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Expansion of the Jensen $(\Gamma_1, \Gamma_2)-$functional inequatities based on Jensen type (η, λ)-functional equation with $3k$-Variables in complex Banach space

Ly Van An[1]

[1] Faculty of Mathematics Teacher Education, Tay Ninh University, Tay Ninh, Vietnam
* Correspondence: lyvanan145@gmail.com, lyvananvietnam@gmail.com

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Abstract: In this paper, we work on expanding the Jensen (Γ_1, Γ_2)-function inequalities by relying on the general Jensen (η, λ)-functional equation with $3k$-variables on the complex Banach space. That is the main result of this.

Keywords: Generalized Jensen type (Γ_1, Γ_2)-functional inequality; Generalized Jensen type (η, λ)-functional equations; Hyers-Ulam-Rassias stability; Complex Banach space; Complex normed vector spaces

MSC: 39B22, 39B82, 46S10.

1. Introduction

Let $\mathbf{X}$ and $\mathbf{Y}$ be normed spaces over the same field $\mathbb{K}$, and let $f : \mathbf{X} \to \mathbf{Y}$ be a mapping. We denote the norms on $\mathbf{X}$ and $\mathbf{Y}$ as $\|\cdot\|_{\mathbf{X}}$ and $\|\cdot\|_{\mathbf{Y}}$, respectively. In this paper, we investigate the relationship between Jensen-type functional equations and Jensen-type (Γ_1, Γ_2)-function inequalities when $(\mathbf{X}, \|\cdot\|_{\mathbf{X}})$ is a complex normed vector space, and $(\mathbf{Y}, \|\cdot\|_{\mathbf{Y}})$ is a complex normed vector Banach space.

When $\mathbf{X}$ is a complex normed vector space and $\mathbf{Y}$ is a complex Banach space, we analyze and establish the Hyers-Ulam stability concerning the following relationship between Jensen-type (Γ_1, Γ_2)-function inequalities and Jensen-type functional equations:

$$\begin{aligned}
&\left\| f\left(\sum_{i=1}^k x_i + \lambda\sum_{i=1}^k y_i + \eta\sum_{i=1}^k z_i\right) - f\left(\sum_{i=1}^k x_i - \eta\sum_{i=1}^k z_i\right) - \lambda\sum_{i=1}^k f(y_i) - 2\sum_{i=1}^k f(\eta z_i)\right\|_{\mathbf{Y}} \\
&\leq \left\|\Gamma_1\left(f\left(\sum_{i=1}^k x_i + \eta\sum_{i=1}^k z_i\right) - \sum_{i=1}^k f(x_i) - \sum_{i=1}^k f(\eta z_i)\right)\right\|_{\mathbf{Y}} \\
&+ \left\|\Gamma_2\left(f\left(\sum_{i=1}^k x_i + \lambda\sum_{i=1}^k y_i - \eta\sum_{i=1}^k z_i\right) - \sum_{i=1}^k f(x_i) - \lambda\sum_{i=1}^k f(y_i) + \sum_{i=1}^k f(\eta z_i)\right)\right\|_{\mathbf{Y}}
\end{aligned} \tag{1}$$

and

$$\begin{aligned}
&\left\| f\left(\sum_{i=1}^k x_i + \sum_{i=1}^k y_i + \eta\sum_{i=1}^k z_i\right) - f\left(\sum_{i=1}^k x_i + \sum_{i=1}^k y_i - \eta\sum_{i=1}^k z_i\right) - 2\sum_{i=1}^k f(y_i) - 2\sum_{i=1}^k f(z_i)\right\|_{\mathbf{Y}} \\
&\leq \left\|\Gamma_1\left(f\left(\sum_{i=1}^k x_i + \sum_{i=1}^k y_i + \eta\sum_{i=1}^k z_i\right) - \sum_{i=1}^k f(x_i + y_i) - \sum_{i=1}^k f(\eta z_i)\right)\right\|_{\mathbf{Y}} \\
&+ \left\|\Gamma_2\left(f\left(\sum_{i=1}^k x_i + \sum_{i=1}^k y_i - \eta\sum_{i=1}^k z_i\right) + \sum_{i=1}^k f(-x_i) + \sum_{i=1}^k f(\eta z_i - y_i)\right)\right\|_{\mathbf{Y}}
\end{aligned} \tag{2}$$

These inequalities are based on the following Jensen-type (η, λ)-functional equations involving $3k$ variables:

$$f\left(\sum_{i=1}^{k}x_i+\lambda\sum_{i=1}^{k}y_i+\eta\sum_{i=1}^{k}z_i\right)-f\left(\sum_{i=1}^{k}x_i-\eta\sum_{i=1}^{k}z_i\right)-\lambda\sum_{i=1}^{k}f(y_i)-2\sum_{i=1}^{k}f(\eta z_i)=0 \quad (3)$$

and

$$f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i+\sum_{i=1}^{k}\eta z_i\right)+f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i-\eta\sum_{i=1}^{k}z_i\right)$$
$$-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)=0 \quad (4)$$

Note: Here, k is a positive integer, Γ_1 and Γ_2 are fixed complex numbers, and η and λ are real numbers.

The exploration of functional equation stability originated from S.M. Ulam's question [1] regarding the stability of group homomorphisms. The central inquiry involves determining whether for a given $\epsilon > 0$, there exists a $\delta > 0$ such that if a function $f : \mathbb{G} \to \mathbb{G}'$ satisfies:

$$d\Big(f(x*y), f(x)\circ f(y)\Big) < \delta$$

for all $x, y \in \mathbb{G}$, then there exists a homomorphism $h : \mathbb{G} \to \mathbb{G}'$ such that:

$$d\Big(f(x), h(x)\Big) < \epsilon$$

for all $x \in \mathbb{G}$. In case the answer is affirmative, we would affirm that the equation of the homomorphism $h(x*y) = h(x)\circ h(y)$ is stable. The concept of stability for a functional equation emerges when we substitute the functional equation with an inequality that acts as a perturbation of the equation. Consequently, the question of stability for functional equations pertains to understanding how the solutions of the inequality differ from those of the original functional equation.

Hyers [2] provided the first affirmative answer to Ulam's question in the following manner:

Let E_1 be a normed space and E_2 a Banach space. Suppose the mapping $f : E_1 \to E_2$ satisfies the inequality:

$$\|f(x+y)-f(x)-f(y)\| \leq \epsilon,$$

for all $x, y \in E_1$, where $\epsilon \geq 0$ is a constant. Then, the limit $T(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ exists for each $x \in E_1$, and T is the unique additive mapping such that:

$$\|f(x)-T(x)\| \leq \epsilon, \quad \forall x \in E_1.$$

Moreover, if the functional $t \to f(xt)$ from $\mathbb{R}$ to E_2 is continuous for each x in $\mathbb{E}_1$, and f is continuous at a single point of $\mathbb{E}_1$, then T is continuous everywhere in $\mathbb{E}_1$.

Rassias [3] later generalized Hyers' Theorem as a special case. Suppose $\mathbb{E}$ and $\mathbb{E}'$ are normed spaces, with $\mathbb{E}'$ being a complete normed space. Let $f : \mathbb{E} \to \mathbb{E}'$ be a mapping such that for each fixed $x \in E$, the mapping $t \to f(xt)$ is continuous on $\mathbb{R}$. Assume there exist $\epsilon > 0$ and $p \in [0, 1]$ such that:

$$\|f(x+y)-f(x)-f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p), \quad \forall x, y \in \mathbb{E}.$$

Then there exists a unique linear $L : \mathbb{E} \to \mathbb{E}'$ that satisfies:

$$\|f(x)-L(x)\| \leq \frac{\epsilon}{1-2^{1-p}}\|x\|^p, \quad x \in \mathbb{E}.$$

The existence of a unique linear mapping was initially obtained by Aoki [4]. However, Aoki's claim of the existence of a unique linear mapping is not valid because he did not assume certain continuity conditions for the mapping f. Rassis [5], who independently introduced the unbounded difference, was the first to prove the existence of a unique linear mapping T satisfying:

$$\|f(x)-T(x)\| \leq \frac{\epsilon}{1-2^{1-p}}\|x\|^p, \quad x \in \mathbb{E}.$$

In 1990, Rassias [6] posed the question of whether such a theorem could be proved for $p \geq 1$. In 1991, Z. Gajda [7] provided an affirmative solution to this question for $p > 1$. However, it was shown by Gajda [8], as well as by Rassias and P. Semrl [8], that a Rassias-type theorem cannot be proven when $p = 1$. Subsequently, in 1994, P. Găvruta [9] generalized Rassias' theorem further by replacing the bounded expression $\epsilon(\|x\|^p + \|y\|^p)$ with a general control function $\psi(x, y)$ for the existence of a unique linear mapping.

In a broader scope, we have established the Jensen-type functional inequality relationship with the multivariable Jensen equation on a complex Banach space. This extension aims to enhance the classical Jensen equation by involving a larger number of variables, where the unique solution remains a general additive function.

Recently, the author has formulated general inequalities on spaces such as Banach spaces and non-Archimedean Banach spaces (see [10,11]). We have derived and proven the Hyers-Ulam type stability for functional equations (1) and (2), which involve $3k$ variables. This stability is achieved under suitable assumptions on spaces **X** and **Y**, thereby establishing that the mappings satisfy the functional equations (1) or (2). Our results in this paper generalize those in [10] and [11] for functional equations with $3k$ variables.

The paper is organized as follows: In the preliminaries section, we review some basic notations from [12], such as solutions of the inequalities. Section 3 addresses the stability of the Jensen-type (Γ_1, Γ_2)-functional inequalities (1) associated with the (η, λ)-functional equation of (3). Additionally, in Section 4, we explore the stability of the Jensen-type (Γ_1, Γ_2)-functional inequalities (2) associated with the (η, λ)-functional equation of (4).

2. Preliminaries

2.1. Solutions of the inequalities.

The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the Cauchuy equation. In particular, every solution of the Cauchuy equation is said to be an additive mapping. The functional equations

$$f(\frac{x+y}{2}) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$$

is called the Jensen equations. In particular, every solution of the Jensen equation is said to be a Jensen additive mapping.

3. Stability (Γ_1, Γ_2)-functional inequalities (1) relative to functional equation (3)

In this section, assume that **X** is a complex normed vector spaces, **Y** is a complex Banach space Γ_1, Γ_2 are the fixed complex numbers with $G(\Gamma_1, \Gamma_2)$ functional inequality and η, λ are the real numbers with $g(\eta, \lambda)$ functional equation. Under this setting, we can show that the mappings satisfying (1) is additive.
Here I assume that:

$$\Big|(2k-1)\Big|\Big|\Gamma_1\Big| + \Big|\Gamma_2\Big|\Big|k\lambda - 1\Big| < k\Big|\lambda + 2\Big|.$$

3.1. Condition for existence of solution of (1)

Lemma 1. *Suppose that $f : \mathbf{X} \to \mathbf{Y}$ be a mapping and it satisfies the functional inequality*

$$\begin{aligned}
&\left\| f\left(\sum_{i=1}^{k}(x_i) + \lambda\sum_{i=1}^{k} y_i + \eta\sum_{i=1}^{k} z_i\right) - f\left(\sum_{i=1}^{k} x_i - \eta\sum_{i=1}^{k} z_i\right) - \lambda\sum_{i=1}^{k} f\Big(y_i\Big) - 2\sum_{i=1}^{k} f\Big(\eta z_i\Big)\right\|_{\mathbf{Y}} \\
&\leq \left\|\Gamma_1\left(f\left(\sum_{i=1}^{k} x_i + \eta\sum_{i=1}^{k} z_i\right) - \sum_{i=1}^{k} f\Big(x_i\Big) - \sum_{i=1}^{k} f\Big(\eta z_i\Big)\right)\right\|_{\mathbf{Y}} \\
&+ \left\|\Gamma_2\left(f\left(\sum_{i=1}^{k} x_i + \lambda\sum_{i=1}^{k} y_i - \eta\sum_{i=1}^{k} z_i\right) - \sum_{i=1}^{k} f\Big(x_i\Big) - \lambda\sum_{i=1}^{k} f\Big(y_i\Big) + \sum_{i=1}^{k} f\Big(\eta z_i\Big)\right)\right\|_{\mathbf{Y}}
\end{aligned} \tag{5}$$

For all $x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k \in \mathbf{X}, i = 1 \to k$ then f is additive..

Proof. We replacing $(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k)$ by $(0, ..., 0, 0, ..., 0, 0, ..., 0)$ in (5), we have

$$\left(k\left|\lambda+2\right|-\left|(2k-1)\right|\left|\Gamma_1\right|-\left|\Gamma_2\right|\left|k\lambda-1\right|\right)\left\|f\left(0\right)\right\|_{\mathbf{Y}}\leq 0,$$

Thus $f\left(0\right)=0$.

Next we replacing $(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k)$ by $(0, ..., 0, 0, ..., 0, z, ..., 0)$ in (5), we have

$$\left\|f\left(\eta z\right)+f\left(-\eta z\right)\right\|\leq\left\|\Gamma_2\left(f\left(\eta z\right)+f\left(-\eta z\right)\right)\right\|_{\mathbf{Y}}$$

equivalent

$$\left\|1-\Gamma_2\right\|\left\|f\left(\eta z\right)+f\left(-\eta z\right)\right\|\leq 0$$

So

$$f\left(-\eta z\right)=-f\left(\eta z\right) \tag{6}$$

or is

$$f\left(-z\right)=-f\left(z\right)$$

It follows that f is an odd mapping.

Next we replacing $(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k)$ by $(x_1, ..., x_k, y_1, ..., y_k, 0, ..., 0)$ in (5), we have

$$\begin{aligned}&\left\|f\left(\lambda y-\alpha z\right)-f\left(-\eta z\right)-\lambda f\left(y\right)-2f\left(\eta z\right)\right\|_{\mathbf{Y}}\\&\leq\left\|\Gamma_2\left(f\left(\lambda y+\eta z\right)-\lambda f\left(y\right)+f\left(\eta z\right)\right)\right\|_{\mathbf{Y}}\end{aligned} \tag{7}$$

for all $y, z \in \mathbf{X}$ From (6) and (7) I have

$$\begin{aligned}&\left\|f\left(\lambda y-\eta z\right)-\lambda f\left(y\right)-f\left(\eta z\right)\right\|_{\mathbf{Y}}\\&\leq\left\|\Gamma_2\left(f\left(\lambda y+\eta z\right)-\lambda f\left(y\right)+f\left(\eta z\right)\right)\right\|_{\mathbf{Y}}\end{aligned} \tag{8}$$

for all $y, z \in \mathbf{X}$.
Next I put $z=-z$ in (8) we have

$$\left\|f\left(\lambda y+\eta z\right)-\lambda f\left(y\right)-f\left(\eta z\right)\right\|_{\mathbf{Y}}\leq 0 \tag{9}$$

Thus

$$f\left(\lambda y+\eta z\right)-\lambda f\left(y\right)-f\left(\eta z\right)=0$$

for all $y, z \in \mathbf{X}$. and so

$$f\left(y+z\right)-f\left(y\right)-f\left(z\right)=0$$

for all $y, z \in \mathbf{X}$. Hence f is additive as we expected. □

Corollary 1. *Suppose that $f : \mathbf{X} \to \mathbf{Y}$ be a mapping satisfying*

$$\begin{aligned}
&\left\| f\left(\sum_{i=1}^{k}(x_i) + \lambda \sum_{i=1}^{k} y_i + \eta \sum_{i=1}^{k} z_i\right) - f\left(\sum_{i=1}^{k} x_i - \eta \sum_{i=1}^{k} z_i\right) - \lambda \sum_{i=1}^{k} f\left(y_i\right) - 2\sum_{i=1}^{k} f\left(\eta z_i\right)\right\|_{\mathbf{Y}} \\
&= \left\| \Gamma_1\left(f\left(\sum_{i=1}^{k} x_i + \eta \sum_{i=1}^{k} z_i\right) - \sum_{i=1}^{k} f\left(x_i\right) - \sum_{i=1}^{k} f\left(\eta z_i\right)\right)\right\|_{\mathbf{Y}} \\
&+ \left\| \Gamma_2\left(f\left(\sum_{i=1}^{k} x_i + \lambda \sum_{i=1}^{k} y_i - \eta \sum_{i=1}^{k} z_i\right) - \sum_{i=1}^{k} f\left(x_i\right) - \lambda \sum_{i=1}^{k} f\left(y_i\right) + \sum_{i=1}^{k} f\left(\eta z_i\right)\right)\right\|_{\mathbf{Y}}
\end{aligned} \tag{10}$$

for all $x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k \in \mathbf{X}$. ,then f is additive.

3.2. Constructing a solution for the function inequality 1

Theorem 2. *Suppose that $f : \mathbf{X} \to \mathbf{Y}$ be a mapping. Let a function $\varphi : \mathbf{X}^{\mathbf{3k}} \to \left[0, \infty\right)$, $\varphi\left(0, ..., 0, 0, ..., 0, 0...., 0\right) = 0$ such that*

$$\begin{aligned}
&\left\| f\left(\sum_{i=1}^{k}(x_i) + \lambda \sum_{i=1}^{k} y_i + \eta \sum_{i=1}^{k} z_i\right) - f\left(\sum_{i=1}^{k} x_i - \eta \sum_{i=1}^{k} z_i\right) - \lambda \sum_{i=1}^{k} f\left(y_i\right) - 2\sum_{i=1}^{k} f\left(\eta z_i\right)\right\|_{\mathbf{Y}} \\
&\leq \left\| \Gamma_1\left(f\left(\sum_{i=1}^{k} x_i + \eta \sum_{i=1}^{k} z_i\right) - \sum_{i=1}^{k} f\left(x_i\right) - \sum_{i=1}^{k} f\left(\eta z_i\right)\right)\right\|_{\mathbf{Y}} \\
&+ \left\| \Gamma_2\left(f\left(\sum_{i=1}^{k} x_i + \lambda \sum_{i=1}^{k} y_i - \eta \sum_{i=1}^{k} z_i\right) - \sum_{i=1}^{k} f\left(x_i\right) - \lambda \sum_{i=1}^{k} f\left(y_i\right) + \sum_{i=1}^{k} f\left(\eta z_i\right)\right)\right\|_{\mathbf{Y}} \\
&+ \varphi(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k)
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&\lim_{j\to\infty} \frac{1}{|1 + k\lambda|^j} \times \\
&\varphi\left((1 + k\lambda)^j x_1, ..., (1 + k\lambda)^j x_k, (1 + k\lambda)^j y_1, ..., (1 + k\lambda)^j y_k, (1 + k\lambda)^j z_1, ..., (1 + k\lambda)^j z_k\right) \\
&= 0
\end{aligned} \tag{12}$$

for all $(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k \in \mathbf{X}$. Then there exisists a unique additive mapping

$$H : \mathbf{X} \to \mathbf{Y}$$

$$\left\| f\left(x\right) - H\left(x\right)\right\|_{\mathbf{Y}} \leq \widetilde{\varphi}\left(x, ..., 0, x, ..., x, 0, ..., 0\right) \tag{13}$$

for all $x \in \mathbf{X}$. Where

$$\begin{aligned}
&\widetilde{\varphi}(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k) \\
&= \frac{1}{|1 + k\lambda|(1 - |\Gamma_1|)} \sum_{j=0}^{\infty} \frac{1}{|1 + k\lambda|^j} \times \\
&\varphi\left((1 + k\lambda)^j x_1, ..., (1 + k\lambda)^j x_k, (1 + k\lambda)^j y_1, ..., (1 + k\lambda)^j y_k, (1 + k\lambda)^j z_1, ..., (1 + k\lambda)^j z_k\right) \\
&< \infty.
\end{aligned} \tag{14}$$

for all $x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k \in \mathbf{X}$

Proof. We replacing $(x_1,...,x_k,y_1,...,y_k,z_1,...,z_k)$ by $(0,...,0,0,...,0,0,...,0)$ in (11), we have

$$\Big(k\Big|\lambda+2\Big|-\Big|(2k-1)\Big|\Big|\Gamma_1\Big|-\Big|\Gamma_2\Big|\Big|k\lambda-1\Big|\Big)\Big\|f\Big(0\Big)\Big\|_{\mathbf{Y}}\leq 0.$$

Thus $f\Big(0\Big)=0$.
Next we replacing $(x_1,...,x_k,y_1,...,y_k,z_1,...,z_k)$ by $(x,...,0,x,...,x,0,...,0)$ in (11), we have

$$\Big\|f\Big((1+k\lambda)x\Big)-(1+k\lambda)f\Big(x\Big)\Big\|_{\mathbf{Y}}\leq\Big|\Gamma_2\Big|\Big\|f\Big((1+k\lambda)x\Big)-(1+k\lambda)f\Big(x\Big)\Big\|_{\mathbf{Y}} + \varphi\Big(x,...,0,x,...,x,0,...,0\Big) \tag{15}$$

for all $x\in\mathbf{X}$. Thus

$$\Bigg\|f\Big(x\Big)-\frac{f\Big((1+k\lambda)x\Big)}{(1+k\lambda)}\Bigg\|_{\mathbf{Y}}\leq\frac{1}{|1+k\lambda|}\cdot\frac{1}{1-\Big|\Gamma_2\Big|}\varphi\Big(x,...,0,x,...,x,0,...,0\Big) \tag{16}$$

for all $x\in\mathbf{X}$. Hence one may have the following formula for positive integer m,l with $m>l$,

$$\Bigg\|\frac{1}{|1+k\lambda|^{l}}f\Big((1+k\lambda)^{l}x\Big)-\frac{1}{|1+k\lambda|^{m}}f\Big((1+k\lambda)^{m}x\Big)\Bigg\|_{\mathbf{Y}} \leq\sum_{j=l}^{m-1}\frac{1}{|1+k\lambda|^{j}}\varphi\Big((1+k\lambda)^{j}x,...,0,(1+k\lambda)^{j}x,...,(1+k\lambda)^{j}x,0...,0\Big) \tag{17}$$

for all $x\in\mathbf{X}$ It folows from (12) that the sequence $\left\{\frac{f\left((1+k\lambda)^n x\right)}{(1+k\lambda)^n}\right\}$ is Cauchy sequence. Since $\mathbf{Y}$ is complete, we conlude that $\left\{\frac{f\left((1+k\lambda)^n x\right)}{(1+k\lambda)^n}\right\}$ is convergent. So one may define the mapping $H:\mathbf{X}\to\mathbf{Y}$ by

$$H(x)=\lim_{n\to\infty}\frac{f\Big((1+k\lambda)^{n}x\Big)}{(1+k\lambda)^{n}},\forall x\in\mathbf{X}. \tag{18}$$

By taking $m=0$ and letting $l\to\infty$ in (17), we get (13)

$$\begin{aligned}
&\left\|H\left(\sum_{i=1}^{k}x_i+\lambda\sum_{i=1}^{k}y_i+\eta\sum_{i=1}^{k}z_i\right)+H\left(\sum_{i=1}^{k}x_i-\eta\sum_{i=1}^{k}z_i\right)-\lambda\sum_{i=1}^{k}H\left(y_i\right)-2\sum_{i=1}^{k}H\left(\eta z_i\right)\right\|_{\mathbf{Y}}\\
&=\lim_{n\to\infty}\frac{1}{|1+k\lambda|^n}\left\|f\left((1+k\lambda)^n\left(\sum_{i=1}^{k}x_i+\lambda\sum_{i=1}^{k}y_i+\eta\sum_{i=1}^{k}z_i\right)\right.\right.\\
&\left.\left.+f\left((1+k\lambda)^n\left(\sum_{i=1}^{k}x_i-\eta\sum_{i=1}^{k}z_i\right)-\lambda\sum_{i=1}^{k}f\left((1+k\lambda)^n y_i\right)-2\sum_{i=1}^{k}f\left((1+k\lambda)^n z_i\right)\right)\right\|_{\mathbf{Y}}\\
&\le\lim_{n\to\infty}\frac{1}{|1+k\lambda|^n}\left\|\Gamma_1\left(f\left((1+k\lambda)^n\left(\sum_{i=1}^{k}x_i-\eta\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}f\left((1+k\lambda)^n x_i\right)\right.\right.\right.\\
&\left.\left.\left.-\sum_{i=1}^{k}f\left((1+k\lambda)^n\eta z_i\right)\right\|_{\mathbf{Y}}+\lim_{n\to\infty}\frac{1}{|1+k\lambda|^n}\right\|\Gamma_2\left(f\left((1+k\lambda)^n\left(\sum_{i=1}^{k}x_i+\lambda\sum_{i=1}^{k}y_i-\eta\sum_{i=1}^{k}z_i\right)\right.\right.\\
&\left.\left.-f\left((1+k\lambda)^n\sum_{i=1}^{k}x_i\right)-\lambda\sum_{i=1}^{k}f\left((1+k\lambda)^n y_i\right)+\sum_{i=1}^{k}f\left((1+\lambda)^n\eta z_i\right)\right)\right\|_{\mathbf{Y}}\\
&\lim_{n\to\infty}\frac{1}{|1+k\lambda|^j}\\
&\varphi\left((1+k\lambda)^j x_1,\ldots,(1+k\lambda)^j x_k,(1+k\lambda)^j y_1,\ldots,(1+k\lambda)^j y_k,(1+k\lambda)^j z_1,\ldots,(1+k\lambda)^j z_k\right)\\
&=\left\|\Gamma_1\left(H\left(\sum_{i=1}^{k}x_i+\eta\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}f\left(x_i\right)-\sum_{i=1}^{k}H\left(\eta z_i\right)\right)\right\|_{\mathbf{Y}}\\
&+\left\|\Gamma_2\left(H\left(\sum_{i=1}^{k}x_i+\lambda\sum_{i=1}^{k}y_i-\eta\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}H\left(x_i\right)-\lambda\sum_{i=1}^{k}H\left(y_i\right)+\sum_{i=1}^{k}H\left(\eta z_i\right)\right)\right\|_{\mathbf{Y}}
\end{aligned}\tag{19}$$

for all $x\in\mathbf{X}$. One can see that that H satisfies the inequalitty (5) and so it is additive by Lemma 1.

Now, I show that the uniqueness of H. Let $T:\mathbf{X}\to\mathbf{Y}$ be another additive mapping satisfying (6) then one has

$$\begin{aligned}
\left\|H\left(x\right)-T\left(x\right)\right\|_{\mathbf{Y}}&=\left\|\frac{1}{(1+k\lambda)^n}H\left((1+k\lambda)^n x\right)-\frac{1}{(1+k\lambda)^n}T\left((1+k\lambda)^n nx\right)\right\|_{\mathbf{Y}}\\
&\le\frac{1}{|1+k\lambda|^n}\left(\left\|H\left((1+k\lambda)^n x\right)-f\left((1+k\lambda)^n x\right)\right\|_{\mathbf{Y}}\right.\\
&\left.+\left\|f\left((1+k\lambda)^n x\right)-T\left((1+k\lambda)^n x\right)\right\|_{\mathbf{Y}}\right)\\
&\le 2\frac{1}{|1+k\lambda|^n}\widetilde{\varphi}\left((1+k\lambda)^n x,\ldots,0,(1+k\lambda)^n x,\ldots,(1+k\lambda)^n x,0\ldots,0\right)
\end{aligned}\tag{20}$$

which tends to zero as $n\to\infty$ for all $x\in\mathbf{X}$. So we can couclude that $H(x)=T(x)$ for all $x\in\mathbf{X}$. □

Corollary 3. *Suppose that $q>1$, θ be nonnegative real number and $f:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfying*

$$\begin{aligned}
&\left\|f\left(\sum_{i=1}^{k}(x_i)+\lambda\sum_{i=1}^{k}y_i+\eta\sum_{i=1}^{k}z_i\right)-f\left(\sum_{i=1}^{k}x_i-\eta\sum_{i=1}^{k}z_i\right)-\lambda\sum_{i=1}^{k}f\left(y_i\right)-2\sum_{i=1}^{k}f\left(\eta z_i\right)\right\|_{\mathbf{Y}}\\
&\le\left\|\Gamma_1\left(f\left(\sum_{i=1}^{k}x_i+\eta\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}f\left(x_i\right)-\sum_{i=1}^{k}f\left(\eta z_i\right)\right)\right\|_{\mathbf{Y}}\\
&+\left\|\Gamma_2\left(f\left(\sum_{i=1}^{k}x_i+\lambda\sum_{i=1}^{k}y_i-\eta\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}f\left(x_i\right)-\lambda\sum_{i=1}^{k}f\left(y_i\right)+\sum_{i=1}^{k}f\left(\eta z_i\right)\right)\right\|_{\mathbf{Y}}
\end{aligned}\tag{21}$$

for all $x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k \in \mathbf{X}$ *with* $|1 + k\lambda| > 1$.
Then there exisists a unique additive mapping $H : \mathbf{X} \to \mathbf{Y}$

$$\left\|f\left(x\right) - H\left(x\right)\right\|_{\mathbf{Y}} \le \frac{2k\theta}{|1+k\lambda| - |1+k\beta|^q} \times \frac{1}{1-|\Gamma_2|}\left\|x\right\|_{\mathbf{X}}^r$$

for all $x \in \mathbf{X}$.

Theorem 4. *Suppose that* $f : \mathbf{X} \to \mathbf{Y}$ *be a mapping. Let a function* $\varphi : \mathbf{X}^{3k} \to \left[0, \infty\right)$, $\varphi\left(0, ..., 0, 0, ..., 0, 0, ..., 0\right) = 0$ *such that*

$$\begin{aligned}
&\left\|f\left(\sum_{i=1}^{k}(x_i) + \lambda\sum_{i=1}^{k}y_i + \eta\sum_{i=1}^{k}z_i\right) - f\left(\sum_{i=1}^{k}x_i - \eta\sum_{i=1}^{k}z_i\right) - \lambda\sum_{i=1}^{k}f\left(y_i\right) - 2\sum_{i=1}^{k}f\left(\eta z_i\right)\right\|_{\mathbf{Y}} \\
&\le \left\|\Gamma_1\left(f\left(\sum_{i=1}^{k}x_i + \eta\sum_{i=1}^{k}z_i\right) - \sum_{i=1}^{k}f\left(x_i\right) - \sum_{i=1}^{k}f\left(\eta z_i\right)\right)\right\|_{\mathbf{Y}} \\
&+ \left\|\Gamma_2\left(f\left(\sum_{i=1}^{k}x_i + \lambda\sum_{i=1}^{k}y_i - \eta\sum_{i=1}^{k}z_i\right) - \sum_{i=1}^{k}f\left(x_i\right) - \lambda\sum_{i=1}^{k}f\left(y_i\right) + \sum_{i=1}^{k}f\left(\eta z_i\right)\right)\right\|_{\mathbf{Y}} \\
&+ \varphi(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k) \qquad (22)
\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{\varphi}(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k) = \\
&\sum_{j=1}^{\infty}|1+k\lambda|^j\varphi\Big(\frac{x_1}{(1+k\lambda)^j}, ..., \frac{x_k}{(1+k\lambda)^j}, \frac{y_1}{(1+k\lambda)^j}, ..., \frac{y_k}{(1+k\lambda)^j}, \frac{z_1}{(1+k\lambda)^j}, ..., \frac{z_k}{(1+k\lambda)^j}\Big) \\
&< \infty \qquad (23)
\end{aligned}$$

for all $(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k \in \mathbf{X}$. *. Then there exisists a unique additive mapping*

$$H : \mathbf{X} \to \mathbf{Y}$$

$$\left\|f\left(x\right) - H\left(x\right)\right\|_{\mathbf{Y}} \le \frac{1}{1-|\Gamma_1|}\widetilde{\varphi}\Big(\frac{x}{(1+k\lambda)^j}, ..., 0, \frac{x}{(1+k\lambda)^j}, ..., \frac{x}{(1+k\lambda)^j}, 0, .., 0\Big) \qquad (24)$$

for all $x \in \mathbf{X}$.

Proof. The proof is similar to the proof of Theorem 7. □

Corollary 5. *Suppose that* $q > 1$, θ *be nonnegative real number and* $f : \mathbf{X} \to \mathbf{Y}$ *be a mapping satisfying*

$$\begin{aligned}
&\left\|f\left(\sum_{i=1}^{k}(x_i) + \lambda\sum_{i=1}^{k}y_i + \eta\sum_{i=1}^{k}z_i\right) - f\left(\sum_{i=1}^{k}x_i - \eta\sum_{i=1}^{k}z_i\right) - \lambda\sum_{i=1}^{k}f\left(y_i\right) - 2\sum_{i=1}^{k}f\left(\eta z_i\right)\right\|_{\mathbf{Y}} \\
&\le \left\|\Gamma_1\left(f\left(\sum_{i=1}^{k}x_i + \eta\sum_{i=1}^{k}z_i\right) - \sum_{i=1}^{k}f\left(x_i\right) - \sum_{i=1}^{k}f\left(\eta z_i\right)\right)\right\|_{\mathbf{Y}} \\
&+ \left\|\Gamma_2\left(f\left(\sum_{i=1}^{k}x_i + \lambda\sum_{i=1}^{k}y_i - \eta\sum_{i=1}^{k}z_i\right) - \sum_{i=1}^{k}f\left(x_i\right) - \lambda\sum_{i=1}^{k}f\left(y_i\right) + \sum_{i=1}^{k}f\left(\eta z_i\right)\right)\right\|_{\mathbf{Y}} \qquad (25)
\end{aligned}$$

for all $x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k \in \mathbf{X}$ *with* $\left|1+k\lambda\right| < 1$.
Then there exisists a unique additive mapping $H : \mathbf{X} \to \mathbf{Y}$

$$\left\|f\left(x\right) - H\left(x\right)\right\|_{\mathbf{Y}} \leq \frac{2k\theta}{|1+k\lambda|^q - |1+k\lambda|} \times \frac{1}{1-|\Gamma_2|}\left\|x\right\|_{\mathbf{X}}^r$$

for all $x \in \mathbf{X}$.

4. Stability (Γ_1, Γ_2)-functional inequalities (2) relative to functional equation (4)

In this section, assume that $\mathbf{X}$ is a complex normed vector spaces, $\mathbf{Y}$ is a complex Banach space Γ_1, Γ_2 are the fixed complex numbers with $G(\Gamma_1, \Gamma_2)$ functional inequality and η, λ are the real numbers with $g(\eta, \lambda)$ functional equation. Under this setting, we can show that the mappings satisfying (1) is additive.

Here we assume that:
$$\left|k\right|\left|\Gamma_1\right| + \left|\Gamma_2\right|\left|2k+1\right| < 2\left|2k-1\right|.$$

4.1. Condition for existence of solution of (1)

Lemma 2. *Suppose that* $f : \mathbf{X} \to \mathbf{Y}$ *be a mapping and it satisfies the functional inequality*

$$\begin{aligned}
&\left\|f\left(\sum_{i=1}^k (x_i) + \sum_{i=1}^k y_i + \eta\sum_{i=1}^k z_i\right) + f\left(\sum_{i=1}^k x_i + \sum_{i=1}^k y_i - \eta\sum_{i=1}^k z_i\right) - 2\sum_{i=1}^k f\left(x_i\right) - 2\sum_{i=1}^k f\left(y_i\right)\right\|_{\mathbf{Y}} \\
&\leq \left\|\Gamma_1\left(f\left(\sum_{i=1}^k x_i + \sum_{i=1}^k y_i + \eta\sum_{i=1}^k z_i\right) - f\left(\sum_{i=1}^k x_i + \sum_{i=1}^k y_i\right) - \sum_{i=1}^k f\left(\eta z_i\right)\right)\right\|_{\mathbf{Y}} \\
&+ \left\|\Gamma_2\left(f\left(\sum_{i=1}^k x_i + \sum_{i=1}^k y_i - \eta\sum_{i=1}^k z_i\right) + \sum_{i=1}^k f\left(-x_i\right) + \sum_{i=1}^k f\left(\eta z_i - y_i\right)\right)\right\|_{\mathbf{Y}}
\end{aligned} \tag{26}$$

For all $x_i, y_i, z_j \in \mathbf{X}, i = 1 \to k$ *then* f *is additive..*

Proof. We replacing $(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k)$ by $(0, ..., 0, 0, ..., 0, 0, ..., 0)$ in (26), we have

$$\left(2\left|2k-1\right| - \left|k\right|\left|\Gamma_1\right| - \left|\Gamma_2\right|\left|2k+1\right|\right)\left\|f\left(0\right)\right\|_{\mathbf{Y}} \leq 0,$$

Thus $f\left(0\right) = 0$.

Next we replacing $(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k)$ by $(0, ..., 0, 0, ..., 0, z, ..., 0)$ in (26), we have

$$\left\|f\left(\eta z\right) + f\left(-z\eta\right)\right\| \leq \left\|\Gamma_2\left(f\left(\eta z\right) + f\left(-\eta z\right)\right)\right\|_{\mathbf{Y}}$$

equivalent
$$\left\|1-\Gamma_2\right\|\left\|f\left(\eta z\right) + f\left(-z\eta\right)\right\| \leq 0$$

So
$$f\left(-\eta z\right) = -f\left(\eta z\right) \tag{27}$$

or is
$$f\left(-z\right) = -f\left(z\right)$$

It follows that f is an odd mapping.

Next we replacing $(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k)$ by $(x_1, ..., x_k, y_1, ..., y_k, 0, ..., 0)$

$$f\left(\sum_{i=1}^k (x_i) + \sum_{i=1}^k y_i\right) - \sum_{i=1}^k f(x_i) - \sum_{i=1}^k f(y_i) = 0 \tag{28}$$

Hence f is additive as we expected. □

Corollary 6. *Suppose that $f:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfying*

$$\begin{aligned}&\left\|f\left(\sum_{i=1}^{k}(x_i)+\sum_{i=1}^{k}y_i+\eta\sum_{i=1}^{k}z_i\right)+f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i-\eta\sum_{i=1}^{k}z_i\right)-2\sum_{i=1}^{k}f\Big(x_i\Big)-2\sum_{i=1}^{k}f\Big(y_i\Big)\right\|_{\mathbf{Y}}\\&=\left\|\Gamma_1\left(f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i+\eta\sum_{i=1}^{k}z_i\right)-f\Big(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i\Big)-\sum_{i=1}^{k}f\Big(\eta z_i\Big)\right)\right\|_{\mathbf{Y}}\\&+\left\|\Gamma_2\left(f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i-\eta\sum_{i=1}^{k}z_i\right)+\sum_{i=1}^{k}f\Big(-x_i\Big)+\sum_{i=1}^{k}f\Big(\eta z_i-y_i\Big)\right)\right\|_{\mathbf{Y}}\end{aligned}\tag{29}$$

for all $x_1,...,x_k,y_1,...,y_k,z_1,...,z_k\in\mathbf{X}$,then f is additive.

4.2. Constructing a solution for the function inequality (2)

Theorem 7. *Suppose that $f:\mathbf{X}\to\mathbf{Y}$ be a mapping. Let a function $\varphi:\mathbf{X}^{\mathbf{3k}}\to\big[0,\infty\big)$, $\varphi(0,0,...,0)=0$ such that*

$$\begin{aligned}&\left\|f\left(\sum_{i=1}^{k}(x_i)+\sum_{i=1}^{k}y_i+\eta\sum_{i=1}^{k}z_i\right)+f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i-\eta\sum_{i=1}^{k}z_i\right)-2\sum_{i=1}^{k}f\Big(x_i\Big)-2\sum_{i=1}^{k}f\Big(y_i\Big)\right\|_{\mathbf{Y}}\\&\le\left\|\Gamma_1\left(f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i+\eta\sum_{i=1}^{k}z_i\right)-f\Big(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i\Big)-\sum_{i=1}^{k}f\Big(\eta z_i\Big)\right)\right\|_{\mathbf{Y}}\\&+\left\|\Gamma_2\left(f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i-\eta\sum_{i=1}^{k}z_i\right)+\sum_{i=1}^{k}f\Big(-x_i\Big)+\sum_{i=1}^{k}f\Big(\eta z_i-y_i\Big)\right)\right\|_{\mathbf{Y}}\\&+\varphi(x_1,...,x_k,y_1,...,y_k,z_1,...,z_k)\end{aligned}\tag{30}$$

and

$$\begin{aligned}&\lim_{j\to\infty}\frac{1}{(2k)^j}\varphi\Big((2k)^jx_1,...,(2k)^jx_k,(2k)^jy_1,...,(2k)^jy_k,(2k)^jz_1,...,(2k)^jz_k\Big)\\&=0\end{aligned}\tag{31}$$

for all $(x_1,...,x_k,y_1,...,y_k,z_1,...,z_k\in\mathbf{X}$. Putting

$$\begin{aligned}\widetilde{\varphi}(x)=&\sum_{j=0}^{\infty}\frac{1}{(2k)^{j+1}}\times\frac{1}{2-\Big|\Gamma_2\Big|}\Bigg(\varphi\Big(x,...,x,x,...,x,0,...,0\Big)\\&+\frac{2k\Big|\Gamma_2\Big|}{1-\Big|\Gamma_2\Big|}\varphi\Big(0,...,0,0,...,0,\frac{x}{\eta},...,0\Big)\Bigg)<\infty.\end{aligned}\tag{32}$$

Then there exisists a unique additive mapping

$$H:\mathbf{X}\to\mathbf{Y}$$

$$\Big\|f\Big(x\Big)-H\Big(x\Big)\Big\|_{\mathbf{Y}}\le\widetilde{\varphi}\Big(x\Big)\tag{33}$$

for all $x\in\mathbf{X}$.

Proof. We replacing $\big(x_1,...,x_k,y_1,...,y_k,z_1,...,z_k\big)$ by$\big(0,...,0,0,...,0,0,...,0\big)$ in (26), we have

$$\Big(2\Big|2k-1\Big|-\Big|k\Big|\Big|\Gamma_1\Big|-\Big|\Gamma_2\Big|\Big|2k+1\Big|\Big)\Big\|f\Big(0\Big)\Big\|_{\mathbf{Y}}\le 0,$$

Thus $f\Big(0\Big)=0$.

Next we replacing $(x_1,...,x_k,y_1,...,y_k,z_1,...,z_k)$ by $(0,...,0,0,...,0,z,...,0)$ in (26), we have

$$\left\|f\left(\eta z\right)+f\left(-z\eta\right)\right\|\leq\left\|\Gamma_2\left(f\left(\eta z\right)+f\left(-\eta z\right)\right)\right\|_{\mathbf{Y}}+\varphi\left(0,...,0,0,...,0,z,...,0\right)$$

equivalent

$$\left\|1-\Gamma_2\right\|\left\|f\left(\eta z\right)+f\left(-\eta z\right)\right\|\leq\varphi\left(0,...,0,0,...,0,z,...,0\right), \tag{34}$$

if we replace z by $\frac{z}{\alpha}$ in (36), we have

$$\left\|f\left(z\right)+f\left(-z\right)\right\|\leq\frac{\varphi\left(0,...,0,0,...,0,\frac{z}{\alpha},...,0\right)}{\left\|1-\Gamma_2\right\|}. \tag{35}$$

Next we replacing $(x_1,...,x_k,y_1,...,y_k,z_1,...,z_k)$ by $(x,...,x,x,...,x,0,...,0)$ in (11), we have

$$\begin{aligned}\left\|2f\left(2kx\right)-4kf\left(x\right)\right\|_{\mathbf{Y}}\leq\left|\Gamma_2\right|\left\|f(2kx)+2kf(-x)\right\|_{\mathbf{Y}}\\+\varphi\left(x,...,x,x,...,x,0,...,0\right)\end{aligned} \tag{36}$$

for all $x\in\mathbf{X}$. Thus

$$\begin{aligned}\left\|2f\left(2kx\right)-4kf\left(x\right)\right\|_{\mathbf{Y}}\leq\left|\Gamma_2\right|\left\|f(2kx)-2kf(x)\right\|_{\mathbf{Y}}+2k\left|\Gamma_2\right|\left\|f(x)+f(-x)\right\|_{\mathbf{Y}}\\+\varphi\left(x,...,x,x,...,x,0,...,0\right)\end{aligned} \tag{37}$$

for all $x\in\mathbf{X}$. Therefore

$$\begin{aligned}\left\|f\left(2kx\right)-2kf\left(x\right)\right\|_{\mathbf{Y}}\leq&\frac{2k\left|\Gamma_2\right|}{2-\left|\Gamma_2\right|}\left\|f(x)+f(-x)\right\|_{\mathbf{Y}}\\&+\frac{1}{2-\left|\Gamma_2\right|}\varphi\left(x,...,x,x,...,x,0,...,0\right)\end{aligned} \tag{38}$$

for all $x\in\mathbf{X}$. From (37) and (38) we have

$$\begin{aligned}\left\|f\left(x\right)-\frac{1}{2k}f\left(2kx\right)\right\|_{\mathbf{Y}}\leq\frac{1}{2k}\frac{1}{2-\left|\Gamma_2\right|}\Bigg(&\varphi\left(x,...,x,x,...,x,0,...,0\right)\\&+\frac{2k\left|\Gamma_2\right|}{1-\left|\Gamma_2\right|}\varphi\left(0,...,0,0,...,0,\frac{x}{\eta},...,0\right)\Bigg)\end{aligned} \tag{39}$$

for all $x\in\mathbf{X}$. Hence one may have the following formula for positive integer m, l with $m>l$,

$$\begin{aligned}&\left\|\frac{1}{(2k)^l}f\left((2k)^lx\right)-\frac{1}{(2k)^m}f\left((2k)^mx\right)\right\|_{\mathbf{Y}}\\&\leq\sum_{j=l}^{m-1}\frac{1}{(2k)^{j+1}}\times\frac{1}{|2-\Gamma_2|}\varphi\left((2k)^jx,...,(2k)^jx,(2k)^jx,...,(2k)^jx,0...,0\right)\\&+\sum_{j=l}^{m-1}\frac{1}{(2k)^{j+1}}\times\frac{1}{|2-\Gamma_2|}\times\frac{2k\Gamma_2}{1-|\Gamma_2|}\varphi\left(0,...,0,0,...,0,\frac{(2k)^jx}{\eta},...,0\right)\end{aligned} \tag{40}$$

for all $x \in \mathbf{X}$ It folows from (32) that the sequence $\left\{ \frac{f\left((2k)^n x\right)}{(2k)^n} \right\}$ is Cauchy sequence. Since $\mathbf{Y}$ is complete, we conlude that $\left\{ \frac{f\left((2k)^n x\right)}{(2k)^n} \right\}$ is convergent. So one may define the mapping $H : \mathbf{X} \to \mathbf{Y}$ by

$$H(x) = \lim_{n\to\infty} \frac{f\left((2k)^n x\right)}{(2k)^n}, \forall x \in \mathbf{X}. \tag{41}$$

By taking $m = 0$ and letting $l \to \infty$ in (40), we get (33)

$$\begin{aligned}
&\left\| H\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i + \eta \sum_{i=1}^{k} z_i\right) + H\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} x_i - \eta \sum_{i=1}^{k} z_i\right) - 2\sum_{i=1}^{k} H\left(x_i\right) - 2\sum_{i=1}^{k} H\left(y_i\right)\right\|_{\mathbf{Y}} \\
&= \lim_{n\to\infty} \frac{1}{|2k|^n} \left\| f\left((2k)^n \left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i + \alpha \sum_{i=1}^{k} z_i\right)\right.\right. \\
&\left.+ f\left((2k)^n \left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i - \eta \sum_{i=1}^{k} z_i\right) - 2\sum_{i=1}^{k} f\left((2k)^n x_i\right) - 2\sum_{i=1}^{k} f\left((2k)^n y_i\right)\right\|_{\mathbf{Y}} \\
&\le \lim_{n\to\infty} \frac{1}{|2k|^n} \left\| \Gamma_1 \left(f\left((2k)^n \left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i + \eta \sum_{i=1}^{k} z_i\right) - \sum_{i=1}^{k} f\left((2k)^n \left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i\right)\right)\right.\right. \\
&\left.\left. - \sum_{i=1}^{k} f\left((2k)^n \eta z_i\right)\right\|_{\mathbf{Y}} + \lim_{n\to\infty} \frac{1}{|1+k\beta|^n} \left\| \Gamma_2 \left(f\left((2k)^n \left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i - \eta \sum_{i=1}^{k} z_i\right)\right.\right.\right. \\
&\left.\left.\left. - f\left(-(2k)^n \sum_{i=1}^{k} x_i\right) - \sum_{i=1}^{k} f\left((2k)^n (y_i - \eta z_i)\right)\right)\right\|_{\mathbf{Y}} \\
&+ \lim_{n\to\infty} \frac{1}{|2k|^j} \varphi\left((2k)^j x_1, ..., (2k)^j x_k, (2k)^j y_1, ..., (2k)^j y_k, (2k)^j z_1, ..., (2k)^j z_k\right) \\
&= \left\| \Gamma_1 \left(f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i + \eta \sum_{i=1}^{k} z_i\right) - f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i\right) - \sum_{i=1}^{k} f\left(\eta z_i\right)\right)\right\|_{\mathbf{Y}} \\
&+ \left\| \Gamma_2 \left(f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i - \eta \sum_{i=1}^{k} z_i\right) + \sum_{i=1}^{k} f\left(-x_i\right) + \sum_{i=1}^{k} f\left(\eta z_i - y_i\right)\right)\right\|_{\mathbf{Y}}
\end{aligned} \tag{42}$$

for all $x \in \mathbf{X}$. One can see that that H satisfies the inequalitty (26) and so it is additive by Lemma 2

Now, I show that the uniqueness of H. Let $T : \mathbf{X} \to \mathbf{Y}$ be another additive mapping satisfying (30) then one has

$$\begin{aligned}
&\left\| H\left(x\right) - T\left(x\right)\right\|_{\mathbf{Y}} = \left\| \frac{1}{(2k)^n} H\left((2k)^n x\right) - \frac{1}{(2k)^n} T\left((2k)^n x\right)\right\|_{\mathbf{Y}} \\
&\le \frac{1}{|2k|^n} \left(\left\| H\left((2k)^n x\right) - f\left((2k)^n x\right)\right\|_{\mathbf{Y}} \right. \\
&\left. + \left\| f\left((2k)^n x\right) - T\left((2k)^n x\right)\right\|_{\mathbf{Y}} \right) \\
&\le \frac{1}{(2k)^n} \widetilde{\phi}((2k)^n x) \\
&= \sum_{j=l}^{m-1} \frac{1}{(2k)^{j+1}} \times \frac{1}{|2-\Gamma_2|} \varphi\left((2k)^j x, ..., (2k)^j x, (2k)^j x, ..., (2k)^j x, 0..., 0\right) \\
&+ \sum_{j=l}^{m-1} \frac{1}{(2k)^{j+1}} \times \frac{1}{|2-\Gamma_2|} \times \frac{2k\Gamma_2}{1-|\Gamma_2|} \varphi\left(0, ..., 0, 0, ..., 0, \frac{(2k)^j x}{\alpha}, ..., 0\right)
\end{aligned} \tag{43}$$

which tends to zero as $n \to \infty$ for all $x \in \mathbf{X}$. So we can couclude that $H(x) = T(x)$ for all $x \in \mathbf{X}$. □

Corollary 8. *Suppose that $q > 1$, θ be nonnegative real number and $f : \mathbf{X} \to \mathbf{Y}$ be a mapping satisfying*

$$\begin{aligned}
&\left\| f\left(\sum_{i=1}^{k}(x_i) + \sum_{i=1}^{k} y_i + \eta \sum_{i=1}^{k} z_i\right) + f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i - \eta \sum_{i=1}^{k} z_i\right) - 2\sum_{i=1}^{k} f\left(x_i\right) - 2\sum_{i=1}^{k} f\left(y_i\right)\right\|_{\mathbf{Y}} \\
&\leq \left\| \Gamma_1\left(f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i + \eta \sum_{i=1}^{k} z_i\right) - f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i\right) - \sum_{i=1}^{k} f\left(\eta z_i\right)\right)\right\|_{\mathbf{Y}} \\
&+ \left\| \Gamma_2\left(f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i - \eta \sum_{i=1}^{k} z_i\right) + \sum_{i=1}^{k} f\left(-x_i\right) + \sum_{i=1}^{k} f\left(\eta z_i - y_i\right)\right)\right\|_{\mathbf{Y}} \\
&+ \theta\left(\sum_{i=1}^{k} \|x_i\|^q + \sum_{i=1}^{k} \|y_i\|^q + \sum_{i=1}^{k} \|z_i\|^q\right) \qquad (44)
\end{aligned}$$

for all $x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k \in \mathbf{X}$, then there exisists a unique additive mapping $H : \mathbf{X} \to \mathbf{Y}$

$$\left\| f\left(x\right) - H\left(x\right)\right\|_{\mathbf{Y}} \leq \frac{2k\theta}{(2k - (2k)^q)} \times \frac{1}{(1 - |\Gamma_2|)(2 - |\Gamma_2|)} \left\|x\right\|_{\mathbf{X}}^q$$

for all $x \in \mathbf{X}$.

Theorem 9. *Suppose that $f : \mathbf{X} \to \mathbf{Y}$ be a mapping. Let a function $\varphi : \mathbf{X}^{3\mathbf{k}} \to [0, \infty)$, $\varphi(0, 0, ..., 0) = 0$ such that*

$$\begin{aligned}
&\left\| f\left(\sum_{i=1}^{k}(x_i) + \sum_{i=1}^{k} y_i + \eta \sum_{i=1}^{k} z_i\right) + f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i - \eta \sum_{i=1}^{k} z_i\right) - 2\sum_{i=1}^{k} f\left(x_i\right) - 2\sum_{i=1}^{k} f\left(y_i\right)\right\|_{\mathbf{Y}} \\
&\leq \left\| \Gamma_1\left(f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i + \eta \sum_{i=1}^{k} z_i\right) - f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i\right) - \sum_{i=1}^{k} f\left(\eta z_i\right)\right)\right\|_{\mathbf{Y}} \\
&+ \left\| \Gamma_2\left(f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i - \eta \sum_{i=1}^{k} z_i\right) + \sum_{i=1}^{k} f\left(-x_i\right) + \sum_{i=1}^{k} f\left(\eta z_i - y_i\right)\right)\right\|_{\mathbf{Y}} \\
&+ \varphi(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k) \qquad (45)
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{j\to\infty} (2k)^j \varphi\left(\frac{x_1}{(2k)^j}, ..., \frac{x_k}{(2k)^j}, \frac{y_1}{(2k)^j}, ..., \frac{y_k}{(2k)^j}, \frac{z_1}{(2k)^j}, ..., \frac{z_k}{(2k)^j}\right) \\
&= 0 \qquad (46)
\end{aligned}$$

for all $(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k \in \mathbf{X}$. Putting

$$\begin{aligned}
\widetilde{\varphi}(x) = &\sum_{j=0}^{\infty} \frac{1}{(2k)^j} \times \frac{1}{2 - \left|\Gamma_2\right|} \left(\varphi\left(\frac{x}{(2k)^{j+1}}, ..., \frac{x}{(2k)^{j+1}}, \frac{x}{(2k)^{j+1}}, ..., \frac{x}{(2k)^{j+1}}, 0, ..., 0\right) \right. \\
&\left. + \frac{2k\left|\Gamma_2\right|}{1 - \left|\Gamma_2\right|} \varphi\left(0, ..., 0, 0, ..., 0, \frac{x}{\eta(2k)^{j+1}}, ..., 0\right)\right) < \infty, \qquad (47)
\end{aligned}$$

then there exisists a unique additive mapping

$$H : \mathbf{X} \to \mathbf{Y}$$

$$\left\| f\left(x\right) - H\left(x\right)\right\|_{\mathbf{Y}} \leq \widetilde{\varphi}\left(x\right) \qquad (48)$$

for all $x \in \mathbf{X}$.

The proof is similar to the proof of theorem 2.

Corollary 10. *Suppose that* $q > 1$, θ *be nonnegative real number and* $f : \mathbf{X} \to \mathbf{Y}$ *be a mapping satisfying*

$$\begin{aligned}
&\left\| f\left(\sum_{i=1}^{k}(x_i) + \sum_{i=1}^{k} y_i + \eta \sum_{i=1}^{k} z_i\right) + f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i - \eta \sum_{i=1}^{k} z_i\right) - 2\sum_{i=1}^{k} f\left(x_i\right) - 2\sum_{i=1}^{k} f\left(y_i\right)\right\|_{\mathbf{Y}} \\
&\leq \left\| \Gamma_1\left(f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i + \eta \sum_{i=1}^{k} z_i\right) - f\Big(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i\Big) - \sum_{i=1}^{k} f\left(\eta z_i\right)\right)\right\|_{\mathbf{Y}} \\
&+ \left\| \Gamma_2\left(f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i - \eta \sum_{i=1}^{k} z_i\right) + \sum_{i=1}^{k} f\Big(-x_i\Big) + \sum_{i=1}^{k} f\Big(\eta z_i - y_i\Big)\right)\right\|_{\mathbf{Y}} \\
&+ \theta\left(\sum_{i=1}^{k} \|x_i\|^q + \sum_{i=1}^{k} \|y_i\|^q + \sum_{i=1}^{k} \|z_i\|^q\right) \qquad (49)
\end{aligned}$$

for all $x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k \in \mathbf{X}$, *then there exisists a unique additive mapping* $H : \mathbf{X} \to \mathbf{Y}$

$$\left\| f\Big(x\Big) - H\Big(x\Big)\right\|_{\mathbf{Y}} \leq \frac{(2k)^{1+q}\theta}{((2k)^q - 1)} \times \frac{1}{(1 - |\Gamma_2|)(2 - |\Gamma_2|)} \left\|x\right\|_{\mathbf{X}}^q$$

for all $x \in \mathbf{X}$.

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