_1 X_t dt$ and the part containing the diffusion $\sigma(t, X_t) = \theta_2 X_t dB_t$ remains unchanged.

Calculating $\phi(\Delta t)$ without diffusion according to [20], we have

$$\phi(\Delta t) = \frac{e^{\theta_1\Delta t}-1}{\theta_1}\,. \tag{20}$$

Carrying (20) in (19), the Non-standard Euler-Maruyama scheme gives,

$$X_{k+1}^{EMNS} = X_k\left(1+\theta_1\left(\frac{e^{\theta_1\Delta t}-1}{\theta_1}\right)\right) + \theta_2 X_k\sqrt{\Delta t} Z_k.$$

This gives us,

$$X_{k+1}^{EMNS} = e^{\theta_1\Delta t} X_k + \theta_2 X_k\sqrt{\Delta t} Z_k\,. \tag{21}$$

Let us evaluate the stabilities of the expression (21).

In the following paragraph, we state and prove the mean and mean-square stability conditions of the Non-standard Euler-Maruyama scheme. Let us start with the mean stability of the scheme.

3.2.1. Mean stability of the non-standard Euler-Maruyama scheme

Theorem 7. *(Mean stability of the non-standard Euler-Maruyama scheme) The non-standard Euler-Maruyama scheme of the model (15) is asymptotically stable in mean if*

$$E\left(X_{k+1}^{EMNS}\right) = \left(e^{\theta_1\Delta t}\right)^{k+1} E\left(X_0\right)\,,$$

with $\left|e^{\theta_1\Delta t}\right| < 1;\quad \theta_1 < 0$ *and*

$$\lim_{\Delta t\to 0}\left(\lim_{x\to\infty} E\left(X_{k+1}^{EMNS}\right)\right) = 0\,.$$

Proof. Let's calculate the mean of the non-standard Euler-Maruyama scheme.

$$\begin{aligned}
E\left(X_{k+1}^{EMNS}\right) &= E\left[\left(e^{\theta_1\Delta t} + \theta_2\sqrt{\Delta t} Z_k\right) X_k\right] \\
&= E\left(e^{\theta_1\Delta t} X_k\right) + E\left(\theta_2\sqrt{\Delta t} X_k\right) E\left(Z_k\right) \quad \text{avec} \quad E\left(Z_k\right) = 0 \\
&= e^{\theta_1\Delta t} E\left(X_k\right) \\
&= e^{\theta_1\Delta t}\left(e^{\theta_1\Delta t}\left(e^{\theta_1\Delta t}\left(e^{\theta_1\Delta t}\left(\cdots e^{\theta_1\Delta t} E\left(X_0\right)\right)\right)\right)\right)\,.
\end{aligned}$$

Continuing with the iterations, we get

$$E\left(X_{k+1}^{EMNS}\right)=\left(e^{\theta_1\Delta t}\right)^{k+1}E\left(X_0\right). \tag{22}$$

If we assume that

$$\left|e^{\theta_1\Delta t}\right|<1 \quad \text{with} \quad \theta_1<0,$$

passing to the limit for $\Delta t \to 0$ and $k \to +\infty$ we get the sought result i.e.,

$$\lim_{\Delta t\to 0}\left(\lim_{x\to\infty}E\left(X_{k+1}^{EMNS}\right)\right)=0.$$

□

3.2.2. Mean-square stability of the non-standard Euler-Maruyama scheme

Theorem 8. *(Mean-square stability of the EMNS scheme) The non-standard Euler-Maruyama scheme of the model* (15) *is asymptotically stable in mean-square if*

$$E\left(\left|X_{k+1}^{EMNS}\right|^2\right)=\left(\left(e^{\theta_1\Delta t}\right)+\left(\left|\theta_2\sqrt{\Delta t}\right|\right)\right)^{2(k+1)}\left(E\left|X_0\right|^2\right),$$

with

$$\left|e^{\theta_1\Delta t}+\theta_2\sqrt{\Delta t}\right|<1,$$

and

$$\lim_{\Delta t\to 0}\left(\lim_{x\to\infty}E\left(X_{k+1}^{EMNS}\right)\right)=0.$$

Proof. Let us evaluate the mean-square of the Non-standard Euler-Maruyama scheme (21) of the Eq. (15)

$$\begin{aligned}
E\left(\left|X_{k+1}^{EMNS}\right|^2\right)&=E\left(\left|\left(e^{\theta_1\Delta t}+\theta_2\sqrt{\Delta t}Z_k\right)X_k\right|^2\right)=E\left(\left|e^{\theta_1\Delta t}+\theta_2\sqrt{\Delta t}Z_k\right|^2\right)E\left(\left|X_k\right|^2\right)\\
&=\left(\left(e^{\theta_1\Delta t}\right)^2+\left|\theta_2\sqrt{\Delta t}\right|^2\right)E\left(\left|X_k\right|^2\right)=\left(\left(e^{\theta_1\Delta t}\right)+\left|\theta_2\sqrt{\Delta t}\right|\right)^2E\left(\left|X_k\right|^2\right)\\
&=\left(e^{\theta_1\Delta t}+\left(\theta_2\sqrt{\Delta t}\right)^2E\left(\left|X_k\right|^2\right)\right)\\
&=\left(e^{\theta_1\Delta t}+\theta_2\sqrt{\Delta t}\right)^2\left(\left(e^{\theta_1\Delta t}+\theta_2\sqrt{\Delta t}\right)^2E\left(\left|X_{k-1}\right|^2\right)\right)\\
&\;\vdots\\
&=\left(\left(e^{\theta_1\Delta t}\right)+\left(\left|\theta_2\sqrt{\Delta t}\right|\right)\right)^{2(k+1)}\left(E\left|X_0\right|^2\right)\\
&=\left(e^{\theta_1\Delta t}+\theta_2\sqrt{\Delta t}\right)^2\left(\left(e^{\theta_1\Delta t}+\theta_2\sqrt{\Delta t}\right)^2E\left(\left|X_{k-1}\right|^2\right)\right)\\
&\;\vdots\\
&=\left(e^{\theta_1\Delta t}+\left|\theta_2\sqrt{\Delta t}\right|\right)^{2(k+1)}\left(E\left|X_0\right|^2\right).
\end{aligned}$$

Continuing with the iterations, we get

$$E\left(\left|X_{k+1}^{EMNS}\right|^2\right)=\left(\left(e^{\theta_1\Delta t}\right)+\left(\left|\theta_2\sqrt{\Delta t}\right|\right)\right)^{2(k+1)}\left(E\left|X_0\right|^2\right). \tag{23}$$

If we assume that

$$\left|e^{\theta_1\Delta t}+\theta_2\sqrt{\Delta t}\right|<1$$

from the fact that the sequence is geometric, going to the limit with the fact that the quantity

$$E\left(\left|X_{k+1}^{EMNS}\right|\right)=0.$$

For $\Delta t \to 0$ and $k \to +\infty$, we find that

$$\lim_{\Delta t \to 0} \left(\lim_{x \to \infty} E\left(X_{k+1}^{EMNS}\right)\right) = 0.$$

□

Theorem 9. *Under the assumptions, the non-standard Euler-Maruyama scheme in mean and mean-square is asymptotically stable if*

$$\left|e^{\theta_1 \Delta t}\right| < 1 \quad with \quad \theta_1 < 0, \quad \theta_2 > 0$$

respectively

$$\left|e^{\theta_1 \Delta t} + \theta_2 \sqrt{\Delta t}\right| < 1.$$

4. Numerical simulations of the models

This section presents some numerical simulations of Vasicek and geometric Brownian motion models. We treat the stability and instability cases and the increasing and decreasing cases. We will associate with each model the calculation of the residuals.

Let's start with the simulation of the first model, the Vasicek's one.

4.1. Vasicek model simulation

In this subsection, we consider the Vasicek equation, we present the different simulations of this model.

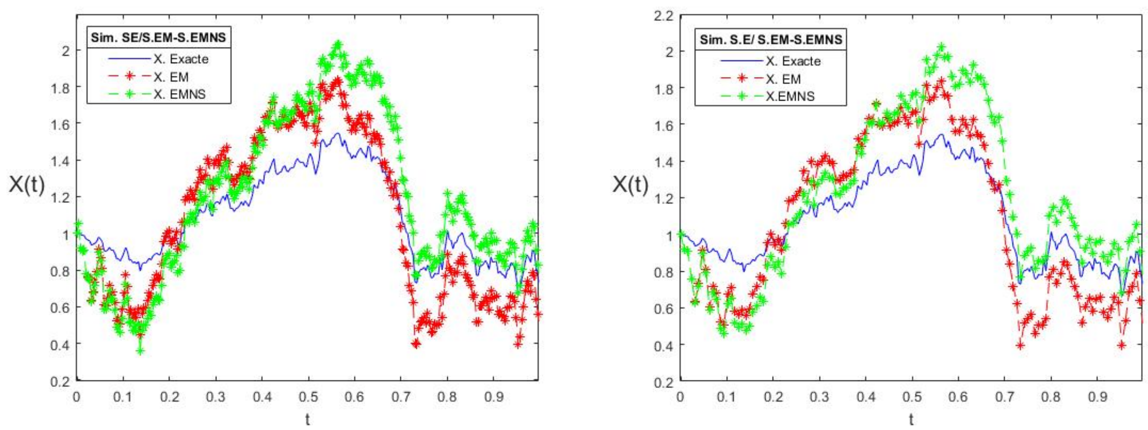

Figure 1. Stability of the Vasicek model using the EMNS scheme

4.2. Geometric Brownian motion model simulation

In this subsection, we present the different numerical simulations of geometric Brownian motion.

4.3. Interpretations of the results

4.3.1. Vasicek's Model

The two figures in Figure 1 show that there is stability of the non-standard Euler-Maruyama scheme for the Vasicek model. Moreover, the errors are smaller, namely Emnserr $= 0.0338$ resp. (Emnserr $= 0.0354$) compared to the errors obtained by using the Euler-Maruyama scheme Emerr $= 0.2280$ resp. (Emerr $= 0.2286$) of the same model for a $\Delta t = 0.004$ resp.($\Delta t = 0.015$).

On the other hand the two other figures in Figure 2 show that there is stability resp.(instability) of the non-standard Euler-Maruyama scheme for the Vasicek model, because the errors are smaller namely

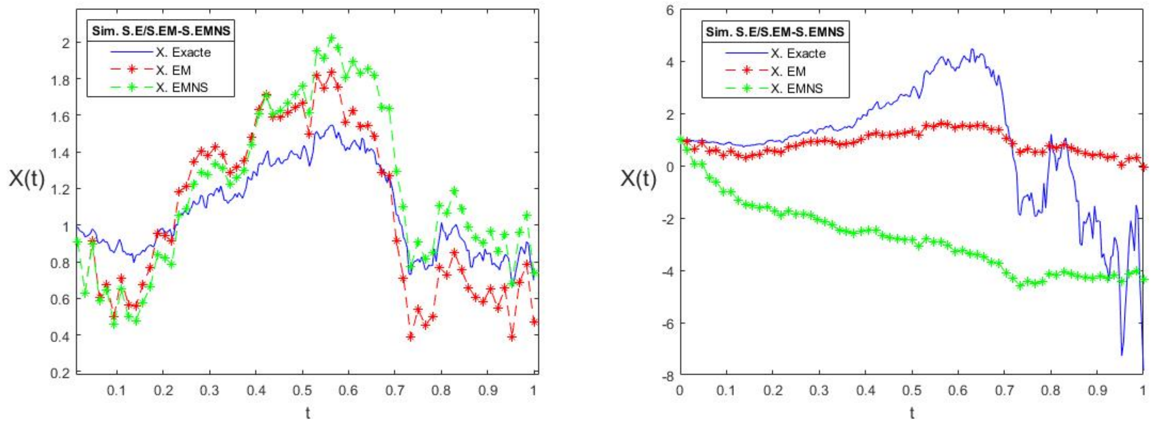

Figure 2. Stability and instability of the Vasicek model using the EMNS scheme

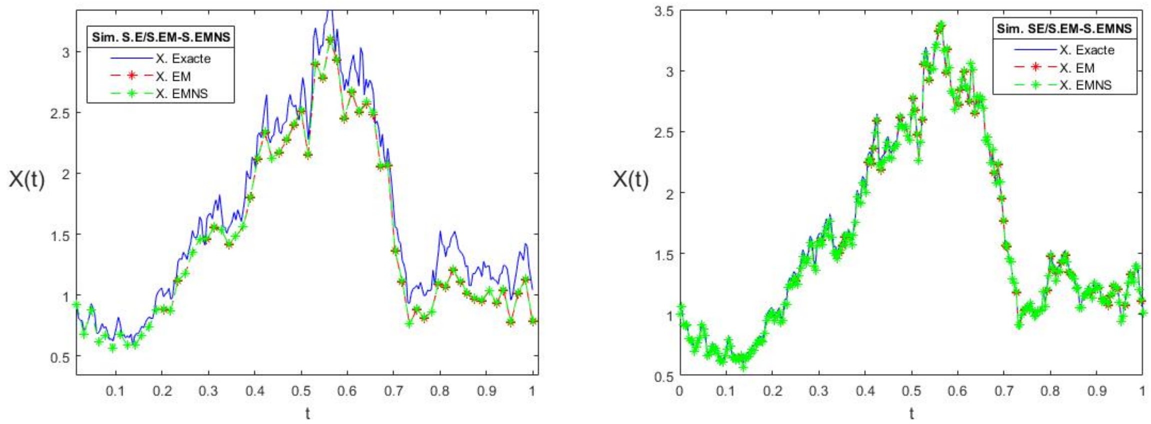

Figure 3. Increasing case of geometric Brownian motion

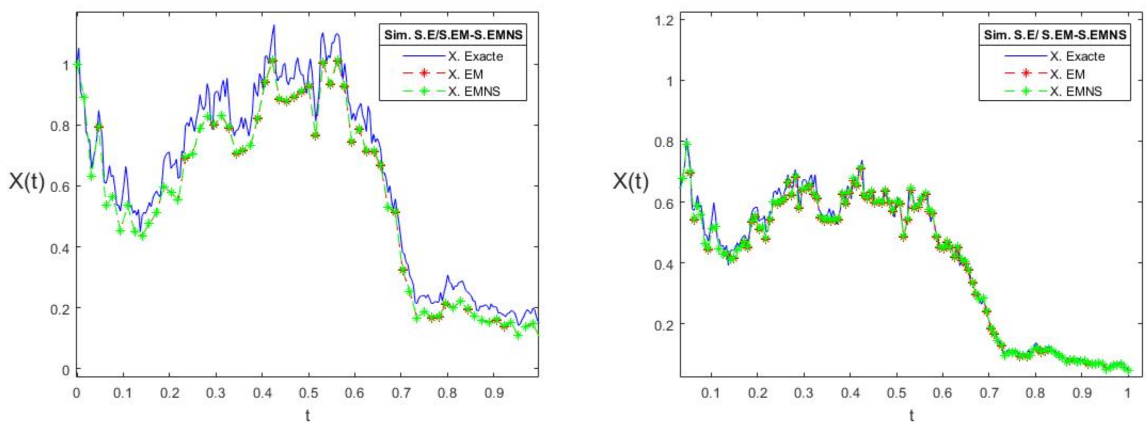

Figure 4. decreasing case of geometric Brownian motion

Emnserr $= 0.0377$ resp. (Emnserr $= 3.4794$) compared to those obtained with the Euler-Maruyama scheme Emerr $= 0.2298$ resp. (Emerr $= 7.7772$) of the same model for a $\Delta t = 0.015$.

Although the same discretization step was considered, the instability is because the value of the parameter $\theta_2 = -4$ i.e., negative, does not correspond to the right value for this parameter.

4.3.2. Geometric Brownian motion model

The two figures, i.e., Figure 3 and Figure 4 present the stability of the model of geometric Brownian motion; the first two figures in Figure 3 are elaborated with a positive θ_1 that is why the figures express the growth while in the two figures of Figure 4 with a negative θ_1 that is why the figures express the decrease.

However, in both figures in Figure 3 the errors are smaller i.e. Emnserr $= 0.2548$ respectively (Emnserr $=$ 0.0283) compared to the Euler-Maruyama scheme Emerr $= 0.2611$ resp. (Emerr $= 0.0303$) of this model for a $\Delta t = 0.015$ resp. ($\Delta t = 0.004$) for a $\theta_1 = 1$.

On the other hand, in the two other figures of Figure 4 the errors of the non-standard Euler-Maruyama scheme are smaller Emnserr $= 0.0415$ resp. (Emnserr $= 0.0046$) compared to the errors obtained with the Euler-Maruyama scheme Emerr $= 0.0423$) resp. (Emerr $= 0.0054$) for a $\Delta t = 0.015$ resp.($\Delta t = 0.008$) for a $\theta_1 = -1$ resp.($\theta_1 = -2$).

Remark 6. In line with the results obtained in the residual calculations for the two stochastic differential equation models treated, we find that the non-standard Euler-Maruyama scheme is better than the Euler-Maruyama scheme, as the latter improves on the Euler-Maruyama scheme by best approximating the exact solutions.

5. Conclusion

This paper presents the mean and mean-square stability of the non-standard Euler-Maruyama scheme using Vasicek and the geometric Brownian motion models. Both models are linear in dimension one; the Vasicek model is additive noise while the geometric Brownian motion is of multiplicative type; In these models we have established the conditions of numerical stabilities of the Non-standard Euler-Maruyama scheme, and a comparison has been made with the Euler-Maruyama scheme.

These conditions are stated as theorems, and we prove these results using the classical approach. The results found were supported by numerical simulations and the determination of residuals. It should be noted that the Non-standard Euler-Maruyama scheme improves the Euler-Maruyama scheme. In future work, we will focus on a Non-standard scheme of non-linear and higher dimensional SDEs.

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