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A class of power series based modified Newton method with high precision for solving nonlinear models

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Abstract: This manuscript proposed high-precision methods for obtaining solutions for nonlinear models. The method uses the Newton method as its predictor and an iterative function that involves the perturbed Newton method with the quotient of two power series as the corrector function. The theoretical analysis of convergence indicates that the methods class is of convergence order four, requiring three functions evaluation per cycle. The computation performance comparison with some existing methods shows that the developed methods class has perfect precision.

Keywords: Nonlinear models; Iterative method; Newton method; Power series.

MSC: 49M15, 90C53

1. Introduction

Many physical sciences and engineering problems are reduced to nonlinear models (NLM) of $f(s) = 0$, where $s \in R$. Their solutions are often required to understand better and study these problems. Because there is no generic analytic formula for solving NLM, iterative methods (IM) are utilized. An iterative approximation technique is the use of a recursion formula or procedure to approximate the solution $\sigma \in R$ of the target NLM such that for every subsequent evaluation of the complete cycle of the recursion formula, a better and improved approximation of σ is achieved until exact or desired solution is attained. Since the discovery of the classical convergent order (CO), two Newton iterative method (NIM) [1] that is put forward as:

$$s_{j+1} = s_j - t(s_j); \quad j = 0, 1, 2, \cdots , \tag{1}$$

where $t(s_j) = \frac{f(s_j)}{f'(s_j)}$, the development of its modifications has attracted the interest of several scholars. These modifications are either one, two or more point iteration methods developed with the primary motivation of scaling the convergence order (CO) of the NIM from two to higher orders and, in another case, improving its efficiency index. For instance, in Weerakoon and Fernando [2], the function derivative $f'(s_j)$ was replaced by the arithmetic mean of $f'(s_j)$ and $f'\left(s_j - \frac{f(s_j)}{f'(s_j)}\right)$ to obtain a CO three method with better efficiency index compared to the NIM. One can find many modified forms of the NIM where different types of means were used other than the arithmetic mean in the literature [3–7] and some reference therein. Different weight functions were utilized in another modification form to develop several one-point and multi-point modified NIM. For instance, Chun in [8] used a power series-like expression attached to the second step of modified NIM to develop a family of two-point iterative methods (TPIM) of CO four. Also, Khattri and Abbasbandy [9] used another type of power series involving a function different from the one used in Chun [8] in the second step of the modified NIM to develop a class of TPIM. Babajee and Khatrri in [10] further studied the dynamic behaviour of the generalized class of methods put forward in Khattri and Abbasbandy [9]. In the works of Ahmad [11], Babajee [12] and Mahdu [13], the power series of different forms were utilized as weight functions in the modification of the NIM to develop CO eight iterative methods.

In line with the research trend that involves the use of the power series to modify the NIM, this work put forward a class of two-point modified NIM of CO four that is an improvement of Khattri and Abbasbandy

[9] with the advantage of high precision for approximation of NLM solution. In the succeeding sections, the formation of the method, its convergence investigation and the applicability of its concrete members are presented.

2. Method Formation and Convergence Investigation

We begin by acknowledging the two-step power series based class of CO four modified NIM presented in Khattri and Abbabandy [9] as:

$$\begin{aligned} y_j &= s_j - \frac{2}{3} t\left(s_j\right), \\ s_{j+1} &= y_j - t(s_j)\left[1 + \sum_{i=1}^{\infty} \alpha_i v^i\right]; \end{aligned} \tag{2}$$

where $v = \frac{f'(y)}{f'(s)}$ and $\alpha_i \in R$ are free parameters. To establish the convergence of a sequence of approximations of (2), the case of $m = 4$ was considered. This resulted in constructing a class of four CO methods that are very effective for solving NLM. In like manner, Chun in [8] considered the method:

$$\begin{aligned} y_j &= s_j - t\left(s_j\right), \\ s_{j+1} &= y_j - \frac{f(y_j)}{f'(s_j)}\left[1 + \sum_{i=1}^{\infty} \alpha_i u^i\right]; \end{aligned} \tag{3}$$

where $u_j = \frac{f(y_j)}{f(s_j)}$. Chun utilized a few first terms of the power series in (3) to construct good CO order four methods. Recently, Ogbereyivwe and Izevbizua [14] contributed to the literature by putting forward an extended family of Chun [8], Ahmad[11], Babajee[12] and Jarratt [15] as

$$s_{k+1} = s_k - t(s_j)\left[\frac{(1 + \sum_{i=1}^{m} \tau_i (v(s_k) - 1)^i)}{(1 + \sum_{i=1}^{m} \lambda_i (v(s_k) - 1)^i)}\right], \tag{4}$$

where τ_i and $\lambda_i, i = 1, 2, 3, \ldots$ are real, free and dispensable parameters.

Motivated by the works above, a class of methods similar to the methods in Chun [8] and Khattri and Abbabandy [9] but better in precision is put forward as:

$$\begin{aligned} y_j &= s_j - t\left(s_j\right), \\ s_{j+1} &= s_j - t(s_j)\left[\frac{1 + \sum_{i=1}^{m} \alpha_i u^i}{1 + \sum_{i=1}^{m} \beta_i u^i}\right]; \end{aligned} \tag{5}$$

where $\sum_{i=1}^{m} \alpha_i u^i$ and $\sum_{i=1}^{m} \beta_i u^i$ are power series that are converging, α_j, $\beta_j \in R$ are free parameters to be determined and are responsible for ensuring the IM's sequence of approximations converge to the solution of NLM with a desired order. We note here that for $\beta_i = 0, \forall i$, a variant class of the TPIM in Chun [8] is obtained. The TPIM (5) require three functions evaluations in an iteration cycle and if it converge with order four for any choice of $m(i \leq m \leq \infty)$, then its efficiency index E_{eff} is 1.5874. The next theorem establishes the conditions imposed on the parameters α_j and β_j so as the method in (5) and for $m = 3$, its sequence of approximations converge to σ the solution of NLM with CO four.

The following definitions of basic concepts are required in the proof of the next theorem.

Definition 1. (*Asymptotic error and constant and Convergence order*) Let $d_j = s_j - \sigma$ be IM error at jth iteration and suppose an equation $d_{j+1} = \eta e_j^{\rho} + O(d_j^{\rho+1})$ is obtained from an IM by using the Taylor expansions on the functions $f(\cdot)$, then d_{j+1} is referred to as Asymptotic error equation, η is Asymptotic constant and ρ is Convergence order.

Definition 2. (*Efficiency*) Suppose the equation $d_{j+1} = \eta d_j^{\rho} + O(d_j^{\rho+1})$ holds for an IM as described in Definition 1, then the value $E_{eff} = \rho^{\frac{1}{\tau}}$ (where τ is total number of distinct functions $f(\cdot)$ in one IM cycle) is called the Efficiency index of the IM.

Theorem 1. *Suppose $f : D \subset R \rightarrow R$ is a real valued function that is at least three times differentiable in the domain D, such that $\sigma \subset D$ and $|f'(\cdot)| \neq 0$ in D. If d_0 is close to σ and $m = 3$, then the sequence $\{s_j\}_{j\geq 0}, (s_j \in D)$ of approximations, generated by the class of IM in (5) converges to σ with CO four so long the free parameters α_i and β_i satisfies the conditions $\beta_1 - \alpha_1 + 1 = 0, \beta_2 - \alpha_2 + \alpha_1 + 1 = 0$ and $\beta_3 - \alpha_3 + \alpha_2 + \alpha_1 + 2 = 0$.*

Proof. Let $s{=}s_j$ in the Taylor series expansion of $f(s)$ and $f'(s)$ about σ, then

$$f(s_j) = f'(\sigma)\,(d_j + \sum_{n=2}^{4} c_n d_j^n + O\left(d_j^5\right)), \tag{6}$$

and

$$f'(s_j) = f'(\sigma)\,(1 + \sum_{n=2}^{4} n c_n d_j^{n-1} + O\left(d_j^5\right)), j{=}0,1,2,\cdots \tag{7}$$

where $c_n = \frac{1}{j!}\frac{f^{(j)}(\delta)}{f'(\delta)}, j \geq 2$.

By substituting (6) and (7) in the first step of (5), the following expression is obtained:

$$y_j = \delta + c_2 d_j^2 + (-2c_2^2 + 2c_3)d_j^3 + (4c_2^3 - 7c_2c_3 + 3c_4)d_j^4 + O(d_j^5) \tag{8}$$

and

$$f(y_j) = c_2 d_j^2 + (-2c_2^2 + 2c_3)d_j^3 + (5c_2^3 - 7c_2c_3 + 3c_4)d_j^4 + O(d_j^5) \tag{9}$$

The expansion of the function $u(s)$ is obtained using (6) and (9) as:

$$u(s_j) = c_2 d_j + (-3c_2^2 + 2c_3)d_j^2 + (8c_2^3 - 10c_2c_3 + 3c_4)d_j^3 + O(d_j^4). \tag{10}$$

The Taylor expansion of the quotient power series in the second step of (5) is obtained as:

$$\begin{aligned}\frac{1+\sum_{i=1}^{3}\alpha_i u^i}{1+\sum_{i=1}^{3}\beta_i u^i} = {} & 1 + (\alpha_1 c_2 - \beta_1)d_j + (2\alpha_1 c_3 + c_2^2(-3\alpha_1 + \alpha_2) + 3\alpha_1 c_2^2 - 2\beta_1 \\ & - \alpha_1\beta_1 + \beta_1^2 - \beta_2 c_2^2)d_j^2 + (3\alpha_1 c_4 + c_2 c_3(-10\alpha_1 + 4\alpha_2) \\ & + c_2^3(8\alpha_1 - 6\alpha_2 + \alpha_3) - 8c_2^3\beta_1 + 10\beta_1 c_2 c_3 - 3\beta_1 c_4 \\ & - c_2(2\alpha_1 c_3 + c_2^2(-3\alpha_1 c_3 + \alpha_2))\beta_1 - 6\beta_1^2 c_2^3 + 4\beta_1^2 c_2 c_3 \\ & - \beta_1^3 c_2^3 + 6\beta_2 c_2^3 - 4\beta_2 c_2 c_3 + 2\beta_1\beta_2 c_2^3 + \alpha_1 c_2(3\beta_1 c_2^2 \\ & - 2\beta_1 c_3 + \beta_1^2 c_2^2 - \beta_2 c_2^2) - \beta_3 c_2^3)d_j^2 + O(d_j^4)\end{aligned} \tag{11}$$

The substitution of (11) in the second step of (5), resulted to the expression

$$\begin{aligned}s_{j+1} = {} & \delta + c_2(1 - \alpha_1 + \beta_1)d_j^2 + (2c_3(1 - \alpha_1 + \beta_1) + c_2^2(-2 - \alpha_2 - 4\beta_1 \\ & - \beta_1^2 + \alpha_1(4 + \beta_1) + \beta_2))d_j^3 + (3c_4(1 - \alpha_1 + \beta_1) + c_2c_3(-7 - 4\alpha_2 \\ & - 1 + \alpha_1 - 4\alpha_1^2 + 2\alpha_1(7 + 2\alpha_1) + 4\beta_2) + c_2^3(4 - \alpha_3 + 13\beta_1 + 7\beta_1^2 + \beta_1^3 \\ & + \alpha_2(7 + \alpha_1) - \alpha_1(13 + 7\beta_1 + \beta_1^2\beta - \beta_2) - 7\beta_2 - 2\beta_1\beta_2 + \beta_3))d_1^4 \\ & + O\left(d_j^5\right).\end{aligned} \tag{12}$$

From the error equation (12), the iterative scheme (5) converge to δ with order four if the coefficients of d_j, d_j^2 and d_j^3 vanishes. To achieve this, the following system of equations must be satisfied.

$$\begin{aligned}\beta_1 &= \alpha_1 - 1; \\ \beta_2 &= \alpha_2 - \alpha_1 - 1; \\ \beta_3 &= \alpha_3 - \alpha_2 - \alpha_1 - 2.\end{aligned} \tag{13}$$

By substituting (13) into (12), the expression below is obtained.

$$s_{j+1} = \delta - c_2 c_3 d_j^4 + O(d_j^5). \tag{14}$$

From Definition 1, the equation in (14) is the asymptotic error of the iterative scheme (5) and its CO is four. This end the proof.
□

Remark 1. By substituting the values of the parameters β_j in terms of free parameters α_j in (5) as satisfied in the conditions (13), a class of TPIM of CO four can be obtained. Consequently, the following TPIM of CO four is obtained.

$$y_j = s_j - t(s_j), \tag{15}$$

$$s_{j+1} = s_j - \frac{f(x_j)}{f'(s_j)}\left[\frac{1+\sum_{i=1}^{3}\alpha_i u^i}{1+(\alpha_1-1)\,u+(\alpha_2-\alpha_1-1)\,u^2+(\alpha_3-\alpha_2-\alpha_1-2)\,u^3}\right]. \tag{16}$$

3. Formation of some concrete members of the method

In this section, some members of the class of the TPIM put forward in (15) that are of CO four are presented for illustration. In fact, some existing IM are concrete members of the class (15). In presenting the concrete methods, the denotation M_i^4 is used to read "Method j with CO 4".

Method 1 (M_1^4): For instance, when $\alpha_1 = -1, \alpha_2 = 0$ and $\alpha_3 = 2$ in (15), the following method is constructed :

$$z_{j+1} = s_j - \frac{f(s_j)}{f'(s_j)}\left[\frac{1-\frac{f(y_j)}{f(s_j)}+2\left(\frac{f(y_j)}{f(s_j)}\right)^3}{1-2\frac{f(y_j)}{f(s_j)}+\left(\frac{f(y_j)}{f(s_j)}\right)^3}\right]. \tag{17}$$

Method 2 (M_2^4): For $\alpha_1 = \alpha_2 = \alpha_3 = 0$ in (15), the method denoted M_2^4 is constructed as:

$$z_{j+1} = s_j - \frac{f(s_j)}{f'(s_j)}\left[\frac{1}{1-\frac{f(y)}{f(s)}-\left(\frac{f(y)}{f(s)}\right)^2-2\left(\frac{f(y)}{f(s)}\right)^3}\right]. \tag{18}$$

Method 3 (M_3^4): By substituting $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = 2$ in (15), the following IM is obtained:

$$z_{j+1} = s_j - \frac{f(s_j)}{f'(s_j)}\left[\frac{1+2\left(\frac{f(y_j)}{f(s_j)}\right)^3}{1-\frac{f(y_j)}{f(s_j)}-\left(\frac{f(y_j)}{f(s_j)}\right)^2}\right]. \tag{19}$$

Method 4 (M_4^4): For $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = 4$ in (15), an IM is designed as:

$$z_{j+1} = s_j - \frac{f(s_j)}{f'(s_j)}\left[\frac{1+\frac{f(y_j)}{f(s_j)}+\left(\frac{f(y_j)}{f(s_j)}\right)^2+4\left(\frac{f(y_j)}{f(s_j)}\right)^3}{1-\left(\frac{f(y_j)}{f(s_j)}\right)^2}\right]. \tag{20}$$

Method 5 (M_5^4): For $\alpha_1 = \frac{2}{3}, \alpha_2 = -\frac{2}{3}$ and $\alpha_3 = \frac{1}{3}$ in (15), the IM is obtained:

$$z_{j+1} = s_j - \frac{f(s_j)}{f'(s_j)}\left[\frac{1+\frac{2}{3}\frac{f(y_j)}{f(s_j)}-\frac{2}{3}\left(\frac{f(y_j)}{f(s_j)}\right)^2+\frac{1}{3}\left(\frac{f(y_j)}{f(s_j)}\right)^3}{1-\frac{1}{3}\frac{f(y_j)}{f(s_j)}-\frac{5}{3}\left(\frac{f(y_j)}{f(s_j)}\right)^2-\frac{5}{3}\left(\frac{f(y_j)}{f(s_j)}\right)^3}\right]. \tag{21}$$

Method 6 (M_6^4): For $\alpha_1 = 1, \alpha_2 = 0$, and $\alpha_3 = 3$ in (15), the method denoted M_2^4 is constructed as:

$$z_{j+1} = s_j - \frac{f(s_j)}{f'(s_j)}\left[\frac{1+\frac{f(y_j)}{f(s_j)}+3\left(\frac{f(y_j)}{f(s_j)}\right)^3}{1-2\left(\frac{f(y_j)}{f(s_j)}\right)^2}\right]. \tag{22}$$

Remark 2. For $\alpha_1 = 1$, $\alpha_2 = 2$ and $\alpha_3 = 1$ in (15), we obtain the method in Kuo et al., [16] (Equation (27)).

Remark 3. We note here that if $m = 1$ and $\alpha_1 = -1$ in (3), then the famous Ostrowski method [17] is rediscovered as follows:

$$\begin{aligned} y_j &= s_j - t_j, \\ s_{j+1} &= s_j - \frac{f(x_j)}{f'(s_j)} \left[\frac{1 - \frac{f(y_j)}{f(s_j)}}{1 - 2\frac{f(y_j)}{f(s_j)}} \right]; \end{aligned} \tag{23}$$

Remark 4. Again, for $m = 2$ in (15), a subset class of the TPIM (15) is obtained as following:

$$\begin{aligned} y_j &= s_j - t_j, \\ s_{j+1} &= s_j - \frac{f(x_j)}{f'(s_j)} \left[\frac{1 + \sum_{i=1}^{2} \alpha_i u^i}{1 + (\alpha_1 - 1)\, u + (\alpha_2 - \alpha_1 - 1)\, u^2} \right]; \end{aligned} \tag{24}$$

with asymptotic error equation

$$s_{j+1} = \delta - \left(c_2 c_3 - c_2^3 \left(2 + \alpha_1 + \alpha_2\right) \right) d_j^4 + O\left(d_j^5\right). \tag{25}$$

Some new methods can be formed by arbitrarily assigning real values to $\alpha_i, i = 1, 2, \cdots$ in (23). To illustrate this, the following examples were considered.

Method 7 (M_7^4): For $\alpha_1 = \alpha_2 = 0$ in (23) produced the IM:

$$z_{j+1} = s_j - \frac{f(s_j)}{f'(s_j)} \left[\frac{1}{1 - \frac{f(y_j)}{f(s_j)} - \left(\frac{f(y_j)}{f(s_j)}\right)^2} \right]. \tag{26}$$

Method 8 (M_8^4): For $\alpha_1 = 0$ and $\alpha_2 = -2$ in (23) yields the IM:

$$z_{j+1} = s_j - \frac{f(s_j)}{f'(s_j)} \left[\frac{1 - 2\left(\frac{f(y_j)}{f(s_j)}\right)^2}{1 - \frac{f(y_j)}{f(s_j)} - 3\left(\frac{f(y_j)}{f(s_j)}\right)^2} \right]. \tag{27}$$

Remark 5. When $\alpha_1 = 1$ and $\alpha_3 = 2$ in (23), a method in Kuo et al., [16] (Equation 25) is rediscovered.

Method 9 (M_8^4): For $\alpha_1 = 0$ and $\alpha_2 = 1$ in (23), the method in Kuo et al., [16] (Equation 21) given as:

$$z_{j+1} = s_j - \frac{f(s_j)}{f'(s_j)} \left[\frac{1 + \left(\frac{f(y_j)}{f(s_j)}\right)^2}{1 - \frac{f(y_j)}{f(s_j)}} \right], \tag{28}$$

is rediscovered.

4. Numerical test on the method

This section of the work presents the numerical test conducted on the concrete members of the class of methods put forward herein, with the motivation of establishing its potency in determining the solution δ of NLM. Also, their performance were compared with some existing methods in literature that are of equal CO. Some of these methods includes:

Chun method (CM1) [8],

$$z_{j+1} = y_j - \frac{f(y_j)}{f'(s_j)}\left[1 + 2\frac{f(y_j)}{f(s_j)} + \left(\frac{f(y_j)}{f(s_j)}\right)^2\right]. \tag{29}$$

Chun method (CM2) [8],

$$z_{j+1} = y_j - \frac{f(y_j)}{f'(s_j)}\left[1 + 2\frac{f(y_j)}{f(s_j)}\right]. \tag{30}$$

Khattri and Abbasbandy (KAM) [9],

$$\begin{aligned} y_j &= s_j - \frac{2}{3}t(s_j), \\ s_{j+1} &= s_j - \frac{f(s_j)}{f'(s_j)}\left[1 + \frac{21}{8}\frac{f'(y)}{f'(s_j)} - \frac{9}{2}\left(\frac{f'(y)}{f'(s_j)}\right)^2 + \frac{15}{8}\left(\frac{f'(y)}{f'(s_j)}\right)^3\right]. \end{aligned} \tag{31}$$

Maheshwari method (MM) [18]

$$s_{j+1} = s_j - \frac{f(s_j)}{f'(s_j)}\left[\left(\frac{f(y_j)}{f(s_j)}\right)^2 + \frac{f(s_j)}{f(s_j) - f(y_j)}\right]. \tag{32}$$

The nonlinear models taking from [3,7,19] and used for the test are:

Model 1: (System projectile motion).
$f(s) = s^3 - 9s + 1$, with required root $\sigma \approx 2.942820057795838\ldots$

Model 2: (Concentration of pollutant bactaria).
$f(s) = 2s - lns - 7$, with required root $\sigma \approx 4.21990648378038\ldots$

Model 3: (Anti-symmetric buckling of a beam).
$f(s) = e^s + s - 20$, with required root $\sigma \approx 2.842438953784447\ldots$

Model 4: (Mass of a jumper).
$f(s) = sin(s) - s + 1$, with required root $\sigma \approx 1.934563210752024\ldots$

Model 5: (Gas volume depending on temperature).
$f(s) = e^s - 4s$, with required root $\sigma \approx 0.357402956181389\ldots$

Model 6: (Population growth equation).
$f(s) = 1586000 - \frac{435000}{s}(e^s - 1) - 1000000e^s$,
with required root $\sigma \approx 2.620641345791234\ldots$

Model 7: (Real gas Van der Waals equation).
$f(s) = 0.986s^3 - 5.181s^2 + 9.067s - 5.289$,
with required root $\sigma \approx 1.929846242847862\ldots$

Model 8: (Mixed reactor chemical concentration equation).
$f(s) = f(s) = 1 - 0.75e^{-0.05s}$, with required root $\sigma \approx 5.753641449035618\ldots$

All computations herein were done using the software MAPLE 2017 version, with 2000 digits numeric precision on 2GB RAM processor, Intel Celeron(R) with CPU 1.6GHz. The stopping criteria $|f(s_j)| \leq 10^{-200}$ was used for all programs. The measures used for comparison are: Number of Iterations required by method to achieve convergence (IT), norm of functions of each iteration point value $|f(s_j)|$ and function of last iteration value $|f(s_{j+1})|$ and the convergence order ρ_{co} due to Petkovic [20] given as:

$$\rho_{co} = \frac{log\left|f(s_{j+1})\right|}{log\left|f(s_j)\right|}. \tag{33}$$

The compared numerical results are presented in Table 1-3. Observe that for all the NLM used for the test, the developed methods obtained their solutions with higher precision than the existing methods compared.

Table 1. Methods results comparison for Models 1-3

Methods	*Models*	s_0	*IT*	$\left|f\left(s_{k+1}\right)\right|$	ρ_{co}
CM1			5	$4.26e-640$	4.03
CM2			5	$7.26e-591$	4.02
KAM			5	$7.5e-472$	4.03
MM			5	$9.11e-633$	4.01
M_1^4			4	$1.16e-235$	4.05
M_2^4			5	$5.25e-762$	4.02
M_3^4	1	2.7	4	$3.69e-215$	4.06
M_4^4			5	$7.58e-786$	4.03
M_5^4			5	$5.31e-800$	4.02
M_6^4			5	$1.27e-803$	4.02
M_7^4			5	$5.25e-762$	4.01
$M4_8$			5	$2.07e-624$	4.03
CM1			6	$1.57e-581$	4.03
CM2			6	$7.15e-531$	4.02
KAM			6	$2.47e-767$	4.02
MM			6	$8.19e-488$	4.03
M_1^4			5	$4.98e-201$	4.10
M_2^4			7	$1.65e-261$	4.08
M_3^4	2	0.1	6	$2.33e-767$	4.02
M_4^4			5	$2.47e-342$	4.02
M_5^4			7	$3.11e-228$	4.07
M_6^4			6	$6.01e-405$	4.05
M_7^4			7	$1.65e-261$	4.07
$M4_8$			6	$3.14e-497$	4.04
CM1			4	$3.63e-374$	4.00
CM2			4	$4.97e-364$	4.01
KAM			4	$4.69e-344$	4.00
MM			4	$2.70e-372$	4.00
M_1^4			4	$3.54e-455$	4.00
M_2^4			4	$8.33e-394$	4.02
M_3^4	3	2.0	4	$3.19e-450$	4.02
M_4^4			4	$5.31e-446$	4.02
M_5^4			4	$4.66e-399$	4.00
M_6^4			4	$1.19e-447$	4.00
M_7^4			4	$8.33e-394$	4.02
$M4_8$			4	$2,54e-454$	4.02

Table 2. Methods results comparison for Models 4-6

Methods	s_0	$\lvert f(s_1)\rvert$	$\lvert f(s_2)\rvert$	$\lvert f(s_3)\rvert$	$\lvert f(s_4)\rvert$	$\lvert f(s_5)\rvert$
CM1		$3.1e-2$	$6.2e-8$	$1.1e-30$	$9.4e-122$	$5.5e-482$
CM2		$4.8e-2$	$4.1e-7$	$2.5e-27$	$3.2e-108$	$9.8e-432$
KAM		$1.2e-1$	$2.6e-5$	$7.5e-20$	$5.0e-78$	$1.0e-310$
MM		$3.5e-2$	$9.6e-8$	$6.2e-30$	$1.1e-48$	$9.5e-474$
M_1^4		$1.6e-2$	$7.6e-10$	$3.4e-39$	$1.5e-156$	$4.8e-626$
M_2^4		$1.4e-2$	$1.5e-9$	$1.7e-37$	$3.6e-149$	$6.4e-596$
M_3^4	1.5	$1.4e-2$	$1.9e-10$	$9.0e-42$	$4.1e-167$	$1.6e-668$
M_4^4		$3.2e-2$	$4.7e-9$	$3.0e-36$	$5.1e-145$	$4.1e-580$
M_5^4		$1.0e-2$	$4.4-10$	$1.2e-39$	$8.2e-158$	$1.5e-630$
M_6^4		$9.2e-2$	$5.5e-6$	$9.7e-23$	$9.3e-90$	$7.8e-358$
M_7^4		$1.4e-2$	$1.5e-9$	$1.7e-37$	$3.5e-149$	$6.4e-596$
$M4_8$		$5.0e-3$	$3.8e-12$	$1.3e-48$	$1.6e-194$	$4.0e-778$
$M1$		$5.8e-4$	$7.5e-16$	$2.0e-63$	$1.2e-253$	-
CM2		$6.6e-4$	$1.5e-15$	$3.9e-62$	$1.9e-248$	-
KAM		$1.0e-3$	$1.6e-14$	$8.9e-58$	$8.4e-231$	-
MM		$5.8e-4$	$7.2e-16$	$1.8e-63$	$7.1e-254$	-
M_1^4		$1.2e-4$	$5.6e-20$	$2.6e-81$	$1.1e-326$	-
M_2^4		$4.0e-4$	$1.0e-16$	$4.1e-67$	$1.2e-278$	-
M_3^4	0.1	$2.3e-4$	$3.9e-18$	$3.8e-73$	$3.2e-293$	-
M_4^4		$2.5e-4$	$5.8e-18$	$1.7e-72$	$1.2e-290$	-
M_5^4		$3.7e-4$	$6.5e-17$	$6.5e-68$	$6.9e-272$	-
M_6^4		$7.3e-4$	$2.5e-15$	$3.7e-61$	$1.7e-244$	-
M_7^4		$3.7e-4$	$1.0e-16$	$4.1e-67$	$1.2e-268$	-
$M4_8$		$2.1e-4$	$2.7e-18$	$8.9e-74$	$9.5e-296$	-
CM1		228585.6	186.6	$1.3e-10$	$3.6e-59$	$1.9e-253$
CM2		511703.9	3368.7	$1.8e-5$	$1.6e-38$	$1.1e-170$
KAM		2.3	229265.6	445.6	$1.2e-8$	$5.2e-51$
MM		325544	624	$1.7e-8$	$8.8e-51$	$6.5e-220$
M_1^4		192415.7	134.9	$1.8e-11$	$6.4e-63$	$9.7e-269$
M_2^4		100212.4	37.0	$7.8e-18$	$1.7e-88$	$3.8e-371$
M_3^4	2.0	248231.8	266.5	$1.1e-10$	$3.6e-60$	$3.8e-258$
M_4^4		192415.7	134.9	$1.8e-11$	$6.5e-63$	$9.7e-269$
M_5^4		82375.9	1.3	$9.7e-20$	$2.9e-96$	$2.3e-402$
M_6^4		717725.9	9829.5	$1.6e-3$	$1.2e-30$	$3.5e-139$
M_7^4		100212.4	3.7	$7.9e-18$	$1.7e-88$	$3.8e-371$
$M4_8$		156023	20.5	$4.0e-15$	$5.5e-78$	$2.1e-329$

Table 3. Methods results comparison for Models 7-8

Methods	s_0	$\|f(s_1)\|$	$\|f(s_2)\|$	$\|f(s_3)\|$	$\|f(s_4)\|$	$\|f(s_5)\|$
CM1		$5.1e-2$	$5.3e-3$	$1.2e-4$	$3.4e-10$	$2.2e-32$
CM2		$4.9e-2$	$4.6e-3$	$7.8e-5$	$4.6e-11$	$6.0e-36$
KAM		$6.4e-2$	$9.0e-3$	$5.4e-4$	$1.9e-78$	$4.2e-21$
MM		$4.7e-2$	$4.4e-3$	$6.7e-5$	$2.5e-11$	$4.8e-37$
M_1^4		$2.1e-2$	$2.4e-4$	$1.3e-9$	$1.2e-30$	$8.4e-115$
M_2^4		$3.7e-2$	$2.4e-3$	$6.1e-6$	$8.2e-16$	$2.7e-55$
M_3^4	1.5	$3.0e-2$	$1.3e-3$	$5.3e-8$	$8.5e-25$	$5.6e-92$
M_4^4		$3.6e-2$	$2.2e-3$	$1.8e-6$	$1.2e-18$	$2.1e-67$
M_5^4		$3.5e-2$	$2.0-3$	$3.2e-6$	$4.8e-17$	$2.4e-60$
M_6^4		$5.1e-2$	$5.2e-3$	$1.3e-4$	$4.4e-10$	$5.4e-32$
M_7^4		$3.7e-2$	$2.3e-3$	$6.1e-6$	$8.2e-16$	$2.7e-55$
$M4_8$		$2.0e-2$	$2.7e-4$	$4.2e-10$	$3.4e-33$	$1.4e-125$
CM1		$5.6e-3$	$4.1e-10$	$1.2e-38$	$7.5e-153$	$1.3e-609$
CM2		$8.4e-3$	$2.7e-9$	$2.8e-35$	$3.2e-139$	$5.7e-555$
KAM		$2.1e-2$	$2.0e-7$	$1.7e-27$	$9.6e-108$	$9.5e-429$
MM		$6.1e-3$	$5.6e-10$	$4.1e-38$	$1.2e-150$	$9.6e-601$
M_1^4		$3.6e-3$	$3.6e-11$	$3.7e-43$	$3.8e-171$	$4.2e-683$
M_2^4		$2.1e-3$	$3.3e-12$	$2.1e-47$	$3.1e-188$	$1.5e-751$
M_3^4	0.3	$2.7e-3$	$4.6e-12$	$3.9e-47$	$2.0e-187$	$1.5e-748$
M_4^4		$5.0e-3$	$5.6e-11$	$8.4e-43$	$4.1e-170$	$2.0e-679$
M_5^4		$1.5e-3$	$6.1-13$	$1.7e-50$	$1.1e-200$	$1.9e-801$
M_6^4		$1.4e-2$	$2.4e-8$	$2.2e-31$	$1.7e-123$	$5.4e-492$
M_7^4		$2.1e-3$	$3.3e-12$	$2.1e-47$	$3.1e-188$	$1.5e-751$
$M4_8$		$1.5e-3$	$3.9e-13$	$1.9e-51$	$1.3e-204$	$-$

5. Conclusion

This manuscript has successfully put forward a class of CO four IM for determining the solution of scalar nonlinear models. The class of methods was constructed using the classical NIM as a predictor and corrector iterative function involving two power series' quotients. The famous Ostrowski method [17] and many methods in Kuo et al., [16] are concrete members of the class of methods put forward herein. The numerical experimentation with the developed methods on some standard NLM obtained from recent literature indicates that they have better precision than many other optimal CO four methods. For this reason, the developed methods can be used as good predictor iterative functions in developing multi-point, high-precision methods with higher CO. This can be considered for future work.

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References

[1] Traub, J. F. (1964). *Iterative methods for the solution of equations prentice-hall*. Englewood Cliffs, New Jersey.

[2] Weerakoon, S., & Fernando, T. (2000). A variant of Newton's method with accelerated third-order convergence. *Applied mathematics letters, 13*(8), 87-93.

[3] Ogbereyivwe, O., & Ojo-Orobosa, V. (2021). Families of Means-Based Modified Newtons Method for Solving Nonlinear Models. *Punjab University Journal of Mathematics,53*(11),779-791.

[4] Herceg, D., & Herceg, D. (2013). Third-order modifications of Newton's method based on Stolarsky and Gini means. *Journal of Computational and Applied Mathematics, 245*, 53-61.

[5] Herceg, D., & Herceg, D. (2013). Means based modifications of Newton's method for solving nonlinear equations. *Applied Mathematics and Computation, 219*(11), 6126-6133.

[6] Lukić, T., & Ralević, N. M. (2008). Geometric mean Newton's method for simple and multiple roots. *Applied Mathematics Letters, 21*(1), 30-36.

[7] Ogbereyivwe, O., & Ojo-Orobosa, V. (2021). Family of optimal two-step fourth order iterative method and its extension for solving nonlinear equations. *Journal of Interdisciplinary Mathematics, 24*(5), 1347-1365.

[8] Chun, C. (2008). Some fourth-order iterative methods for solving nonlinear equations. *Applied Mathematics and Computation*, *195*(2), 454-459.
[9] Khattri, S. K., & Abbasbandy, S. (2011). Optimal fourth order family of iterative methods. *Matematicki Vesnik, 63*(243), 67-72.
[10] Babajee, D. K. R., & Khratti, S. K. (2013). Dynamic behaviour of a unified two-point fourth order family of iterative methods. Annual Review of Chaos. *Bifurcations and Dynamical System*, *4*, 16-29.
[11] Ahmad, F. (2016). Comment on: On the Kung-Traub Conjecture for Iterative Methods for Solving Quadratic Equations. *Algorithms, 9*(2), 30.
[12] Babajee, D. K. R. (2015). On the Kung-Traub conjecture for iterative methods for solving quadratic equations. *Algorithms, 9*(1), 30.
[13] Madhu, K. (2018). Two-point iterative methods for solving quadratic equations and its applications. *Mathematical Sciences and Applications E-Notes*, *6*(2), 66-80.
[14] Ogbereyivwe, O., & Izevbizua, O. (2023). A three-free-parameter class of power series based iterative method for approximation of nonlinear equations solution. *Iranian Journal of Numerical Analysis and Optimization, 13*(2), 157-169.
[15] Sargent, R. W. H. (1980). A review of methods for solving nonlinear algebraic equations. Foundations of Computer Aided Chemical Process Design.
[16] Kou, J., Li, Y., & Wang, X. (2007). Fourth-order iterative methods free from second derivative. *Applied mathematics and computation, 184*(2), 880-885.
[17] Ostrowski, A. M. (1960). *Solutions of Equations and System Equations*. Academic Press, Cambridge, MA, USA.
[18] Maheshwari, A. K. (2009). A fourth order iterative method for solving nonlinear equations. Applied mathematics and computation, 211(2), 383-391.
[19] Qureshi, U. K., Shaikh, A. A., & Jamali, S. (2020). Sixth Order Numerical Iterated Method of Open Methods for Solving Nonlinear Applications Problems: Sixth Order Numerical Iterated Method of Open Methods for Solving Nonlinear Applications. *Proceedings of the Pakistan Academy of Sciences: A. Physical and Computational Sciences, 57*(2), 35-40.
[20] Petkovic, M. S. (2011). Remarks on "On a general class of multipoint root-finding methods of high computational efficiency". *SIAM journal on numerical analysis, 49*(3), 1317-1319.