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Coefficient bounds for p-valent functions

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Abstract: This present paper introduces two new subclasses of p-valent functions. The coefficient bounds and Fekete-Szego inequalities for the functions in these classes are also obtained.

Keywords: Coefficient bounds; Fekete-Szego inequalities; p-valent functions

MSC: 42C15,42C40, 47A05, 54A35.

1. Introduction

Let A_p denote the class of all p-valent functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k,\ p \in N \tag{1}$$

which are analytic in the open unit disk $U = \{z \in C : |z| < I\}$

Definition 1. Salagean [1] introduced the operator $D^\lambda f$, called the Salagean operator. Later on, Shenan *et al.* [2] extended the definition of the Salagean operator $D^\lambda f$ for p-valent functions $f \in T_p$ defined as follows:

$$\begin{aligned} D_*^0 f(z) &= f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \\ D_*^1 f(z) &= \frac{z}{p} f'(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k}{p}\right) a_k z^k \\ D_*^2 f(z) &= D(Df(z)) = \frac{z^2}{p^2} f''(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k}{p}\right)^2 a_k z^k \end{aligned}$$

and

$$D_*^\delta f(z) = D(D^{\delta-1} f(z)) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k}{p}\right)^\delta a_k z^k, p \in N, \delta \in NU\{0\} \tag{2}$$

Remark 1. For $p = 1$, $D_*^\delta f(z) = D^\delta f(z)$ by Salagean [1]

Definition 2. Let Ω denote the class of functions $w(z)$ which are analytic in U with $w(0) = 0$ and $|w(z)| < 1$. Let f_1^* and f_2^* be analytic in U. We say that f_1^* is a subordinate to f_2^* written as $f_1^*(z) \prec f_2^*(z)$ if there exists a function $w(z) \in \Omega$ in U such that $f_1^*(z) = f_2^*(w(z))$, see [3].

Definition 3. Let φ be an analytic function with positive real part in U with $\varphi(0) = 1$, $\varphi'(0) > 0$ andφ maps the disk U unto a region starlike with respect to 1 and symmetric with respect to the real axis. The Taylor's series expansion of such function is of the form $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ...$ with $B_i > 0$,$i \in N$

Definition 4. A function $f(z)$ given by (2) is said to be in the class $H_\delta(p,t,\varphi)$ if

$$(1-t)\frac{D_*^{\delta+1}f(z)}{D_*^{\delta}f(z)}+t\frac{D_*^{\delta+2}f(z)}{D_*^{\delta+1}f(z)}\prec\varphi(z),t\geq 0,\delta\in N_0,\ \ z\in U \tag{3}$$

Remark 2. When $t=0$ and $\delta=0$, we obtain the class $S^*(\varphi)$ and when $t=1$ and $\delta=0$ we obtain the class $C(\varphi)$. The two classes were studied by Ma and Minda [4]

Definition 5. A function $f(z)$ given by (2) is said to be in the class $H_\delta(p,t,\beta)$ if:

$$(1-t)\frac{D_*^{\delta+1}f(z)}{D_*^{\delta}f(z)}+t\frac{D_*^{\delta+2}f(z)}{D_*^{\delta+1}f(z)}>\beta,z\in U,t\in[0,1],0\leq\beta<1 \tag{4}$$

Remark 3. Putting $t=0$, and $p=1$, we have the class studied by Kadioglu [5]. For $p=1$ and $\delta=0$, the class reduces to the class M^α of alpha-convex functions by Moccanu [6].

Several authors, for example [7–9] have obtained the Fekete-Szego inequalities for functions in various subclasses of analytic,p-valent functions.

Here we state the necessary lemmas to support our work.

Lemma 1. *[10] If $p(z)=1+c_1z+c_2z^2+c_3z^3+...$is an analytic function in U with positive real part, Re $p(z)>0$, and $p(0)=1$,then*

$$|c_i|\leq 2,(i\in N=\{1,2,3...\})$$

and (1.2)

$$\left|c_2-\frac{c_1^2}{2}\right|\leq 2-\frac{|c_1^2|}{2}$$

Lemma 2. *[11] If $p(z)=1+c_1z+c_2z^2+c_3z^3+...$is an analytic function in U with positive real part and $p(0)=1$,then for any complex number v we have*

$$\left|c_2-vc_1^2\right|\leq 2max\{|1,2v-1|\}$$

The result is sharp for the function $p(z)$ given by $p(z)=\frac{1+z}{1-z}$ and $p(z)=\frac{1+z^2}{1-z^2}$

2. Coefficient bounds

Theorem 1. *Let $f(z)$ be given by (1) be in the class $H_\delta(p,\lambda,\varphi)$. Then*

$$|a_{p+1}|\leq\frac{B_1}{(t-1)(\frac{p+1}{p})^\delta+(1-2t)(\frac{p+1}{p})^{\delta+1}+t(\frac{p+1}{p})^{\delta+2}}$$

and

$$|a_{p+2}|\leq\frac{B_2\left[(t-1)(\frac{p+1}{p})^\delta+(1-2t)(\frac{p+1}{p})^{\delta+1}+t(\frac{p+1}{p})^{\delta+2}\right]^2}{R^*}$$

$+$

$$\frac{B_1^2\left[t(\frac{p+1}{p})^{2\delta+3}+(t-1)(\frac{p+1}{p})^{2\delta}+(1-t)(\frac{p+1}{p})^{2\delta+1}-t(\frac{p+1}{p})^{2\delta+2}\right]}{R^*}$$

where

$$R^*=\left[(t-1)(\frac{p+2}{p})^\delta+(1-2t)(\frac{p+2}{p})^{\delta+1}+t(\frac{p+2}{p})^{\delta+2}\right]\times\left[(t-1)(\frac{p+1}{p})^\delta+(1-2t)(\frac{p+1}{p})^{\delta+1}+t(\frac{p+1}{p})^{\delta+2}\right]^2$$

Proof. Let $f(z) \in H_\delta(p,t,\varphi)$, then there exists the analytic functions $\omega(z)$, $\omega(z): U \longrightarrow U$ with $|\omega(z)| < 1$ satisfying the following subordination:

$$(1-t)\frac{D_*^{\delta+1}f(z)}{D_*^{\delta}f(z)} + t\frac{D_*^{\delta+2}f(z)}{D_*^{\delta+1}f(z)} = \varphi(\omega(z)) \tag{5}$$

Using the operator $D_*^\delta f(z)$ defined in (2) in (5),we have

$$(1-t\left\{\left(1+\sum_{k=p+1}^{\infty}\left(\frac{k}{p}\right)^{\delta+1}a_k z^{k-p}\right)\left(1+\sum_{k=p+1}^{\infty}\left(\frac{k}{p}\right)^{\delta}a_k z^{k-p}\right)^{-1}\right\}$$

$$+t\left\{\left(1+\sum_{k=p+1}^{\infty}\left(\frac{k}{p}\right)^{\delta+2}a_k z^{k-p}\right)\left(1+\sum_{k=p+1}^{\infty}\left(\frac{k}{p}\right)^{\delta+1}a_k z^{k-p}\right)^{-1}\right\} = \varphi(\omega(z)) \tag{6}$$

Using binomial expansion and rearrangement, the left hand side of (6) gives

$$1+\left[(t-1)\left(\frac{p+1}{p}\right)^{\delta}+(1-2t)\left(\frac{p+1}{p}\right)^{\delta+1}+t\left(\frac{p+1}{p}\right)^{\delta+2}\right]a_{p+1}z+$$
$$\left[(t-1)\left(\frac{p+2}{p}\right)^{\delta}+(1-2t)\left(\frac{p+2}{p}\right)^{\delta+1}+t\left(\frac{p+1}{p}\right)^{\delta+2}\right]a_{p+2}z^2+... \tag{7}$$

Now express the right hand side of (6) in the series form.

Let $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...$ be a Caratheodory function and the function $w(z)$ be a Schwartz function. Then

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... \prec \frac{1+z}{1-z} \tag{8}$$

Since $\omega(z)$ is a Schwartz function, then from (8)

$$\omega(z) = \frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \frac{1}{2}\left[c_3 + \left(\frac{c_1^3}{4} - c_1 c_2\right)\right]z^3 + ...$$

Substituting $\omega(z)$ into $\varphi(z)$, it follows that

$$\varphi(\omega(z)) = 1 + \frac{1}{2}B_1 c_1 z + \frac{1}{2}\left[B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{2}B_2 c_1^2\right]z^2 + ... \tag{9}$$

Comparing the coefficients of z and z^2 in (7) and (9)
we have

$$a_{p+1} = \frac{B_1 c_1}{2\left[(t-1)\left(\frac{p+1}{p}\right)^{\delta} + (1-2t)\left(\frac{p+1}{p}\right)^{\delta+1} + t\left(\frac{p+1}{p}\right)^{\delta+2}\right]} \tag{10}$$

Taking modulus on both sides of (10) and applying $|c_i| \le 2, \forall i \ge 1$, we get

$$|a_{p+1}| \le \frac{B_1}{(t-1)(\frac{p+1}{p})^{\delta} + (1-2t)(\frac{p+1}{p})^{\delta+1} + t(\frac{p+1}{p})^{\delta+2}}$$

Also from (7) and (9), we have

$$a_{p+2} = \frac{2\left[(t-1)\left(\frac{p+1}{p}\right)^{\delta} + (1-2t)\left(\frac{p+1}{p}\right)^{\delta+1} + t\left(\frac{p+1}{p}\right)^{\delta+2}\right]^2\left[B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{2}B_2 c_1^2\right]}{4R^*}$$

$+$

$$\frac{B_1^2 c_1^2\left[t(\frac{p+1}{p})^{2\delta+3} + (t-1)(\frac{p+1}{p})^{2\delta} + (1-t)(\frac{p+1}{p})^{2\delta+1} - t(\frac{p+1}{p})^{2\delta+2}\right]}{4R^*} \tag{11}$$

Taking modulus on both sides of(11) and applying Lemma 1, we get

$$|a_{p+2}| \leq \frac{B_2\left[(t-1)(\frac{p+1}{p})^{\delta}+(1-2t)(\frac{p+1}{p})^{\delta+1}+t(\frac{p+1}{p})^{\delta+2}\right]^2}{R^*}$$

+

$$\frac{B_1^2\left[t(\frac{p+1}{p})^{2\delta+3}+(t-1)(\frac{p+1}{p})^{2\delta}+(1-t)(\frac{p+1}{p})^{2\delta+1}-t(\frac{p+1}{p})^{2\delta+2}\right]}{R^*}$$

Where

$$R^* = \left[(t-1)(\frac{p+2}{p})^{\delta}+(1-2t)(\frac{p+2}{p})^{\delta+1}+t(\frac{p+2}{p})^{\delta+2}\right] \times \left[(t-1)(\frac{p+1}{p})^{\delta}+(1-2t)(\frac{p+1}{p})^{\delta+1}+t(\frac{p+1}{p})^{\delta+2}\right]^2 \quad (12)$$

Taking $\delta = 0$ and $p = 1$ in Theorem □

Corollary 2. *is obtained*

$$|a_2| \leq \frac{B_1}{t+1}$$

and

$$|a_3| \leq \frac{B_2(t+1)^2 + B_1^2(3t+1)}{(4t+2)(t+1)^2}$$

Remark 4. Taking $t = 0$ and $B_1 = B_2 = 2$ in Corollary 2, we have $|a_2| \leq 2$, and $|a_3| \leq 3$ which are known bounds for the class of starlike functions are obtained.
Taking $t = 1$ and $B_1 = B_2 = 2$ in Corollary 2, we have $|a_2| \leq 1$, and $|a_3| \leq 1$ which are known bounds for the class of convex functions are obtained.

Theorem 3. *Let $f(z)$ be given by (1) be in the class $H_n(p,t,\beta)$. Then*

$$|a_{p+1}| \leq \frac{2(1-\beta)}{(t-1)(\frac{p+1}{p})^{\delta}+(1-2t)(\frac{p+1}{p})^{\delta+1}+t(\frac{p+1}{p})^{\delta+2}}$$

and

$$|a_{p+2}| \leq \frac{2(1-\beta)\left[(t-1)(\frac{p+1}{p})^{\delta}+(1-2t)(\frac{p+1}{p})^{\delta+1}+t(\frac{p+1}{p})^{\delta+2}\right]^2}{R^*}$$

+

$$\frac{4(1-\beta)^2\left[t(\frac{p+1}{p})^{2\delta+3}+(t-1)(\frac{p+1}{p})^{2\delta}+(1-t)(\frac{p+1}{p})^{2\delta+1}-t(\frac{p+1}{p})^{2\delta+2}\right]}{R^*}$$

where R^*as stated in (12)

Proof. Expressing $p(z) = \beta + (1-\beta)\left(1+q_1z+q_2z^2+q_3z^3..\right)$ in series form,where $p(z)$ is a Caratheodory functions with condition Re$p(z) > 0$ and $p(0) = 1$,then we have

$$p(z) = \beta + (1-\beta) + (1-\beta)q_1z + (1-\beta)q_2z^2 + (1-\beta)q_3z^3 + ... \quad (13)$$

Equating the coefficients of z and z^2 in (7) and (13), we have

$$a_{p+1} = \frac{(1-\beta)q_1}{(t-1)(\frac{p+1}{p})^{\delta}+(1-2t)(\frac{p+1}{p})^{\delta+1}+t(\frac{p+1}{p})^{\delta+2}} \quad (14)$$

Taking modulus on both sides of(14) and applying Lemma 1, we get

$$|a_{p+1}| \leq \frac{2(1-\beta)}{(t-1)(\frac{p+1}{p})^{\delta}+(1-2t)(\frac{p+1}{p})^{\delta+1}+t(\frac{p+1}{p})^{\delta+2}}$$

Also from (7) and (13)we get

$$a_{p+2} = \frac{(1-\beta)q_2\left[(t-1)(\frac{p+1}{p})^{\delta} + (1-2t)(\frac{p+1}{p})^{\delta+1} + t(\frac{p+1}{p})^{\delta+2}\right]^2}{R^*}$$

+

$$\frac{(1-\beta)^2 q_1^2\left[t(\frac{p+1}{p})^{2\delta+3} + (t-1)(\frac{p+1}{p})^{2\delta} + (1-t)(\frac{p+1}{p})^{2\delta+1} - t(\frac{p+1}{p})^{2\delta+2}\right]}{R^*} \tag{15}$$

where R^* ia as stated above

Taking modulus on both sides of(15) and applying Lemma 1

$$|a_{p+2}| \leq \frac{2(1-\beta)\left[(t-1)(\frac{p+1}{p})^{\delta} + (1-2t)(\frac{p+1}{p})^{\delta+1} + t(\frac{p+1}{p})^{\delta+2}\right]^2}{R^*}$$

+

$$\frac{4(1-\beta)^2\left[t(\frac{p+1}{p})^{2\delta+3} + (t-1)(\frac{p+1}{p})^{2\delta} + (1-t)(\frac{p+1}{p})^{2\delta+1} - t(\frac{p+1}{p})^{2\delta+2}\right]}{R}$$

where R^* is as stated in (12) □

Taking $p = 1$ and $\delta = 0$ in Theorem **??**,we have

Corollary 4. $|a_2| \leq \frac{2(1-\beta)}{t+1}$ *and* $|a_3| \leq \frac{2(1-\beta)^2(3(t+1)-2\beta)}{(2t+1)(t+1)^2}$

Remark 5. When $\delta = 0, t = 0, p = 1$ and $\beta = 0$ in Theorem **??**, we have $|a_2| \leq 2$ and $|a_3| \leq 6$ which are the coefficient bounds for $|a_2|$ and $|a_3|$ for the classes of starlike functions. Also When $\delta = 0, t = 1, p = 1$ and $\beta = 0$ in Theorem **??**, we have $|a_2| \leq 1$ and $|a_3| \leq 1$ which are the coefficient bounds for $|a_2|$ and $|a_3|$ for all the classes of convex functions.

3. Fekete-Szego Inequalities

Theorem 5. *If $f(z) \in H_{\delta}(p,t,\varphi)$ and ρ is any complex number, then*

$$\left|a_{p+2} - \rho a_{p+1}^2\right| \leq \frac{1}{2A} \times Max\left\{1, \left|\left(Q - \frac{(B_2 - B_1)}{MB_1}\right) - 1\right|\right\}$$

Where

$$Q = \frac{B_1^2\left(t(\frac{p+1}{p})^{2\delta+3} + (t-1)(\frac{p+1}{p})^{2\delta} + (1-t)(\frac{p+1}{p})^{2\delta+1} - t(\frac{p+1}{p})^{2\delta+2}\right)}{\left[(t-1)(\frac{p+1}{p})^{\delta} + (1-2t)(\frac{p+1}{p})^{\delta+1} + t(\frac{p+1}{p})^{\delta+2}\right]^2}$$

+

$$\frac{B_1^2\rho\left[(t-1)(\frac{p+2}{p})^{\delta} + (1-2t)(\frac{p+2}{p})^{\delta+1} + t(\frac{p+2}{p})^{\delta+2}\right]}{\left[(t-1)(\frac{p+1}{p})^{\delta} + (1-2t)(\frac{p+1}{p})^{\delta+1} + t(\frac{p+1}{p})^{\delta+2}\right]^2}$$

$$M = \left[(t-1)(\frac{p+1}{p})^{\delta} + (1-2t)(\frac{p+1}{p})^{\delta+1} + t(\frac{p+1}{p})^{\delta+2}\right] \tag{16}$$

and $A = R^*$

Proof. From (10) and (11), we get

$$a_{p+2} - \rho a_{p+1}^2 = \frac{2B_1 c_2\left[(t-1)\left(\frac{p+1}{p}\right)^{\delta} + (1-2t)\left(\frac{p+1}{p}\right)^{\delta+1} + t\left(\frac{p+1}{p}\right)^{\delta+2}\right]^2}{4R^*}$$

$$+\frac{B_1^2c_1^2\left\{\left[t(\frac{p+1}{p})^{2\delta+3}+(t-1)(\frac{p+1}{p})^{2\delta}+(1-t)(\frac{p+1}{p})^{2\delta+1}-t(\frac{p+1}{p})^{2\delta+2}\right]-\rho K\right\}}{4R^*}$$

$$+\frac{c_1^2(B_2-B_1)\left[(t-1)(\frac{p+1}{p})^{\delta}+(1-2t)(\frac{p+1}{p})^{\delta+1}+t(\frac{p+1}{p})^{\delta+2}\right]}{4R^*}$$

Thus

$$a_{p+2}-\rho a_{p+1}^2\leq\frac{1}{A}\left\{c_2-\left[\frac{B_1^2(T+\rho K)}{2T^*}-\frac{(B_2-B_1)}{2MB_1}\right]c_1^2\right\}$$

where T^*

$$=\left[(t-1)(\frac{p+1}{p})^{\delta}+(1-2t)(\frac{p+1}{p})^{\delta+1}+t(\frac{p+1}{p})^{\delta+2}\right]^2$$

$$T=\left(t(\frac{p+1}{p})^{2\delta+3}+(t-1)(\frac{p+1}{p})^{2\delta}+(1-t)(\frac{p+1}{p})^{2\delta+1}-t(\frac{p+1}{p})^{2\delta+2}\right)$$

$$K=\left[(t-1)(\frac{p+2}{p})^{\delta}+(1-2t)(\frac{p+2}{p})^{\delta+1}+t(\frac{p+2}{p})^{\delta+2}\right]$$

$$M=\left[(t-1)(\frac{p+1}{p})^{\delta}+(1-2t)(\frac{p+1}{p})^{\delta+1}+t(\frac{p+1}{p})^{\delta+2}\right]$$

which implies that

$$a_{p+2}-\rho a_{p+1}^2=\frac{1}{A}\left[c_2-vc_1^2\right] \tag{17}$$

Taking modulus on (17), we have

$$\left|a_{p+2}-\rho a_{p+1}^2\right|=\left|\frac{1}{A}\right|\left|c_2-vc_1^2\right|$$

Using Lemma 2, we have

$$\left|a_{p+2}-\rho a_{p+1}^2\right|\leq\frac{2}{A}max\left\{1,|2v-1|\right\}$$

which gives

$$\left|a_{p+2}-\rho a_{p+1}^2\right|\leq\frac{1}{2A}\times Max\left\{1,\left|\left(Q-\frac{(B_2-B_1)}{MB_1}\right)-1\right|\right\}$$

where $v=\left(Q-\frac{(B_2-B_1)}{MB_1}\right)$, Q and M as stated in (16) □

Theorem 6. *If $f(z)\in H_n(p,\lambda,\beta)$ and ρ is any complex number, then*

$$\frac{2}{T}Max\left\{1,|2K-1|\right\}$$

where

$$K=\frac{(1-\beta)\left\{\left(t(\frac{p+1}{p})^{2\delta+3}+(t-1)(\frac{p+1}{p})^{2\delta}+(1-t)(\frac{p+1}{p})^{2\delta+1}-t(\frac{p+1}{p})^{2\delta+2}\right)+\rho M\right\}}{\left[(t-1)(\frac{p+1}{p})^{\delta}+(1-2t)(\frac{p+1}{p})^{\delta+1}+t(\frac{p+1}{p})^{\delta+2}\right]^2}$$

$$M=(t-1)(\frac{p+2}{p})^{\delta}+(1-2t)(\frac{p+2}{p})^{\delta+1}+t(\frac{p+2}{p})^{\delta+2}$$

and $T=R^*$as stated above

Proof. From (14) and (15),we obtain $a_{p+2}-\rho a_{p+1}^2=$

$$\frac{(1-\beta)q_2\left[(t-1)(\frac{p+1}{p})^{\delta}+(1-2t)(\frac{p+1}{p})^{\delta+1}+t(\frac{p+1}{p})^{\delta+2}\right]^2}{R^*}$$

$$+\frac{q_1^2(1-\beta)^2\left\{\left(t(\frac{p+1}{p})^{2\delta+3}+(t-1)(\frac{p+1}{p})^{2\delta}+(1-t)(\frac{p+1}{p})^{2\delta+1}-t(\frac{p+1}{p})^{2\delta+2}\right)-\rho M\right\}}{R^*} \tag{18}$$

Which gives

$$\frac{1}{T}\left[q_2-\frac{(1-\beta)\left\{\left(t(\frac{p+1}{p})^{2\delta+3}+(t-1)(\frac{p+1}{p})^{2\delta}+(1-t)(\frac{p+1}{p})^{2\delta+1}-t(\frac{p+1}{p})^{2\delta+2}\right)+\rho M\right\}q_1^2}{\left[(t-1)(\frac{p+1}{p})^{\delta}+(1-2t)(\frac{p+1}{p})^{\delta+1}+t(\frac{p+1}{p})^{\delta+2}\right]^2}\right]$$

Which implies that

$$a_{p+2}-\rho a_{p+1}^2=\frac{1}{T}\left[c_2-vc_1^2\right] \tag{19}$$

Taking modulus on (17), and by applying Lemma 2,we get

$$\left|a_{p+2}-\rho a_{p+1}^2\right|\leq\frac{1}{T}\left[c_2-vc_1^2\right]$$

Which gives

$$\left|a_{p+2}-\rho a_{p+1}^2\right|\leq\frac{2}{T}Max\left\{1,|2v-1|\right\}$$

where

$$v=\frac{(1-\beta)\left\{\left(t(\frac{p+1}{p})^{2\delta+3}+(t-1)(\frac{p+1}{p})^{2\delta}+(1-t)(\frac{p+1}{p})^{2\delta+1}-t(\frac{p+1}{p})^{2\delta+2}\right)+\rho M\right\}}{\left[(t-1)(\frac{p+1}{p})^{\delta}+(1-2t)(\frac{p+1}{p})^{\delta+1}+t(\frac{p+1}{p})^{\delta+2}\right]^2}$$

$$M=(t-1)(\frac{p+2}{p})^{\delta}+(1-2t)(\frac{p+2}{p})^{\delta+1}+t(\frac{p+2}{p})^{\delta+2}$$

and $T=R^*$ as stated above □

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