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Limit cycles obtained by perturbing a degenerate center

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Abstract: This paper deals with the maximum number of limit cycles bifurcating from the degenerate centre

$$\dot{x} = -y(3x^2 + y^2),\ \dot{y} = x(x^2 - y^2),$$

when we perturb it inside a class of all homogeneous polynomial differential systems of degree 5. Using averaging theory of second order, we show that, at most, five limit cycles are produced from the periodic orbits surrounding the degenerate centre under quintic perturbation. In addition, we provide six examples that give rise to exactly $5, 4, 3, 2, 1$ and 0 limit cycles.

Keywords: Limit cycles; Averaging theory; Polynomial differential systems; Degenerate center.

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1. Introduction

The limit cycle of a planar differential system was defined for the first time by Henri Poincare [1] as a periodic orbit of the differential system isolated in the set of all periodic orbits. The main interest in studying the limit cycles of the planar polynomial differential systems is due to the 16-th Hilbert problem, which proposed to find a uniform upper bound for the number of limit cycles of all polynomial differential systems of a given degree see, for instance ([2], [3]). It has yet to prove the existence of such a uniform upper bound, and the problem remains unsolved even for planar polynomial systems of degree 2. An easier problem than Hilbert's 16th problem is the study of the number of limit cycles that can bifurcate from the periodic orbits surrounding a centre of a polynomial differential system, see, for instance, [4]. A centre of an analytic differential system in $\mathbb{R}^2$ is called linear, nilpotent or degenerate if, after an affine change of variables and a rescaling, it can be written as $\dot{x} = -y + p(x,y),\ \dot{y} = x + q(x,y),\ \dot{x} = y + p(x,y),\ \dot{y} = q(x,y)$ or $\dot{x} = p(x,y),\ \dot{y} = q(x,y)$, respectively, where p and q are polynomials without constant and linear terms.
There are many known methods that the researchers have followed to study the limit cycles that can bifurcate from the periodic orbits of a centre, such as the Poincaré return map (see [5]), Poincaré Pontryagin-Melnikov-functions, (see [6],[7], [8]), inverse integrating factor [9] and the averaging theory (see [10],[11]). The last method has played a crucial role in the study of limit cycles of differential systems.

In the last decades, several works about the bifurcation of limit cycles in planar differential systems having a linear centre have been published see, for instance ([12], [13], [14]) and references cited therein. For the degenerate centre, the results related to it are very slight due to the extreme difficulty of the computations; for instance, we can mention ([15], [16]).
Our main objective will be to solve the problem of limit cycles that can bifurcate from the periodic solutions of a degenerate centre of the form

$$\begin{cases} \dot{x} = -y(3x^2 + y^2) \\ \dot{y} = x(x^2 - y^2), \end{cases} \tag{1}$$

when it is perturbed inside the class of all polynomial differential systems of degree 5.
Note that system (1) has a global center at the origin with non rational-first integral $H(x,y) =$

$(x^2+y^2)\exp(\frac{-2x^2}{x^2+y^2})$.

In [17], using averaging theory of first order the authors proved that there are at most $[\frac{n-1}{2}]$ limit cycles bifurcating from periodic orbits of the degenerate center (1) perturbed inside the class of all polynomial differential systems of degree n.

Using the averaging theory of second order, the authors of [18] studied the perturbation of the degenerate center (1) inside the polynomial differential systems of degree 3 and obtained that the number of limit cycles that bifurcate from the period orbits is 2. Moreover in [19], The authors improved this result and they proved that there are at most 3 limit cycles bifurcating from the global center of (1) by using other version of second order averaging method.

Therefore, inspired by the above mentioned papers, we decided to study the perturbation of the degenerate center $\dot{x} = -y(3x^2+y^2)$, $\dot{y} = x(x^2-y^2)$, inside the class of all fifth polynomial differential systems. More precisely, In this paper we provide lower bounds for the maximum number of limit cycles of the system

$$\begin{cases} \dot{x} &= -y(3x^2+y^2)+\epsilon \sum\limits_{0\le i+j\le 5} a_{i,j}x^iy^j+\epsilon^2 \sum\limits_{0\le i+j\le 5} b_{i,j}x^iy^j \\ \dot{y} &= x(x^2-y^2)+\epsilon \sum\limits_{0\le i+j\le 5} d_{i,j}x^iy^j+\epsilon^2 \sum\limits_{0\le i+j\le 5} e_{i,j}x^iy^j, \end{cases} \tag{2}$$

using the averaging method of second order.

The next theorem is the main result of this paper.

Theorem 1. *For $\epsilon \neq 0$ sufficiently small, the maximum number of the limit cycles of the perturbed system* (2) *bifurcating from the periodic orbits of the global center* (1) *using averaging theory of second order is at most five.*

Theorem 1 is proved in section 3. Its proof is based on the averaging theory of second order, see section 2. Six examples are given in the last section to illustrate the established results.

Remark 1. Note that Theorem 1 improves the result of Theorem 3 of [19] by providing 2 more limit cycles in the perturbation of system (1) with homogeneous polynomial differential systems of degree 5. All our calculations have been verified by the help of the algebric manipulator Maple.

2. Preliminary results

Firstly, we present some preliminariers about the averaging theory of second order that we followed to prove our main results. In short, the method gives a quantitative relation between the solutions of a non-autonomous periodic system and the solutions of its averaged system, which is autonomous. It is summarized as follows.

2.1. Averaging theory of second order

Consider the differential equation:

$$\frac{dy}{d\theta} = \sum_{k=0}^{k=2} \epsilon^k F_k(\theta, y) = F_0(\theta, y) + \epsilon F_1(\theta, y) + \epsilon^2 F_2(\theta, y), \tag{3}$$

where $y \in \mathbb{R}$, $\theta \in S^1$ and $\epsilon \neq 0$ a positive real number sufficiently small.
We assume that:

i) The functions $F_k(\theta, y)$ where $k \in \{0,1,2\}$ are $2\pi-$ periodic functions in variable θ.
ii) The solution of the unperturbed system of (3) $y_u(\theta, y_0)$ is $2\pi-$ periodic function in variable θ satisfying $y_0(0, y_0) = y_0 \in \mathcal{J} \subset \mathbb{R}$ ($\mathcal{J}$ is a real open interval).

iii) Consider the following variational equation:

$$\frac{\partial v}{\partial \theta} = \frac{\partial F_0}{\partial y}\left(\theta, y_u\left(\theta, y_0\right)\right).v. \tag{4}$$

The solution $v = v\left(\theta, y_0\right)$ of equation (4) satifyies $v\left(0, \theta\right) = 1$.

We mention that $y_\epsilon\left(\theta, y_0\right)$ the solution of equation (3) with $y_\epsilon\left(0, y_0\right) = y_0$.
Now, the functions $F_{j0}\left(y_0\right)$ with $j \in \{1;2\}$ are defined by:

$$F_{10}\left(y_0\right) = \int_0^{2\pi} \frac{F_1\left(\theta, y_u\left(\theta, y_0\right)\right)}{v\left(\theta, y_0\right)} d\theta, \tag{5}$$

$$F_{20}\left(y_0\right) = \int_0^{2\pi} \left(\frac{F_2(\theta, y_u(\theta, y_0))}{v(\theta, y_0)} + \frac{dF_1}{dy}\left(y\left(\theta, y_0\right)\right).v_1\left(\theta, y\left(\theta, y_0\right)\right)\right) d\theta$$
$$+ \frac{1}{2}\int_0^{2\pi} \frac{\partial^2 F_0}{\partial y^2}(\theta, y_u(\theta, y_0)).(v_1(\theta, y_u(\theta, y_0)))^2 d\theta, \tag{6}$$

where $v_1\left(\theta, y_0\right) = \int\limits_0^{\theta} \frac{F_1\left(\phi, y_u\left(\phi, y_0\right)\right)}{v\left(\phi, y_0\right)} d\theta.$

Theorem 2. *Assume that the hypothesis i)and ii) are verified; then, if $F_{10}\left(y_0\right)$ is identically zero in $\mathcal{J}$ and $F_{20}\left(y_0\right)$ is not identically zero in $\mathcal{J}$. Then for each simple zero y^* of $F_{20}\left(y_0\right)$, there exists a peridic solution $y_\epsilon\left(\theta, y_0\right)$ of (3) such that $y_\epsilon\left(\theta, y_0\right) \to y^*$ when ϵ tends to zero.*

2.2. Descartes Theorem

We recall the Descartes Theorem about the number of zeros of certain real polynomial (for a proof see for instance [20]).

Theorem 3. *Consider the polynomial $p\left(y\right) = a_{i_1}y^{a_{i_1}} + a_{i_2}y^{a_{i_2}} + \cdots + a_{i_n}y^{a_{i_n}}$ with $0 \le i_1 < i_2 < \cdots < i_n$ and the real constant $a_{i_k} \neq 0 \;\; \forall k \in \{1, 2, \ldots, n\}$. When $a_{i_k}a_{i_{k+1}} < 0$, we say that a_{i_k} and $a_{i_{k+1}}$ have a variation of sign. If the number of the variation is $m \in \mathbb{N}$, then $p\left(y\right)$ has at most m positive real roots. Moreover, it is always possible to choose the coeffecient of $p\left(y\right)$ in such a way that $p\left(y\right)$ has exactly $n-1$ positive real roots.*

3. Proof of Theorem 1

Before we start the proof, we define the following numbers

$$\begin{aligned}
J_1 &= \int_0^{2\pi} e^{2\sin(\theta)^2} d\theta = 21.62373221\ldots, \\
J_2 &= \int_0^{2\pi} \cos\left(2\,\theta\right) e^{2\sin(\theta)^2} d\theta = -9.652617083\ldots, & J_6 &= \int_0^{2\pi} \cos\left(4\,\theta\right) e^{-2\sin(\theta)^2} d\theta = 0.3137745891\ldots, \\
J_3 &= \int_0^{2\pi} \cos\left(4\,\theta\right) e^{2\sin(\theta)^2} d\theta = 2.318498042\ldots, & J_7 &= \int_0^{2\pi} \cos\left(6\,\theta\right) e^{-2\sin(\theta)^2} d\theta = 0.05124130979\ldots, \\
J_4 &= \int_0^{2\pi} e^{-2\sin(\theta)^2} d\theta = 2.926453923\ldots, & J_8 &= \int_0^{2\pi} \cos\left(8\,\theta\right) e^{-2\sin(\theta)^2} d\theta = 0.006326729219\ldots. \\
J_5 &= \int_0^{2\pi} \cos\left(2\,\theta\right) e^{-2\sin(\theta)^2} d\theta = 1.306339667\ldots,
\end{aligned}$$

We consider the system

$$\begin{aligned} \dot{x} &= -y(3x^2 + y^2), \\ \dot{y} &= x(x^2 - y^2). \end{aligned} \tag{7}$$

By using the polar coordinates (r, θ) where $x = r\cos\theta$ and $y = r\sin\theta$, the system (7) can be transformed into the following form

$$\frac{dr}{d\theta} = -r\sin\left(2\theta\right). \tag{8}$$

The solution $r_u(\theta, r_0) = r_0 e^{-\sin^2\theta}$ of (8) is $2\pi-$periodic in variable θ satisfying that $r_u(0, r_0) = r_0$.
Let the following variational equation

$$\frac{\partial v}{\partial \theta} = \frac{\partial F_0}{\partial r}(\theta, r_u(\theta, r_0)).v,$$

its solution is $v(\theta, r_0) = \exp(\sin^2\theta)$ satidfying $v(0, r_0) = 1$.
Now, we shall perturb the degenerate center (1) inside the class of all fifth degree polynomial system as in (2). With the aim of applying the averaging method, the system (2) should be transformed to the canonical form (3). After taking θ as an independent variable and the alteration of polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, the system (2) could be written in the following form

$$\frac{dr}{d\theta} = F_0(\theta, y) + \epsilon F_1(\theta, y) + \epsilon^2 F_2(\theta, y), \tag{9}$$

with

$$F_0(\theta, r) = -2r\sin(\theta)\cos(\theta), \tag{10}$$

$$F_1(\theta, r) = \frac{1}{r^2} f_2(\theta) + \frac{1}{r} f_1(\theta) + f_0(\theta) + r\tilde{f}_1(\theta) + r^2\tilde{f}_2(\theta) + r^3\tilde{f}_3(\theta), \tag{11}$$

$$\begin{aligned} F_2(\theta, r) = {} & \frac{1}{r^5} g_5(\theta) + \frac{1}{r^4} g_4(\theta) + \frac{1}{r^3} g_3(\theta) + \frac{1}{r^2} g_2(\theta) + \frac{1}{r} g_1(\theta) + g_0(\theta) \\ & + r\tilde{g}_1(\theta) + r^2\tilde{g}_2(\theta) + r^3\tilde{g}_3(\theta) + r^4\tilde{g}_4(\theta) + r^5\tilde{g}_5(\theta), \end{aligned} \tag{12}$$

where

$$\begin{aligned}
f_2(\theta) = {} & \frac{1}{2} d_{0,0}\sin(3\theta) + \frac{3}{2} d_{0,0}\sin(\theta) + \frac{1}{2} a_{0,0}\cos(\theta) + \frac{1}{2} a_{0,0}\cos(3\theta), \\
f_1(\theta) = {} & \frac{1}{4}(d_{1,0} + a_{0,1})\sin(4\theta) + \frac{1}{4}(a_{1,0} - d_{0,1})\cos(4\theta) + \frac{1}{2}(a_{1,0} - d_{0,1})\cos(2\theta) \\
& + d_{1,0}\sin(2\theta) + \frac{1}{4}(a_{1,0} + 3d_{0,1}), \\
f_0(\theta) = {} & \frac{1}{8}(-d_{0,2} + a_{1,1} + d_{2,0})\sin(5\theta) + \frac{1}{8}(-d_{0,2} + a_{1,1} + 5d_{2,0})\sin(3\theta) + \frac{1}{2}(2d_{0,2} \\
& + d_{2,0})\sin(\theta) + \frac{1}{8}(a_{2,0} - d_{1,1} - a_{0,2})\cos(5\theta) + \frac{1}{8}(-3d_{1,1} + 3a_{2,0} + a_{0,2})\cos(3\theta) \\
& + \frac{1}{2}(d_{1,1} + a_{2,0})\cos(\theta), \\
\tilde{f}_1(\theta) = {} & \frac{1}{16}(-a_{0,3} + a_{2,1} + d_{3,0} - d_{1,2})\sin(6\theta) + \frac{1}{8}(a_{0,3} - d_{1,2} + 3d_{3,0} + a_{2,1})\sin(4\theta) \\
& + \frac{1}{16}(9d_{3,0} - a_{0,3} + 7d_{1,2} + a_{2,1})\sin(2\theta) + \frac{1}{16}(a_{3,0} - a_{1,2} - d_{2,1} + d_{0,3})\cos(6\theta) \\
& + \frac{1}{4}(a_{3,0} - d_{2,1})\cos(4\theta) + \frac{1}{16}(a_{1,2} + d_{2,1} + 7a_{3,0} - 9d_{0,3})\cos(2\theta) \\
& + \frac{1}{4}(2d_{0,3} + a_{3,0} + d_{2,1}), \\
\tilde{f}_2(\theta) = {} & \frac{1}{32}(d_{4,0} - d_{2,2} - a_{1,3} + d_{0,4} + a_{3,1})\sin(7\theta) + \frac{1}{32}(a_{1,3} + 7d_{4,0} - d_{0,4} + 3a_{3,1} \\
& - 3d_{2,2})\sin(5\theta) + \frac{1}{32}(3a_{3,1} - 9d_{0,4} + 5d_{2,2} + a_{1,3} + 15d_{4,0})\sin(3\theta) + \frac{1}{32}(9d_{4,0} \\
& + 7d_{2,2} + a_{3,1} - a_{1,3} + 25d_{0,4})\sin(\theta) + \frac{1}{32}(a_{0,4} - d_{3,1} + d_{1,3} + a_{4,0} - a_{2,2})\cos(7\theta) \\
& + \frac{1}{32}(-5d_{3,1} + 5a_{4,0} - a_{2,2} - 3a_{0,4} + d_{1,3})\cos(5\theta) + \frac{1}{32}(-3d_{3,1} + a_{2,2} + 11a_{4,0} \\
& + 3a_{0,4} - 9d_{1,3})\cos(3\theta) + \frac{1}{32}(9d_{3,1} + 7d_{1,3} + 15a_{4,0} + a_{2,2} - a_{0,4})\cos(\theta),
\end{aligned}$$

$$\begin{aligned}
\tilde{f}3\,(\theta) &= \frac{1}{64}\,(d_{5,0}-d_{3,2}+a_{0,5}-a_{2,3}+d_{1,4}+a_{4,1})\sin(8\,\theta)+\frac{1}{16}(2\,d_{5,0}+a_{4,1}-d_{3,2}\\
&-a_{0,5})\sin(6\,\theta)+\frac{1}{32}\,(11\,d_{5,0}-5\,d_{1,4}+a_{2,3}+3\,a_{0,5}+d_{3,2}+3\,a_{4,1})\sin(4\,\theta)\\
&+\frac{1}{16}\,(-a_{0,5}+6\,d_{5,0}+4\,d_{1,4}+a_{4,1}+3\,d_{3,2})\sin(2\,\theta)+\frac{1}{64}(d_{2,3}-a_{3,2}+a_{5,0}\\
&+a_{1,4}-d_{4,1}-d_{0,5})\cos(8\,\theta)+\frac{1}{32}\,(d_{0,5}-3\,d_{4,1}+3\,a_{5,0}-a_{1,4}-a_{3,2}+d_{2,3})\cos(6\,\theta)
\end{aligned}$$

$$\begin{aligned}
&+\frac{1}{8}\,(2\,a_{5,0}-d_{4,1}-d_{2,3}+d_{0,5})\cos(4\,\theta)+\frac{1}{32}(13\,a_{5,0}+a_{1,4}-d_{2,3}+a_{3,2}-17\,d_{0,5}\\
&+3\,d_{4,1})\cos(2\,\theta)+\frac{1}{64}\,(15\,a_{5,0}+9\,d_{4,1}+1\,a_{3,2}+7\,d_{2,3}-\,a_{1,4}+25\,d_{0,5})\,.
\end{aligned}$$

Clearly, the functions $F_k\,(\theta, r_0)$ with $k \in \{0,1,2\}$ are $2\pi-$ periodic functions in the variable θ.
To apply the averaging method of second order, we need the expression of $F_{10}(r_0)$. Form (5), we have that

$$\begin{aligned}
F_{10}(r_0) &= \int_0^{2\pi} \frac{F_1(\theta, r_u(\theta, r_0))}{v(\theta, r_0)}d\theta\\
&= \frac{1}{r_0^2}\int_0^{2\pi} f_2(\theta)e^{3\sin^2\theta}d\theta + \frac{1}{r_0}\int_0^{2\pi} f_1(\theta)e^{2\sin^2\theta}d\theta + \int_0^{2\pi} f_0(\theta)e^{\sin^2\theta}d\theta\\
&+ r_0\int_0^{2\pi}\tilde{f}_1(\theta)d\theta + r_0^2\int_0^{2\pi}\tilde{f}_2(\theta)e^{-\sin^2\theta}d\theta + r_0^3\int_0^{2\pi}\tilde{f}_3(\theta)e^{-2\sin^2\theta}d\theta,
\end{aligned}$$

in order to simplify the expression of $F_{10}(r_0)$, we use the trigonometric formulas

$$\cos(x+2k\pi) = \cos(x) \;\text{ with }\; k\in\mathbb{Z}\,,\;\; \cos(x+\pi) = -\cos(x)\,,\;\; \sin(x+\pi) = -\sin(x), \tag{13}$$

So, we have

$$\begin{aligned}
F_{10}(r_0) &= \frac{1}{r_0}\int_0^{2\pi} f_1\,(\theta)\,e^{2\sin^2\theta}d\theta + r_0\int_0^{2\pi}\tilde{f}_1\,(\theta)d\theta + r_0^3\int_0^{2\pi}\tilde{f}_3\,(\theta)e^{-2\sin^2\theta}d\theta\\
&= (\frac{1}{4}(-J_3-2J_2+3J_1)d_{0,1}+\frac{1}{4}(J_3+2J_2+J_1)a_{1,0})\times\frac{1}{r_0}+2\pi(\frac{1}{2}d_{0,3}\\
&+\frac{1}{4}d_{2,1}+\frac{1}{4}a_{3,0})r_0+((-J_8+6J_5-6J_7+9J_4-8J_6)d_{4,1}+(J_8-2J_5\\
&+2J_7+7J_4-8J_6)d_{2,3}+(-J_8-34J_5+2J_7+25J_4+8J_6)d_{0,5}\\
&+(J_8+26J_5+6J_7+15J_4+16J_6)a_{5,0}+(-J_8+2J_5-2J_7+J_4)a_{3,2}\\
&+(J_8+2J_5-2J_7-J_4)a_{1,4}))r_0^3.
\end{aligned}$$

Therefore, $F_{10}\,(r_0)\equiv 0$ if and only if the following conditions are satisfied

$$\begin{aligned}
a_{5,0} &= \frac{-9\,J_4-6\,J_5+8\,J_6+6\,J_7+J_8}{J_8+26\,J_5+6\,J_7+15\,J_4+16\,J_6}d_{4,1}+\frac{-7\,J_4+2\,J_5+8\,J_6-2\,J_7-J_8}{J_8+26\,J_5+6\,J_7+15\,J_4+16\,J_6}d_{2,3}\\
&+\frac{-25\,J_4+34\,J_5-8\,J_6-2\,J_7+J_8}{J_8+26\,J_5+6\,J_7+15\,J_4+16\,J_6}d_{0,5}+\frac{-J_4-2\,J_5+2\,J_7+J_8}{J_8+26\,J_5+6\,J_7+15\,J_4+16\,J_6}a_{3,2}\\
&+\frac{J_4-2\,J_5+2\,J_7-J_8}{J_8+26\,J_5+6\,J_7+15\,J_4+16\,J_6}a_{1,4}.\\
a_{1,0} &= \frac{-3\,J_1+2\,J_2+J_3}{J_1+2\,J_2+J_3}d_{0,1}\,,\quad a_{0,3} = -d_{1,2}-2d_{0,3}.
\end{aligned}$$

In other words, we have

$a_{1,0} = -17.65322449 d_{0,1}$, $a_{0,3} = -d_{1,2} - 2d_{0,3}$ and $a_{5,0} = 0.004927298284 a_{1,4} - 0.3768477366 d_{1,4} - 0.06527160474 a_{3,2} - 0.3768477367 d_{0,5} - 0.1859602196 d_{2,3}$.

On the other hand, we have $\frac{\partial^2 F_0}{\partial r^2}(\theta, r_u(\theta, r_0)) = 0$, then (6) can be written as

$$F_{20}(r_0) = F_{20}^{(1)}(r_0) + F_{20}^{(2)}(r_0), \tag{14}$$

where

$$\begin{aligned} F_{20}^{(1)}(r_0) &= \int_0^{2\pi} \left(\frac{F_2(\theta, r_u(\theta, r_0))}{u(\theta, r_0)}\right) d\theta \qquad (15)\\ &= \frac{1}{r_0^5}\int_0^{2\pi} g_5(\theta) e^{6\sin^2\theta} d\theta + \frac{1}{r_0^4}\int_0^{2\pi} g_4(\theta) e^{5\sin^2\theta} d\theta + \frac{1}{r_0^3}\int_0^{2\pi} g_3(\theta) e^{4\sin^2\theta} d\theta + \frac{1}{r_0^2}\int_0^{2\pi} g_2(\theta) e^{3\sin^2\theta} d\theta \\ &+ \frac{1}{r_0}\int_0^{2\pi} g_1(\theta) e^{2\sin^2\theta} d\theta + \int_0^{2\pi} g_0(\theta) e^{\sin^2\theta} d\theta + r_0\int_0^{2\pi} \tilde{g_1}(\theta) d\theta + r_0^2\int_0^{2\pi} \tilde{g_2}(\theta) e^{-\sin^2\theta} d\theta + r_0^3\int_0^{2\pi} \tilde{g_3}(\theta) e^{-2\sin^2\theta} d\theta \\ &+ r_0^4\int_0^{2\pi} \tilde{g_4}(\theta) e^{-3\sin^2\theta} d\theta + r_0^5\int_0^{2\pi} \tilde{g_5}(\theta) e^{-4\sin^2\theta} d\theta, \end{aligned}$$

and

$$F_{20}^{(2)}(r_0) = \int_0^{2\pi} \left(\frac{\partial F_1}{\partial r}(r_u(\theta, r_0)) . v_1(\theta, r_u(\theta, r_0))\right) d\theta = \frac{1}{r_0^5}\int_0^{2\pi} C_5(\theta) d\theta + \frac{1}{r_0^4}\int_0^{2\pi} C_4(\theta) d\theta + \frac{1}{r_0^3}\int_0^{2\pi} C_3(\theta) d\theta$$

$$+ \frac{1}{r_0}\int_0^{2\pi} C_1(\theta) d\theta + \int_0^{2\pi} C_0(\theta) d\theta + r_0\int_0^{2\pi} \tilde{C_1}(\theta) d\theta + r_0^2\int_0^{2\pi} \tilde{C_2}(\theta) d\theta + r_0^3\int_0^{2\pi} \tilde{C_3}(\theta) d\theta + r_0^4\int_0^{2\pi} \tilde{C_4}(\theta) d\theta + r_0^5\int_0^{2\pi} \tilde{C_5}(\theta) d\theta,$$

with

$$\frac{\partial F_1}{\partial r}(\theta, r_u(\theta, r_0)) = -\frac{2}{r_0^3} f_2(\theta) e^{3\sin^2\theta} - \frac{1}{r_0^2} f_1(\theta) e^{2\sin^2\theta} + \tilde{f_1}(\theta) + 2r_0 \tilde{f_2}(\theta) e^{-\sin^2\theta} + 3r_0^2 \tilde{f_3}(\theta) e^{-2\sin^2\theta},$$

and $v_1(\theta, y) = \int_0^\theta \frac{F_1(\phi, r_u(\phi, r_0))}{v(\phi, r_0)} d\theta$.

Now, by applying the formulas (13), we can simplify $F_{20}^{(1)}(r_0)$ as follows

$$\begin{aligned} F_{20}^{(1)}(r_0) =& \frac{1}{r_0^5}\int_0^{2\pi} g_5(\theta) e^{6\sin^2\theta} d\theta + \frac{1}{r_0^3}\int_0^{2\pi} g_3(\theta) e^{4\sin^2\theta} d\theta + \frac{1}{r_0}\int_0^{2\pi} g_1(\theta) e^{2\sin^2\theta} d\theta \\ &+ r_0\int_0^{2\pi} \tilde{g_1}(\theta) d\theta + r_0^3\int_0^{2\pi} \tilde{g_3}(\theta) e^{-2\sin^2\theta} d\theta + r_0^5\int_0^{2\pi} \tilde{g_5}(\theta) e^{-4\sin^2\theta} d\theta. \end{aligned}$$

Again, by using (13), $F_{20}^2(r_0)$ becomes

$$\begin{aligned} F_{20}^{(2)}(r_0) =& \frac{1}{r_0^5}\int_0^{2\pi} C_5(\theta) d\theta + \frac{1}{r_0^3}\int_0^{2\pi} C_3(\theta) d\theta + \frac{1}{r_0}\int_0^{2\pi} C_1(\theta) d\theta + r_0\int_0^{2\pi} \tilde{C_1}(\theta) d\theta \\ &+ r_0^3\int_0^{2\pi} \tilde{C_3}(\theta) d\theta + r_0^5\int_0^{2\pi} \tilde{C_5}(\theta) d\theta. \end{aligned}$$

Yields

$$F_{20}(r_0) = \omega_0 \frac{1}{r_0^5} + \omega_2 \frac{1}{r3_0} + \omega_4 \frac{1}{r_0} + \omega_6 r_0 + \omega_8 r_0^3 + \omega_{10} r_0^5,$$

where

$$\begin{aligned}
w_0 &= 665.2264929\ldots a_{0,0}d_{0,0},\\
w_2 &= 16.68739735\ldots a_{0,0}d_{2,0} - 7.649140221\ldots a_{0,0}a_{1,1} + 95.95703341\ldots a_{0,0}d_{0,2}\\
&\quad - 31.79348852\ldots d_{0,0}a_{2,0} - 110.9164315\ldots d_{0,0}d_{1,1} + 105.7377762\ldots d_{0,0}a_{0,2}\\
&\quad - 244.7413301\ldots d_{1,0}d_{0,1} + 251.5279874\ldots a_{0,1}d_{0,1} - 0.0000010\ldots d_{0,1}^2,\\
w_4 &= -38.88726631\ldots d_{0,1}d_{3,0} + 38.88726638\ldots d_{0,1}a_{0,3} + 2.223841814\ldots d_{2,1}a_{0,1}\\
&\quad + 3.507120048\ldots a_{0,0}d_{2,2} - 74.40755738\ldots d_{0,0}a_{4,0} + 1.738873532\ldots d_{1,0}a_{1,2}\\
&\quad - 68.30745734\ldots d_{0,1}d_{1,2} + 7.629774867\ldots d_{1,0}d_{0,3} - 1.179588309\ldots a_{0,0}a_{1,3}\\
&\quad + 1.159249020\ldots b_{1,0} + 20.46448319\ldots e_{0,1} + 2.318498043\ldots a_{0,2}d_{2,0}\\
&\quad - 4.731652314\ldots d_{0,2}d_{1,1} + 2.318498043\ldots d_{0,2}a_{2,0} - 1.253905250\ldots a_{1,2}a_{0,1}\\
&\quad + 14.28961316\ldots d_{0,3}a_{0,1} + 14.47892563\ldots a_{0,2}d_{0,2} - 1.253905250\ldots a_{0,2}a_{1,1}\\
&\quad + 0.09465622855\ldots a_{1,1}a_{2,0} - 3.761715750\ldots d_{1,1}d_{2,0} + 0.1893124564\ldots a_{2,0}d_{2,0}\\
&\quad + 2.318498043\ldots a_{1,1}d_{1,1} + 17.33550510\ldots d_{0,0}a_{0,4} - 2.740323153\ldots d_{0,0}a_{2,2}\\
&\quad - 42.35725611\ldots d_{0,0}d_{3,1} - 1.443217705\ldots d_{1,0}d_{2,1} - 3.155691809\ldots d_{0,1}a_{2,1}\\
&\quad + 2.728264900\ldots a_{0,0}d_{4,0} + 14.19972526\ldots a_{0,0}d_{0,4} + 0.5199940616\ldots a_{0,0}a_{3,1}\\
&\quad - 31.00730858\ldots d_{0,0}d_{1,3},\\
w_6 &= 0.2137640471\ldots\ a_{1,4}d_{1,0} + 1.960592990\ldots\ d_{0,5}a_{0,1} + 3.184149598\ldots\ d_{0,5}d_{1,0}\\
&\quad + 2.018194236\ldots\ d_{0,4}a_{0,2} + 0.01176643430\ldots\ a_{2,2}d_{2,0} + 6.395609425\ldots\ a_{0,5}d_{0,1}\\
&\quad + 0.4226848073\ldots\ d_{2,3}d_{1,0} + 0.3951177630\ldots\ d_{2,3}a_{0,1} + 0.2601849590\ldots\ a_{3,2}d_{1,0}\\
&\quad + 1.063795942\ldots\ d_{0,4}d_{1,1} - 0.004353626539\ldots\ a_{3,2}a_{0,1} - 3.534291735\ldots\ d_{3,1}d_{0,2}\\
&\quad - 0.9905725301\ldots\ d_{1,3}d_{2,0} - 0.1441784055\ldots\ d_{2,2}d_{1,1} - 2.748893572\ldots\ d_{1,3}d_{0,2}\\
&\quad + 0.3727810141\ldots\ a_{3,1}a_{2,0} - 0.01486268923\ldots\ a_{1,3}a_{2,0} + 3.141592654\ldots\ e_{0,3}\\
&\quad + 1.570796327\ldots\ e_{2,1} - 0.2051743665\ldots\ a_{0,4}a_{1,1} + 0.2684387459\ldots\ d_{4,0}a_{0,2}\\
&\quad + 0.1934471228\ldots\ d_{4,1}a_{0,1} - 0.1889181963\ldots\ a_{1,3}a_{0,2} + 1.570796327\ldots\ b_{3,0}\\
&\quad + 0.4864614390\ldots\ d_{1,3}a_{1,1} + 0.3056713282\ldots\ a_{1,3}d_{1,1} - 0.1886097603\ldots\ a_{1,4}a_{0,1}\\
&\quad + 0.7158366500\ldots\ d_{2,2}a_{2,0} + 0.009959033800\ldots\ a_{3,1}a_{0,2} - 13.73654973\ldots\ d_{5,0}d_{0,1}\\
&\quad - 5.497787143\ldots\ a_{4,0}d_{0,2} - 0.1537925955\ldots\ d_{4,1}d_{1,0} - 0.3847809687\ldots\ a_{2,3}d_{0,1}\\
&\quad + 0.2112880916\ldots\ a_{3,1}d_{1,1} + 0.2989367237\ldots\ d_{3,1}a_{1,1} - 0.3926990818\ldots\ d_{2,1}d_{3,0}\\
&\quad - 3.540174952\ldots\ a_{4,0}d_{2,0} - 15.30734609\ldots\ d_{1,4}d_{0,1} + 0.1934079318\ldots\ a_{4,0}a_{1,1}\\
&\quad - 13.20767224\ldots\ d_{3,2}d_{0,1} + 0.4274798378\ldots\ d_{2,2}a_{0,2} - 2.151020123\ldots\ d_{3,1}d_{2,0}\\
&\quad + 2.159844950\ldots\ d_{0,3}a_{0,3} + 0.005883217058\ldots\ a_{2,2}a_{1,1} + 1.461398678\ldots\ d_{0,4}a_{2,0}\\
&\quad - 0.1963495409\ldots\ a_{1,2}a_{0,3} + 0.3926990818\ldots\ d_{1,2}a_{1,2} + 0.9817477044\ldots\ d_{3,0}d_{0,3}\\
&\quad + 0.1963495409\ldots\ a_{1,2}d_{3,0} + 0.3926990818\ldots\ d_{2,1}a_{0,3} - 0.7853981635\ldots\ d_{2,1}d_{1,2}\\
&\quad - 8.767640129\ldots\ a_{4,1}d_{0,1} + 0.2485206765\ldots\ d_{4,0}a_{2,0} + 0.3750494305\ldots\ a_{0,4}d_{2,0}\\
&\quad + 2.356194491\ldots\ a_{0,4}d_{0,2} - 1.178097245\ldots\ d_{1,2}d_{0,3} - 0.7754391284\ldots\ d_{4,0}d_{1,1},
\end{aligned}$$

$$
\begin{aligned}
w_8 = & -0.006405163750\ldots\, b_{1,4} + 0.4898773753\ldots\, e_{0,5} + 0.4898773752\ldots\, e_{4,1} \\
& + 0.08484881107\ldots\, b_{3,2} + 0.2417361057\ldots\, e_{2,3} + 1.299934503\ldots\, b_{5,0} \\
& + 0.03710327108\ldots\, d_{1,2}a_{3,2} + 0.06642416099\ldots\, a_{0,4}d_{2,2} - 0.001186261761\ldots\, a_{0,4}a_{3,1} \\
& + 0.06631353951\ldots\, a_{0,3}d_{2,3} + 0.2926356580\ldots\, a_{0,3}d_{0,5} + 0.1442724248\ldots\, d_{1,2}d_{0,5} \\
& - 0.001698899757\ldots\, a_{0,3}a_{3,2} + 0.05232791382\ldots\, d_{1,2}a_{1,4} - 0.03217562874\ldots\, d_{1,2}d_{2,3} \\
& + 0.03808922399\ldots\, a_{0,3}d_{4,1} + 0.01360116867\ldots\, a_{4,0}d_{2,2} - 0.8756904479\ldots\, , d_{4,0}a_{4,0} \\
& + 0.02040175301\ldots\, a_{1,3}a_{4,0} + 0.01136518658\ldots\, a_{2,1}d_{4,1} + 0.01704777983\ldots\, a_{2,1}a_{3,2} \\
& + 0.01672245248\ldots\, d_{3,0}a_{1,4} + 0.1754225661\ldots\, d_{3,0}d_{2,3} + 0.6871190930\ldots\, d_{3,0}d_{0,5} \\
& - 0.1893085336\ldots\, d_{2,1}d_{5,0} + 0.06602874065\ldots\, d_{2,1}a_{0,5} - 0.5192937428\ldots\, d_{0,3}d_{1,4} \\
& + 0.02040175301\ldots\, a_{2,2}a_{3,1} + 0.03843098248\ldots\, d_{1,3}a_{3,1} - 0.1572827152\ldots\, d_{1,3}d_{4,0} \\
& + 0.03843098248\ldots\, d_{3,1}a_{1,3} - 0.09726371801\ldots\, d_{3,1}d_{0,4} - 0.1572827153\ldots\, d_{3,1}d_{2,2} \\
& + 0.06642416099\ldots\, a_{2,2}d_{0,4} - 0.1102740074\ldots\, d_{1,2}d_{4,1} - 0.03808922386\ldots\, d_{3,0}d_{4,1} \\
& + 0.08654771082\ldots\, d_{3,0}a_{3,2} - 0.02953278001\ldots\, a_{0,3}a_{1,4} - 0.001186261762\ldots\, a_{2,2}a_{1,3} \\
& + 0.04902727950\ldots\, a_{1,2}d_{5,0} - 0.5192937424\ldots\, d_{0,3}d_{3,2} + 0.03843098248\ldots\, d_{0,4}a_{4,0} \\
& + 0.06642416098\ldots\, d_{1,3}a_{1,3} - 0.0001977103018\ldots\, a_{1,2}a_{2,3} - 0.1316620603\ldots\, d_{1,3}d_{0,4} \\
& - 0.09726371795\ldots\, d_{1,3}d_{2,2} - 0.01300803776\ldots\, d_{0,3}a_{2,3} - 0.0003954205650\ldots\, d_{2,1}a_{2,3} \\
& - 0.5679256019\ldots\, d_{3,1}d_{4,0} - 0.2941636775\ldots\, d_{2,1}d_{3,2} + 0.3592038668\ldots\, d_{0,3}a_{0,5} \\
& + 0.1300803779\ldots\, a_{3,1}a_{4,0} + 0.3250032343\ldots\, d_{0,4}a_{0,4} + 0.05543244326\ldots\, a_{1,2}d_{3,2} \\
& - 0.03222352905\ldots\, a_{1,2}a_{0,5} - 0.2349355213\ldots\, d_{2,1}a_{4,1} + 0.03843098248\ldots\, a_{0,4}d_{4,0} \\
& - 0.2021188616\ldots\, d_{2,1}d_{1,4} - 0.4964802492\ldots\, d_{0,3}a_{4,1} + 0.3363903734\ldots\, d_{0,3}d_{5,0} \\
& - 0.03083955696\ldots\, a_{0,4}a_{1,3} + 0.02621378586\ldots\, a_{1,2}a_{4,1} + 0.08184393943\ldots\, a_{1,2}d_{1,4} \\
& + 0.04709488526\ldots\, a_{2,1}d_{2,3} + 0.01360116867\ldots\, a_{2,2}d_{4,0} + 0.03843098248\ldots\, a_{2,2}d_{2,2} \\
& - 0.002024227119\ldots\, a_{2,1}a_{1,4} + 0.08340366999\ldots\, a_{2,1}d_{0,5} + 0.01360116867\ldots\, d_{3,1}a_{3,1},
\end{aligned}
$$

$$
\begin{aligned}
w_{10} = & \; 0.001592179550\ldots\, d_{5,0}a_{1,4} + 0.2947572926\ldots\, d_{5,0}d_{0,5} - 0.004852826544\ldots\, a_{1,4}a_{0,5} \\
& + 0.04840928673\ldots\, d_{0,5}a_{0,5} + 0.006676120637\ldots\, a_{2,3}d_{0,5} - 0.04618970216\ldots\, a_{4,1}d_{4,1} \\
& + 0.006010711593\ldots\, d_{4,1}a_{0,5} - 0.02589608392\ldots\, d_{4,1}d_{1,4} - 0.02640370046\ldots\, a_{4,1}d_{0,5} \\
& + 0.04233997947\ldots\, a_{3,2}d_{5,0} - 0.0004746561370\ldots\, a_{3,2}a_{0,5} + 0.006900020517\ldots\, d_{3,2}a_{1,4} \\
& - 0.01162168283\ldots\, d_{3,2}d_{0,5} + 0.006587474736\ldots\, a_{3,2}d_{1,4} + 0.006434781110\ldots\, a_{4,1}a_{3,2} \\
& + 0.01080894622\ldots\, d_{2,3}a_{0,5} + 0.01130732797\ldots\, d_{1,4}a_{1,4} - 0.02441752356\ldots\, d_{1,4}d_{0,5} \\
& + 0.004691871440\ldots\, d_{2,3}a_{2,3} + 0.09426342039\ldots\, d_{2,3}d_{5,0} + 0.001730289429\ldots\, a_{3,2}a_{2,3}
\end{aligned}
$$

$$
\begin{aligned}
& + 0.002963794680\ldots\, a_{4,1}a_{1,4} - 0.01070894437\ldots\, a_{4,1}d_{2,3} - 0.0002494984216\ldots\, a_{2,3}a_{1,4} \\
& - 0.02134227821\ldots\, d_{3,2}d_{2,3} + 0.001153526288\ldots\, d_{4,1}a_{2,3} - 0.02225290313\ldots\, d_{4,1}d_{5,0} \\
& - 0.01752438166\ldots\, d_{1,4}d_{2,3} + 0.006593334965\ldots\, d_{3,2}a_{3,2} - 0.05806429194\ldots\, d_{3,2}d_{4,1}.
\end{aligned}
$$

The coeffecients ω_0, ω_2, ω_4, ω_6, ω_8 and ω_{10} are independent. Using Descartes Theorem (Theorem 3), we choose the coeffecients ω_0, ω_2, ω_4, ω_6, ω_8 and ω_{10} such that $F_{20}(r_0)$ has 5, 4, 3, 2, 1 or 0 simple positive real zeros.

4. Examples

Now, we provide some examples that establish our main result. The computations of this Section have been verified with the help of Maple and Mathematica.

Example 1. **(Example with five limit cycles)**
Consider the following system

$$\begin{cases} \dot{x} & = -y(3x^2+y^2)+\epsilon(y^5+813.8487421x^5)+\epsilon^2(4874.474235x-1925.496694x^3 \\ & +500.3665045x^6), \\ \dot{y} & = x(x^2-y^2)+\epsilon(\frac{1}{2}-20.29310103y^2-4376.46688x^2y^3). \end{cases} \tag{16}$$

For $\epsilon = 0.00001$, the expression of $F_{20}(r_0)$ of (16) is given by

$$F_{20}(r_0) = 665.2264929\frac{1}{r_0^5} - 3894531546\frac{1}{r_0^3} + 5650.729487\frac{1}{r_0} - 3024.563121r_0 + 650.4436819r_0^3 \\ - 47.30499505r_0^5.$$

$F_{20}(r_0)$ has five positive zeros near to: $r_0 = 0.5\,;1\,;1.5\,;2\,;2.5$. This means that the system (16) has five limit cycles bifurcating from the degenerate center of the unperturbed system of (16), see Figure 1.

Example 2. **(Example with four limit cycles)**
We consider the system

$$\begin{cases} \dot{x} & = -y(3x^2+y^2)+\epsilon(1-0.3768477367x^5)+\epsilon^2(-0.8626274275x), \\ \dot{y} & = x(x^2-y^2)+\epsilon(3-7.558400223y^2+y^5+3.91816296x^5)+\epsilon^2(67.27415794y^3 \\ & -63.65366812x^4y). \end{cases} \tag{17}$$

The function $F_{20}(r_0)$ of the previous system with $\epsilon = 0.001$ is given by

$$F_{20}(r_0) = 1995.679479\frac{1}{r_0^5} - 2175.844988\frac{1}{r_0^3} - 1.154907016\frac{1}{r_0} + 211.3480004r_0 - 31.18249186r_0^3 \\ + 1.154907106r_0^5.$$

The equation $F_{20}(r_0) = 0$ has exactly four positive solutions:$r_0 = 1\,;\,2\,;\,3\,;\,4$. As consequence of it, there is 4 limit cycles bifurcating from the global center of the unperturbed (17), see Figure 2.

Example 3. **(Example with three limit cycles)**
For $\epsilon = 0.0001$, the system

$$\begin{cases} \dot{x} & = -y(3x^2+y^2)+\epsilon(4-0.376877367x^5)+\epsilon^2(508.1571728x^3y^2), \\ \dot{y} & = x(x^2-y^2)+\epsilon(\frac{1}{2}-1.179494576y^2+y^5-10.44843456x^5)+\epsilon^2(11.73841899y \\ & -101.9528222x^2y). \end{cases} \tag{18}$$

has the second averaged function

$$F_{20}(r_0) = 332.6132465\frac{1}{r_0^5} - 452.7235855\frac{1}{r_0^3} + 240.220678\frac{1}{r_0} - 160.1471187r_0 + 43.11653195r_0^3 \\ - 3.079752282r_0^5.$$

If we suppose that $F_{20}(r_0) = 0$, we obtain three positive values of r_0 which are: $r_0 = 1$, $r_0 = 2$ and $r_0 = 3$. Then the maximum number of limit cycles bifurcates from the global center of the unperturbed (17) is three, see Figure 3.

Example 4. **(Example with two limit cycles)**
The system

$$\begin{cases} \dot{x} & = -y(3x^2+y^2)+\epsilon(4-0.376877367x^5)+\epsilon^2(508.1571728x^3y^2), \\ \dot{y} & = x(x^2-y^2)+\epsilon(\frac{1}{2}-1.179494576y^2+y^5-10.44843456x^5)+\epsilon^2(11.73841899y \\ & -101.9528222x^2y). \end{cases} \tag{19}$$

with $\epsilon = 0.001$ has

$$F_{20}(r_0) = 332.6132465\frac{1}{r_0^5} - 452.7235855\frac{1}{r_0^3} + 240.220678\frac{1}{r_0} - 160.1471187r_0 + 43.11653195r_0^3 - 3.079752282r_0^5.$$

The equation $F_{20}(r_0) = 0$ has two positive real solutions $r_0 = 1$ and $r_0 = 3$. Hence two limit cycles bifurcated from the global center of the unperturbed system (19), see Figure 4.

Example 5. (Example with one limit cycles)
For $\epsilon = 0.0001$, we consider the system

$$\begin{cases} \dot{x} &= -y(3x^2+y^2) + \epsilon(1 + 152.1930999xy - 0.376877367x^5) + \epsilon^2(5355.864437x), \\ \dot{y} &= x(x^2-y^2) + \epsilon(7 + y^5 - 1316.502754x^5) + \epsilon^2(-988.1581229x^2y \\ & +6421.031503x^2y^3). \end{cases} \tag{20}$$

The function $F_{20}\,(r_0)$ of (20) is

$$F_{20}\,(r_0) = +4656.585450\frac{1}{r_0^5} - 1164.146362\frac{1}{r_0^3} + 6208.780600\frac{1}{r_0} + 1552.195150r_0^3 - 1558.19150r_0 - 388.0487875r_0^5.$$

One limit cycle can bifurcate from the degenerate center of the unperturbed system of (20) because $r_0 = 2$ is the only positive root of $F_{20}\,(r_0)$, see Figure 5.

Example 6. (Example with zero limit cycles)
Consider the system

$$\begin{cases} \dot{x} &= -y(3x^2+y^2) + \epsilon(1 - 0.376877367x^5) + \epsilon^2(2613.379742x^3y^2), \\ \dot{y} &= x(x^2-y^2) + \epsilon(1 + 3.466272714y^2 + y^5 + 376.1436442x^5) + \epsilon^2(43.34185473y \\ & +282.3308924x^2y), \end{cases} \tag{21}$$

with $\epsilon = 0.001$. We have that

$$F_{20}\,(r_0) = 665.2264932 \times \frac{1}{r_0^5} + 332.6132466 \times \frac{1}{r_0^3} + 886.9686576 \times \frac{1}{r_0} + 443.4843288\,r_0 + 221.7421644\,r_0^3 + 110.8710822\,r_0^5.$$

The equation $F_{20}\,(r_0) = 0$ does not have a positive zero, which implies that there is no limit cycle bifurcating from the global center of the unperturbed system (21), see Figure 6.

Figure 1: Five limit cycles (Example 1)

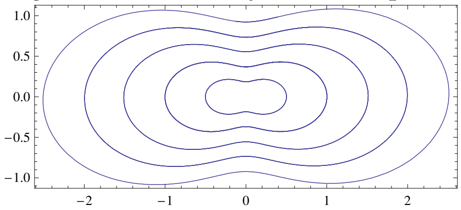


Figure 2: Four limit cycles (Example 2)

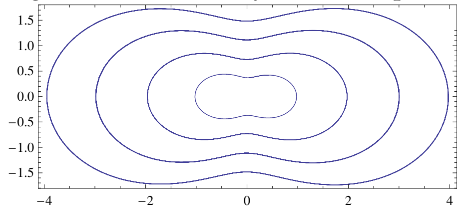


Figure 3: Three limit cycles(Example 3)

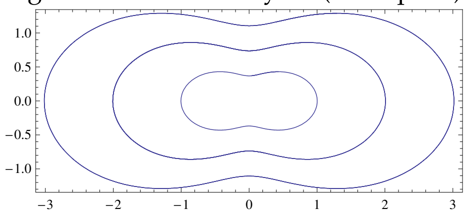


Figure 4: Two limit cycles (Example 4)

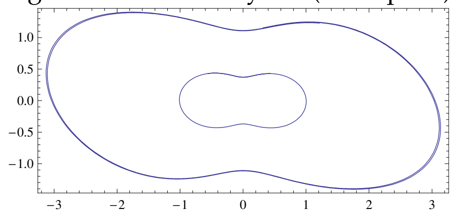


Figure 5: One limit cycle (Example 5)

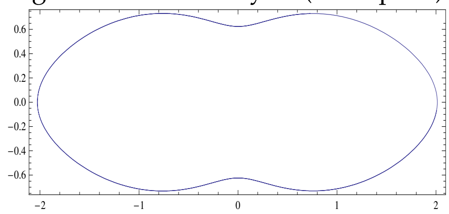
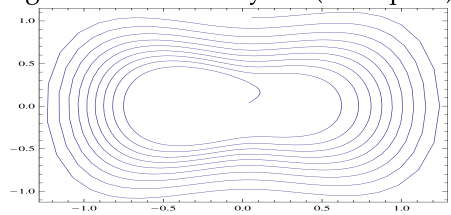


Figure 6: Zero limit cycles (Example 6)



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