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On norms of derivations implemented by self-adjoint operators

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Abstract: In this paper, we concentrate on norms of derivations implemented by self-adjoint operators. We determine the upper and lower norm estimates of derivations implemented by self-adjoint operators.The results show that the knowledge of self-adjoint governs the quantum chemical system in which the eigenvalue and eigenvector of a self-adjoint operator represents the ground state energy and the ground state wave function of the system respectively.

Keywords: Norm; Orthogonality; Self-adjoint operator; Derivation; Linear operator.

MSC: 47B47, 47A30.

1. Introduction

The study of operators has continued to attract the attention of many researchers. Of special interest is the determination of derivations implemented by self-adjoint operators. Let $B(H)$ denote the algebra of all bounded linear operators on an infinite-dimensional complex separable Hilbert space H. For operators A, B in $B(H)$, the generalized derivation $\delta_{A,B}$ on $B(H)$ is given as $\delta_{A,B}(X) = AX - XB$ while the inner derivation is $\delta_A(X) = AX - XA$. Let H be a Hilbert space. We denote its inner product by $\langle ., . \rangle$, which is another common notation for inner products that is often reserved for Hilbert spaces. Therefore, if x, y are vector spaces in a Hilbert space H, then we say that x and y are orthogonal, written as $x \perp y$ if and only if $\langle x, y \rangle = 0$. Two subsets A and B are said to be orthogonal, written $A \perp B$, if $x \perp y$ for every $x \in A$ and $y \in B$. The orthogonal complement $A^\perp$ of a subset A is the set orthogonal to A, written $A^\perp$=$\{x \in H | x \perp y \text{ for all } y \in A\}$, We also define orthogonal direct sum of subspaces of a Hilbert space. If M and N are orthogonal closed linear subspaces of a Hilbert space, then we define orthogonal direct sum of M N by $M \bigoplus N$. If M is a closed subspace of a Hilbert space H, then $H = M \bigoplus M^\perp$. Thus, every closed subspace M of a Hilbert space has a closed complementary subspace $M^\perp$. In a general Banach space, there may be no element of a closed subspace that is closest to a given element of a Banach space, and a closed linear subspace of a Banach space may have no complementary subspace. A subset U of a non-zero vectors in a Hilbert space is orthogonal if any two distinct elements in U are orthogonal. A set of vectors is orthonormal if it is orthogonal and $\|u\| = 1$ for all $u \in U$, in which case the vectors u are said to be normalized. An orthonormal basis of a Hilbert space is an orthonormal set such that every vector in the space can be expanded in terms of the basis. Every Hilbert space has an orthonormal basis, which may be finite, countably infinite, or uncountable. Two Hilbert spaces whose orthonormal bases have the same cardinality are isomorphic. A bounded linear operator $A : H \to H$ on a Hilbert space H is self-adjoint if $A^* = A$. Equivalently, a bounded linear operator A on H is self-adjoint if and only if $\langle x, Ay \rangle = \langle Ax, y \rangle$ for all $x, y \in H$. A linear map on $\mathbb{R}^n$ with the matrix A is self-adjoint if and only if A is symmetric, meaning that $A = A^T$, where A^T is the transpose of A. A linear map $\mathbb{C}^n$ with matrix A is self-adjoint if A is Hermitian. Given a linear operator $A : H \to H$,we define a sesquilinear form $a : H \times H \to \mathbb{C}$ by $a(x, y) = \langle x, Ay \rangle$. If A is self-adjoint, then this form is a Hermitian symmetric, or symmetric, meaning that $a(x, y) = \overline{a(y, x)}$. It follows that the associated quadratic form $q(x) = a(x, x)$, or $q(x) = \langle x, Ax \rangle$, is real valued. We say that A is a nonnegative if it is self-adjoint and $\langle x, Ay \rangle \geq 0$ for all $x \in H$. We say that A is positive or positive definite, if it is self-adjoint and $\langle x, Ax \rangle > 0$ for every nonzero $x \in H$, If A is positive, bounded operator, then $(x, y) = \langle x, Ay \rangle$ defines the inner product on H. If in addition, there is a constant $c > 0$ such that $\langle x, Ax \rangle \geq c\|x\|^2$ for all $x \in H$, then we say that A is bounded from below, and the norm associated with $(., .)$

is equivalent to the norm associated with $\langle ., . \rangle$. The concept of norms of derivation has been studied by quite a number of researchers. This has been done under Elementary operators in which normal derivations belong. For instance, Cabrera and Rodrigues [18] proved that for JD^*-algebras, $\|M_{C,D} + M_{D,C}\| \geq \frac{1}{20412}\|C\|\|D\|$, while Stacho and Zalar [61] proved that for standard operator algebras on Hilbert spaces $\|M_{C,D} + M_{D,C}\| \geq 2(\sqrt{2} - 1)\|C\|\|D\|$. Nyamwala [49] dealt with norm of a C^*-algebra and established that $\|CYD - DYC\| = 2\|C\|\|D\|$. Timoney [67] investigated norms of elementary operators and in [68] he focussed on computing the norm of elementary operators where he showed that $\|M_{C,D} + M_{D,C}\| \geq \|C\|\|D\|$. Mathieu [43] prove that for prime C^*-algebras, $\|M_{C,D} + M_{D,C}\| \geq \frac{2}{3}\|C\|\|D\|$. Seddik [58] used injective norm to characterize nomaloid operators and determined their lower norm estimates as, $\|C\|\|D\| \leq \|CYD + DYC\| \leq 2\|C\|\|D\|$. Okelo, Agure and Ambogo [51] determined the norm of an elementary operator and characterized these norms when they are implemented by norm-attainable operators. In their study they showed that $\|\mathcal{J}_{N,C,D}|B(H)\| \geq \|C\|\|D\|$, in which $C, D \in B(H)$ and $\mathcal{J}_{N,C,D}$ is a norm-attainable Jordan elementary operator. Others who studied this topic include [12,36,48]. Through all these studies, it remains that there is no known formula for computing the norm of a derivation in terms of its coefficients. Orthogonality in normed spaces and derivations is also a concept that has been analyzed through the norm property of elementary operators. In relation to orthogonality involving elementary operators, Anderson [1] studied orthogonality of range and kernel of normal derivations in which he showed that if $A, B \in B(H)$ such that A is normal and $AB = BA$ then for all $Y \in B(H)$, $\|\delta_A(Y) + B\| \geq \|B\|$. Kittaneh [37] established that $\|\delta_A(Y) + B\|_2^2 = \|\delta_A(Y)\|_2^2 + \|B\|_2^2$, for a Hilbert-Schmidt operator.Micheri [44] characterized orthogonality in the sense of Birkhoff and established that for a general bounded linear operator A on a normed linear space Z, $RanA \perp KerA \Longrightarrow \overline{RanA} \bigcap KerA = \{0\}$ and $RanA \bigcap KerA = \{0\}$. Okelo [53] focused on elementary operators and their orthogonality in normed spaces where he showed that for all $A, B, X \in B(H)$ and for a generalized derivation $\delta_{A,B} = AX - XB$, $Ran\delta_{A,B} \perp Ker\delta_{A,B} \Longrightarrow Ran\delta_{A,B} \bigcap Ker\delta_{A,B} = \{0\}$. For details see [6,9,16,31] We investigated lower and upper norm estimates of a derivation. Lastly, we investigated orthogonality of the range and kernel of derivations. Several methods such as numerical ranges, tensor products approach and limits have been employed in attempting to solve the norm and orthogonality problems of derivations.

2. Basic Concepts and Preliminaries

In this section we give basic concepts and definitions useful in the sequel.

Definition 1. A Hilbert space is an inner product space $\langle ., . \rangle$ such that the induced Hilbertian norm is complete.

Definition 2. An operator is a linear map of a Hilbert space onto itself. If T is an operator, then T is such that $T : H \to H$.

Definition 3. Let $A : H \to H$ be a bounded linear operator. The adjoint of A, denoted as A^*, is unique operator $A^* : H \to H$, such that $\langle Ax, y \rangle = \langle x, A^*y \rangle$. The operator A is self-adjoint or Hermitian if $A = A^*$.

Definition 4. A normed vector space is a pair $(X, \|.\|)$ consisting of a vector space X over $\mathbb{R}$ or $\mathbb{C}$ and a norm $\|.\|$ such that

(i) $\|x\| \geq 0$, for all $\lambda \in \mathbb{C}$ if and only if $\|x\| = 0$
(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{C}$ and $x \in X$
(iii) $\|x + y\| \leq \|x\| + \|y\|, \forall\, x, y \in X$ (triangle inequality)

$\Longleftrightarrow$ The mapping $\|.\|$ is called a norm and $\|x\|$ is called the norm of x

Definition 5. If T is an operator on a Hilbert space H then

(i) T is normal if $TT^* = T^*T$
(ii) T is self-adjoint or Hermitian if $T = T^*$
(iii) T is positive if $\langle Tx, x \rangle \geq 0 \,\forall\, x \in H$
(iv) T is unitary if $TT^* = T^*T = I$.

Definition 6. Let $(H, \langle ., . \rangle)$ be an inner product space, we say $x, y \in H$ are orthogonal and write $x \perp y$ if and only if $\langle x, y \rangle = 0$.

Definition 7. An orthogonal projection on a Hilbert space is a linear map $P : H \longrightarrow H$ that satisfies $P^2 = P, \langle Px, y\rangle = \langle x, Py\rangle \forall x, y \in H$. An orthogonal projection is necessarily bounded. If P is non-zero orthogonal projection then $\|P\| = 1$.

Definition 8. The numerical range of an operator T is a complex Hilbert space H given by $W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\}$

Definition 9. A Banach space is a complete normed vector space with respect to the metric $d(x, y) = \|x - y\|$.

Definition 10. A Banach algebra is a complex Banach space A together with an associative and distributive multiplication such that $\lambda(ab) = (\lambda a)b = a(\lambda b)$ and $\|ab\| \leq \|a\|\|b\|, \forall a, b \in \mathbb{C}$.

Definition 11. Let A be a subset of $\mathbb{R}$. We say that $M \in \mathbb{R}$ is an upper bound of A if $x \leq M$ for all $x \in A$, and $m \in \mathbb{R}$ is a lower bound of A if $m \leq x$ for all $x \in A$. The set A is bounded from above if it has an upper bound, bounded from below if it has a lower bound and bounded if it has both an upper and a lower bound.

3. Main results

3.1. Introduction

In this chapter we study norms of derivations implemented by self-adjoint operators. Here we determine the lower norm estimate and upper norm estimates of derivations implemented by self-adjoint operators.

3.2. Norms of Derivations

A derivation on a Banach algebra X is a linear transformation $\delta : X \to X$ which satisfies $\delta(uv) = u\delta(v) + \delta(v)a$ for all $u, v \in X$. If for a fixed $u, \delta : v \to ab - ba$, then δ is called an inner derivation. In [53], Rosenblum determined spectrum of inner derivation. Norm of a derivation has been studied by quite a number of researchers including Anderson, Stampfli and many others. We establish norm of a derivation using the following lemma.

Lemma 1. *Let V be an essential left ideal in C^*-algebra B. Let $p : N \to B$ be a linear mapping defined on a subspace N of B. If, for some derivation $\delta : B \to B$, the identity.*

$$p(a)b = -a\delta(b)(a \in N, b \in V).$$

holds, then p is bounded with a norm atmost $\|\delta\|$.

Proof. Let μ be an irreducible representation of B. By hypothesis,

$$\mu(p(a)h)\mu(r)\mu(b) = -\mu(a)\delta_\mu(\mu(hrb))$$

for all $a \in N, h, r \in B$ and $b \in V$, where δ_μ denotes the induced derivation on $\mu(B)$. Hence

$$\begin{aligned} \|G_\mu(p(a)h)\mu(b)\mu(r)\| &\leq \|\mu(a)\|\|\delta_\mu\|\|\mu(h)\|\|\mu(r)\|\|\mu(b)\| \\ &\leq \|a\|\|\delta\|\|h\|\|\mu(b)\|, \end{aligned}$$

whereby,

$$\|G_\mu(p(a)h)\mu(b)\| \leq \|a\|\|\delta\|\|h\|\|\mu(b)\|,$$

for all $a \in N$ $h \in B$ and $b \in V$. Let J be the closed ideal $\overline{YY^*}$. If $\ker\mu$ does not contain J, there is $b \in V$ such that $\mu(b) \neq 0$.
Then

$$\|G_\mu(p(a)h)\mu(b)\| = \|\mu(p(a)h)\|\|\mu(b)\|,$$

hence the above inequality entails that

$$\|\mu(p(a)h)\| \leq \|a\|\|\delta\|\|h\|.$$

Since each irreducible representation of J extends to an irreducible representation of B not vanishing on J, it follows that

$$\|p(a)h\| \leq \|a\|\|\delta\|\|h\|.$$

for all $a \in N$ and $h \in J$. Since J is essential, we conclude that

$$\|p(a)\| = \sup \|p(a)h\| |h \in J, \|h\| \leq 1 \leq \|\delta\|\|a\|$$

for all $a \in N$ as required. □

Theorem 1. *Let δ be a derivation of a C^*-algebra B. Suppose there exists an essential left ideal J of B and an element $b \in B$ satisfying $b\delta J = 0$ and $(1 - e_b)\delta J = 0$. Then there is $n \in Tl(B)$ such that $\delta = \delta_n, bn = 0, Jn = 0$ and $\|n\| \leq \|\delta\|$.*

Proof. For all $r \in J$ and $d \in B$, we have

$$bd\delta r + b(\delta d)r = b\delta(dr) = 0$$

by assumption, we have

$$G_{b,\delta r} + G_{b,u} \circ \delta = 0, (r \in J) \tag{1}$$

On the ideal $N = BbB$, we define $p : N \to B$ by $\sum_i u_i b v_i \to \sum_i u_i b \delta v_i$ whenever u_i, v_i are finitely many elements in B. Note that,

$$\sum_i u_i b(\delta v_i)r = -\sum_i u_i b v_i \delta r$$

hence

$$p(u)r = -u\delta r(u \in N, r \in J) \tag{2}$$

and

$$p(u)vr = -u\delta(vr)(u \in N, v \in B, r \in J) \tag{3}$$

By (2),

$$(p(u_i + \alpha u_2) - p(u_1) - \alpha p(u_2)r = 0$$

for all $u_i, u_2 \in N, \alpha \in \mathbb{C}$ and $r \in J$ whereby $u = 0$ implies that $p(u)r = 0$ for all $r \in J$. Since J is essential, it follows that p is a well-defined linear mapping on N.
Applying the lemma 5.1 to (2), we conclude that p is bounded with norm atmost $\|\delta\|$. Hence replacing N in B, we may assume that N is closed. Let $N^{\perp}$ be annihilator of N in B If $u_1 \in N$ and $u_2 \in N^{\perp}$, we put $\overline{p}(u_1 + u_2) = p(u_1)$. Then, as $(1 - e_b)\delta J = 0$,

$$\overline{p}(u_1 + u_2)v = p(u_1)v = -u_1\delta r = -(u_1 + u_2)e_b\delta r(r \in J)$$

Hence, replacing N by $N + N^{\perp}$ and p by $\overline{p}$, we may assume that N is essential closed ideal in B.
By (2),

$$(p(vu) - vp(u))r = (vu - vu)\delta r = 0, (u \in N, v \in B, r \in J),$$

hence p is a left B-module map. Put $h = p - \delta$, then

$$\begin{aligned} h(uv)r &= p(uv)r - \delta(uv)r \\ &= uv\delta r - \delta(uv)r \\ &= -\delta(uvr) \\ &= -(\delta u)vr - u\delta(vr) \\ &= h(u)r \end{aligned}$$

for all $u \in N, v \in B$ and $v \in J$ so that h a right B-module map from N into B. Moreover, if $u, v \in N$, then by (3),

$$p(u)vr = -u\delta(vr) = -uv\delta r = u((p) - \delta v)r = uh(v)r, (r \in J),$$

and thus $p(u)v = uh(v)$. As a result, (p,h) is a double centralizer of N represented by an element $a \in G(N)$. By definition, $\delta = p - h = M_a - J_a = \delta_a$ on N. From this, we infer that

$$\begin{aligned} (\delta v)u &= \delta(vu) - v(\delta u) \\ &= p(vu) - h(vu) - vp(u) + vh(h) \\ &= vh(u) - h(vu) \\ &= vau - avu \\ &= [v, a]u \end{aligned}$$

for all $u \in N$ and $v \in B$. Since N is essential, this yields $\delta = \delta_a$ on B. The identity

$$b(vra - avr) = b\delta(vr) = 0$$

implies that

$$G_b, ra = G_{ba}, r \tag{4}$$

Therefore the mapping

$$\sum_i u_1 bv_i + x \to \sum_i u_i bav_i, (u_i, v_i, x \in (BbB)^{\perp}$$

is a well defined B-bimodule map from the essential ideal $BbB + (BbB)^{\perp}$ into B which gives rise to an element $\alpha \in \mathbb{C}$ with the property $\alpha b = ba$. This together with (4) entails that

$$G_{b,ra-\alpha r} = G_{b,ra} - \alpha G_{b,r} = 0$$

hence $0 = e_b r(a - \alpha) = r(a - \alpha)$ as $e_b a = a$ and $e_b \alpha = \alpha$. Replacing a by $a - \alpha$, we thus obtain $\delta = \delta_a$ as well as $ba = 0$ and $Ja = 0$. In particular, $uar = -u\delta r$ for all u in the domain of a and $r \in J$, thus the same reasoning shows that a still bounded with $\|a\| \leq \|\delta\|$. □

Proposition 1. *Let $C \in B(H)$ where H is a complex Hilbert space and let λ_0 be the center of C.*

(i) $\|\delta_C\| = 2\|C - \lambda_0\| = 2inf\|C - \lambda\|, \lambda \in \mathbb{C}$
(ii) if $\beta \in W_0(C)$, then $\|\delta_C\| \geq 2(\|C\|^2 - \beta^2)^{\frac{1}{2}}$.

Proof. (i) If dim $H = 1$ the proof is evident. Suppose dim $H \geq 1$. We establish that
$0 \in W_0(C) \Longleftrightarrow [0$ is the center of

$$C : \|C\| \leq \|C + \lambda\|,$$

for all $\lambda \in \mathbb{C}$. Which is equivalent to

$$\sup \|CY - YC\|, \|y\| = 1 = 2\|C\|.$$

Since

$$\delta_C = \delta_{C-\lambda I},$$

the second equivalence fix the value of $\|\delta_C\|$ with the choice of λ imposed by the first equivalence.
(ii) For $\beta \in W_0(C)$ we associate a sequence $\{y_k\}$ with

$$\|y_k\| = 1, \lim_k \|Cy_k\| = \|C\|, \beta = \lim_k (Cy_k, y_k)$$

and $G_k = Vect\{y_k, y_k'\}$, where y_k, y_k' is an orthonormal basis of G_k and

$$(Cy_k, y_k') \geq 0,$$

where $Cy_k \in G_k$.
Let

$$Y_k = y_k \otimes y_k - y_k' \otimes y_k'.$$

Then

$$(\delta_C Y_k)y_k = Cy_k - (Cy_k, y_k)y_k + (Cy_k, y_k')y_k' = 2(Cy_k, y_k')y_k'$$
$$= 2(\|Cy_k\|^2 - (|(Cy_k, y_k)|^2)^{\frac{1}{2}} y_k'.$$

Hence

$$\|\delta_C\| \geq \lim_k \|(\delta_C Y_k)y_k\| = 2(\|C\|^2 - |\beta|^2)^{\frac{1}{2}}.$$

□

Proposition 2. *Let C, D be two elements of $B(E)$, where E is a complex Hilbert space. Then*

(i) $\|\delta_{C,D}\| = inf\|C - \lambda\| + \|D - \lambda\|, \lambda \in \mathbb{C}$,
(ii) $W_N(C) \bigcup W_N(D) \neq \Phi \Longleftrightarrow \|\delta_{C,D}\| = \|C\| + \|D\|$.

Proof. In the study of $W_N(A)$ we established that

$$\|C\| + \|D\| \leq \|C - \lambda\| + \|D - \lambda\| \Longleftrightarrow \exists\{Y_k\}, \|Y_k\| = 1,$$

such that

$$\lim_k \|CY_k - Y_k D\| = \|C\| + \|D\|.$$

Since

$$\delta_{C,D}(Y) = \delta_{C-\lambda, D-\lambda}$$

hence

$$\|\delta_{C,D}(Y)\| \leq \|C - \lambda\| + \|D - \lambda\|,$$

for all, $Y \in B(H), \|Y\| = 1$.
Then

$$\|\delta_{C,D}\| \leq inf\|C - \lambda\| + \|D - \lambda\|, \lambda \in \mathbb{C}.$$

□

Proposition 3. *Let $C \in B(H)$ Then,*
$W_0(C) = \{z \in \mathbb{C} : z = \lim_k (Cy_k, y_k), \|y_k\| = 1, \lim_k (\|C^*C\| - C^*C)y_k\}$.

Proof. Since

$$\|C^*C\| - C^*C \geq 0, \lim_k (\|C^*C\| - C^*C)y_k = 0 \Leftrightarrow \lim_k (\|C^*C\| - C^*C)y_k, y_k = 0$$

We also know that

$$((\|A^*A\| - A^*A)x_n, x_n) = \|A\|^2 - \|Ax_n\|^2$$

and

$$\lim_k (\|C\|^2 - \|Cy_k\|^2) = 0 \Leftrightarrow \lim_k (\|C\| - \|Cy_k\|) = 0$$

we have

$$\lim_k (\|C^*C\| - C^*C)y_k = 0 \Leftrightarrow \lim_k (\|C\| - \|Cy_k\|) = 0.$$

□

Proposition 4. *$W_0(C)$ is a non empty closed convex set included in $W(C)$.*

Proof. We establish this as follows: (a) There exists $\{y'_k\}$ such that $\|y'_k\| = 1$ and

$$\lim_k (\|C^*C\| - C^*C)y'_k, y'_k = 0.$$

Then sequence (Cy'_k, y'_k) is then bounded sequence in $\mathbb{C}$, it a convergent subsequence $f = \lim_k (Cy_k, y_k)$ and

$$\lim_k (\|C^*C\| - C^*C)y_k = 0.$$

Hence $V(C)$ is a non empty set
(b) We prove that $W_0(C)$ is convex. Let

$$f = \lim_k (Cy_k, y_k), s = \lim_k (Cz_k, z_k)$$

be two disjoint points of $W_0(C)$. For $r \in [0,1]$, we show that

$$rf + (1-r)s \in W_0(C).$$

We construct an associated sequence $\{t_k\}$. We extract two subsequences y_k and z_k. We assume that

$$|(Cy_k, y_k) - (Cz_k, z_k)| \geq \frac{|f-s|}{2}.$$

This implies in particular that y_k and z_k are not collinear. □

Lemma 2. *Let $\alpha \in W_0(A)$. Then $\|\delta_A\| \geq 2(\|A\|^2 - |\alpha|^2)^{\frac{1}{2}}$*

Proof. Note that $\|\delta_A\| = \sup\{\|AX - XA\| : X \in B(H) \text{ and } \|x\| = 1\}$. Since $\alpha \in W_0(A)$, there exists $u_n \in H$ such that $\|u_n\| = 1, \|Au_n\| \to \|A\|$ and $(Au_n, u_n) \to \alpha$. Set $Au_n = \mu u_n + \beta v_n$ where $(u_n, v_n) = 0$. Set $R_n u_n = u_n, R_n v_n = -v_n$ and $R_n = 0$ on $\{u_n, v_n\}$. Then $\|(AR_n - R_nA)u_n\| = 2|\beta_n| \geq 2(\|T\| - |b_n|^2)^{\frac{1}{2}} - \lambda_n$ where $\lambda \to 0$. Since $b_n \to \alpha$ hence the proof. □

Theorem 2. *$\|\delta_A\| = 2\|A\|$ if and only if $0 \in W_0(A)$.*

Proof. It follows from the above lemma that $\|\delta_A\| \geq 2\|A\|$ if $0 \in W_0(A)$. Since $\|\delta_A\| \leq 2\|A\|$ sufficiency is proved. Suppose $\|\delta_A\| \leq 2\|A\|$ and so there exist u_n and X_n such that

$$\|u_n\| = \|X_n\| = 1$$

and

$$\|AX_nu_n\| \to \|A\|.$$

Moreover, since

$$\|(AX_n - X_nA)u_n\| \to 2\|A\|, AX_nu_n = -X_nAu_n + \overline{\lambda}_n$$

where $\|\overline{\lambda}\| \to 0$. Let $(Au_n, u_n) \to \alpha$ by choosing subsequence if necessary i.e $\alpha \in W_0(A)$. Observe that

$$(AX_nu_n, X_nu_n) = -(X_nA, X_n^*X_nu_n) = -(Au_n, u_n) + \lambda'_n.$$

Thus

$$\lim_{n\to\infty} (AX_nu_n, X_nu_n) = -\alpha.$$

Since α and $-\alpha \in W_0(A)$, it implies that $0 \in W_0(A)$. □

Theorem 3. *Let $\|T - A\| \leq \delta$. Then*

$$|C_T - C_A| \leq \frac{(\delta + [\delta^2 + 8\delta] \|T - C_T\|^{\frac{1}{2}})}{2}$$

where C_A is the center of mass of operator A. In this sense, the map $A \to C_A$ is continuous in the uniform operator topology.

Proof. We let $C_A = 0$, then

$$\begin{aligned} \|A\|^2 &\geq |C_A|^2 + \|A - C_A\|^2 \\ &\geq |C_A|^2 + \|T - C_A\|^2 - 2\delta\|T - C_A\| + \delta^2 \\ &\geq 2|C_A|^2 + \|T\|^2 - 2\delta(\|T\| + |C_A|) + \delta^2 \\ &\geq \|A\|^2 + (2|C_A|^2 - 2\delta|C_A| - 4\delta\|T\|). \end{aligned}$$

Solving for C_A in the last expression on the right,we conclude that

$$\frac{(\delta + [\delta^2 + 8\delta\|T\|^{\frac{1}{2}}])}{2}.$$

□

Lemma 3. *$W_0(A) \bigcap W_0(A + \beta) = \phi$, for any $\beta \in \mathbb{C}, \beta = 0$.*

Proof. Let

$$\alpha \in W_0(A) \bigcap W_0(A + \beta).$$

Then

$$\|A\| + |\lambda|^2 + 2Re\overline{\lambda}\beta \leq \|A + \lambda\|$$

for $\lambda \in \mathbb{C}$, and

$$\|A + \beta\|^2 + |\lambda|^2 + 2Re\overline{\lambda}\alpha \leq \|A + \beta + \lambda\|^2, \lambda \in \mathbb{C}.$$

Letting $\lambda = \beta$ in the first inequality, we obtain

$$\|A + \beta\|^2 + |\beta|^2.$$

Let $\lambda = -\beta$ in the second inequality, we obtain

$$\|A + \beta\|^2 + |\beta|^2 - 2Re\overline{\beta}\alpha \leq \|A\|^2.$$

Combining these yields $|\beta|^2 \leq 0$, which completes the proof. □

Theorem 4. *Let δ_A be a derivation on $B(H)$. Then,*

$$\|\delta_A\| = \sup\{\|AX - XA\| : X \in B(H), \|X\| = 1\} = \inf_{\lambda \in \mathbb{C}} 2\|A - \lambda\|.$$

Proof. Since

$$\|AX - XA\| = \|(A - \lambda)X + X(A - \lambda)\| \leq 2\|A - \lambda\|\|X\|,$$

it follows that

$$\|\delta_T\| \leq \inf_{\lambda \in \mathbb{C}} 2\|A - \lambda\|.$$

On the other hand, $\|A - \lambda\|$ is large for λ large, so $\inf \|A - \lambda\|$ must be taken at some point, say s_0. But

$$\|A - s_0\| \leq \|(A - s_0)\| \leq \|(A - s_0) + \lambda\|,$$

for all $\lambda \in \mathbb{C}$ implies that $0 \in W_0(A - s_0)$.
Hence,

$$\|\delta_A\| = \|\delta_{A-s_0}\| = 2\|A - s_0\|.$$

□

Proposition 5. *Let $0 \leq C \leq I$ and $0 \leq D \leq I$. Then $ReCD \geq \frac{-1}{8}$. More generally,*

$$ReCD \geq l_1 l_2 - (L_1 - l_1)(L_2 - l_2)/8$$

for $0 \leq l_1 \leq C \leq L_1$ and $0 \leq l_2 \leq D \leq L_2$.

Proof. Let

$$Cu = \beta u + \lambda v,$$

where $(u, v) = 0$ and $\|u\| = \|v\| = 1$.
Let $(Cv, v) = \mu$. Then, $|\lambda| \leq \beta\mu$, since $C \geq 0$ and

$$|\lambda|^2 \leq (1 - \beta)(1 - \mu),$$

since $I - C \geq 0$. Combining these yields,

$$|\lambda| \leq \beta(1 - \beta).$$

Let

$$Du = \gamma u + \tau s$$

where (u,s)=0.
By a similar argument,

$$|\tau|^2 \leq \gamma(1 - \gamma).$$

Since,

$$(CDu, u) = \beta\gamma + \tau\overline{\lambda}(s, v),$$

it follows that

$$Re(CDu, u) = \beta\gamma - [\beta\gamma(1 - \beta)(1 - \gamma)]^{\frac{1}{2}}$$

and a standard argument shows that the last term has a minimum of $\frac{-1}{8}$
for $0 \leq \beta \leq 1, 0 \leq \gamma \leq 1$.
These estimates are sharp. For instance, if

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 1/4 & \sqrt{3/4} \\ \sqrt{3/4} & 3/4 \end{pmatrix},$$

then $Re(CDu, u) = \frac{-1}{8}$ for suitable chosen u. □

Lemma 4. *$ReW_0(T) \leq b$. Then, given $\delta > 0$, there exists a $\delta > 0$, there a $\varepsilon > 0$, such that*

$$ReW_0(T + \lambda) < b + \delta, |\lambda| < \varepsilon.$$

Proof. Assume, without loss of generality, that $\|T\| = 1$. Let

$$\gamma = \sup\{\|Tu\| : \|u\| = 1, Re(Tu, u) \geq b + \delta\}.$$

It is clear that

$$\|T + \lambda\| \geq 1 - |\lambda|.$$

However, for $v \in H$ when $\|v\| = 1$ and

$$Re(Tv, v) \geq b + \delta,$$

we see that,

$$\|(T + \lambda)v\|^2 \leq \gamma^2 + 2|\lambda| + |\lambda|^2.$$

Thus for

$$|\lambda| < (1-\gamma^2),$$

it follows that

$$ReW_0(T+\lambda) < b+\delta.$$

□

Theorem 5. *Let G be an irreducible C^*-algebra on H. Let $A \in G(H)$. Then*

$$\|\delta_A|G\| = \sup\{\|AX-XA\| : X \in G, \|X\| = 1\} = \inf_{\lambda\in\mathbb{C}} 2\|A-\lambda\|.$$

Proof. We use the fact that $B(H)$ contains an operator T such that $Tu = u, Tv = -v$ and $\|T\| = 1$ for any $u, v \in H$ where $\langle u, v\rangle = 0$. However, if G is an irreducible C^*-algebra then there exists a unitary operator $R \in G$ such that $Ru = u$ and $Rv = -v$ whenever $\langle u, v\rangle = 0$.The rest of the proof carries over with only trivial modifications which we shall omit. □

Corollary 6. *Let G_B be an irreducible C^*-algebra on the Hilbert space H_β for β in the index set N. Let $G = \sum_\beta \oplus G_\beta$ on $H = \sum_\beta \oplus H_\beta$ where $\|X\| = \sup_\beta \|X_\beta\|$ for $X \in G$. for $X \in G$. Let $A \in B(H)$, and let $\delta_A : G - G$. Then*

$$\|\delta_A\| = \sup \|AX-XA\| : X \in H, \|X\| = 1 = \inf\{2\|A-N\| : N \in B(G)\}$$

where $B(G)$ is the centre of G.

Proof. Since $\delta_A : G - G$ then $A = \sum \oplus A_\beta$ where $A \in B(H_\beta)$. For each β choose λ_β such that

$$\|\delta_{A_\beta}\| = 2\|A-\lambda_\beta\|.$$

Then Note that the corollary is not true if we hold our conditions on G. For instance let G contains an operator valued 2×2 matrices on $H \oplus H$ of the form $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, where $X \in B(H)$. Then, $\delta_A : G \to G$. Indeed, $\delta_A = \delta_0$, and so $\|\delta_A\| = 0$. But, $\inf_{\lambda\in\mathbb{C}}\{\|A-N\| : N \in B(G)\} = 1$. □

Lemma 5. *Suppose that neither S nor T is a scalar multiple of the identity. Then*

$$\inf\{\|S-\lambda\| + \|T-\lambda\|\} = \|S-\lambda_0\| + \|T-\lambda_0\|$$

if and only if

$$W_N(S-\lambda_0) \bigcap W_N(-(T-\lambda_0)) \neq \phi.$$

Proof. Let $W_N(S-\lambda_0) \bigcap W_N(-(T-\lambda_0)) \neq \phi$. Then

$$\begin{aligned} \|\delta_{ST}\| &= \|\delta_{(S-\lambda_0),(T-\lambda_0)}\| \\ &= \|S-\lambda_0\| + \|T-\lambda_0\| \end{aligned}$$

Since

$$\begin{aligned} \|SK-KT\| &= \|(S-\lambda)K - K(T-\lambda)\| \\ &\leq \|S-\lambda\| + \|T-\lambda\| \\ &\leq \inf_{\lambda\in\mathbb{C}}\{\|S-\lambda\| + \|T-\lambda\|\} \end{aligned}$$

hence the necessity is shown.
For sufficiency, we assume without loss of generality that $\lambda_0 = 0$. This means there is $\lambda, \varepsilon \geq 0$ such that there exists $u, v \in H$ of unit norm, so that

$$\|(S+\lambda)u\| + \|(T+\lambda)v\| \geq \|S\| + \|T\| - \varepsilon.$$

After some algebra, we find that

$$Re\overline{\lambda}[(Su,u)/\|S\| + (Tv,v)/\|T\|] \leq B(|\lambda|^2 + \varepsilon)$$

where B is a constant, independent of λ and ε. Suppose

$$W_N(S) \bigcap W_N(-T) \neq \phi.$$

Then the distinct

$$[W_N(S), W_N(-T)] = \delta > 0$$

and by continuity,

$$dist[W_N(S+\lambda), W_N(-(T+\lambda))] > \frac{\delta}{2},$$

for small λ. Thus by convexity and continuity, any choice of u, v which satisfies the above conditions, must satisfy the inequality $|(Su,u)/\|S\| + (Tv,v)/\|T\|| \geq \frac{\delta}{4}$ for λ small. But then we are lead to the inequality

$$|\lambda| \leq B(|\lambda|^2 + \varepsilon)$$

for a suitable choice of $\arg\lambda$ and a small $|\lambda|$, which is impossible. Thus it is a contradiction since λ was not minimal, hence the proof. □

Proposition 6. *Let $B \in J(E)$ where E is a complex Hilbert space of* $\dim \geq 2$. *Then*

$$0 \in W_0(B) \Longleftrightarrow \sup\{\|BY - YB\|, \|y\| = 1\} = 2\|B\|.$$

Proof. Let $0 \in W_0$ and let $\{y_n\}$ be a sequence such that

$$y_k = 1, \lim_k \|By_k\| = \|B\|, \lim_k (By_n, y_n) = 0.$$

Associate to each k a subspace F_k of dim $= 2$. Let $\{y_k, y'_k\}$ be orthonormal basis of F_k and

$$Y_k = y_k \otimes y_k - y'_k \otimes y'_k$$

Then we have

$$Y_k y_k = y_k, Y_k y'_k = -y_k, Y_k x = (x, y_k)y_k - (x, y'_k)y_k, \|y_k\| = 1, \forall x \in J.$$

Then

$$(By_k - Y_k B)y_k = 2(By_k - (By_k, y_k)y_k$$

and

$$\sup\{\|BY - YB\|, \|Y\| = 1\} \geq \sup_k \|(BY_k - Y_k B)y_k\| = 2\|B\|.$$

Since

$$\|BY - YB\| \leq 2\|B\|, \forall Y \in J,$$

we have

$$\sup\{\|BY - YB\|, \|Y\| = 1\} = 2\|B\|.$$

Let

$$2\|B\| = \sup\{\|BY - YB\| : \|y\| = 1\}$$

and let $\{X_k\}$ be a normal sequence of $J(E)$ such that

$$0 \leq 2\|B\| - \|BX_k - X_k B\| \leq \frac{1}{k}.$$

For each k there exists $y_n \in H$ with $\|x_k\| = 1$ and

$$0 \leq \|BX_k - X_k B\| - \|(BX_k X_k A)x_k\| \leq \frac{1}{k}.$$

Put

$$r_k = (BX_k)x_k, S_k = (X_kB)x_k.$$

Then

$$\|r_k\| \leq \|B\|, \|s_k\| \leq \|B\|$$

and

$$0 \leq 2\|B\| - \|r_k - s_k\| \leq \frac{2}{k}.$$

It results that

$$\lim_k \|r_k\| = \|B\|, \lim_k \|X_kX_k\| = 1, \lim_k \|s_k\| = \|B\|, \lim_k \|Bx_k\| = 1.$$

Remark that

$$\begin{aligned} \lim_n (r_k, s_k) + (s_k, r_k) &= \lim_k (\|r_k - s_k\|^2 - \|r_k\|^2 - \|s_k\|^2) \\ &= -2\|B\|^2. \end{aligned}$$

We deduce that

$$\lim_k (r_k + s_k) = 0.$$

□

Proposition 7. *Let $C, D \in B(E)$ such that $C \neq 0, D \neq 0$ with $\dim \geq 2$, where E is a Hilbert space. Then the following conditions are equivalent*
(i)$W_N(C) \bigcap W_N(-D) \neq \phi$,
(ii) There exists a sequence of operators $\{U_k\}$ in $B(E)$ such that

$$\|U_k\| = 1, \lim_k \|CU_k - U_kD\| = \|C\| + \|D\|,$$

(iii) $\|C\| + \|D\| \leq \|C + \lambda\|^k + \|D + \lambda\|$.

Proof. (i)⇒ (ii): Let $\beta \in W_N(C) \bigcap W_N(-D)$. Let's consider two sequences u_k, v_k in E satisfying

$$\|u_k\|, \lim_k \|Cu_k\| = \|C\|, \lim_k (Cu_k, u_k) = \beta\|C\|$$

$$\|v_k\| = 1, \lim_k \| - Dv_k\| = \|D\|, \lim_k (-Dv_k, v_k) = \beta\|C\|.$$

We construct normed U_k's of rank atmost equal to two such that

$$\lim_k \|CU_k - U_kD\| = \|C\| + \|D\|$$

by the same way as studied for $W_0(C)$. Let $F_k = Vect\{u_k, x_k\}$, where $\{u_k, x_k\} \geq 0$ and let $J_k = Vect\{v_k, y_k\}$ where $\{v_k, x_k\}$ is an orthonormal sequence,

$$Dv_k \in J_k, (-Dv_k, y_k) \geq 0.$$

If we take $U = u_k \otimes v_k + x_k \otimes y_k$, then

$$\begin{aligned} (CU_k - U_kD)v_k &= Cu_k + ((-Dv_k, v_k)v_k + (-Dv_k, y_k)v_k)) \\ &= ((u_k, u_k) + (-Dv_k, v_k))u_k + (Cu_k, x_k) + (-Dv_k, y_k)x_k. \end{aligned}$$

Since

$$\lim_k (Cu_k, u_k) = \beta\|C\|, \lim_k \|Cu_k\| = \|C\|,$$

hence

$$\lim_k (Cu_k, u_k) = (1 - \beta^2)^{\frac{1}{2}}\|D\|.$$

Then

$$\lim_k \|(CU_k - U_kD)v_k\|^2 = \beta^2(\|C\| + \|D\|)^2 + (1-\beta^2)(\|C\| + \|D\|)^2$$
$$= (\|C\| + \|D\|)^2$$

Therefore

$$\|C\| + \|D\| \le \lim_k \|CU_k - U_kD\| \le \|C\| + \|D\|.$$

Hence

$$\lim_k \|CU_k - U_kD\| = \|C\| + \|D\|.$$

(ii)⇒(iii): Let $\{U_k\}$ be such that

$$\|U_k\| = 1, \lim_k \|CU_k - U_kD\| = \|C\| + \|D\|.$$

Since

$$\|CU_k - U_kD\| = \|(C+\lambda)U_k - U_k(D+\lambda)\|$$
$$\le \|C+\lambda\| + \|D+\lambda\|, \forall \lambda \in \mathbb{C}.$$

we have

$$\|C\| + \|D\| \le \|C+\lambda\| + \|D+\lambda\|, \forall \lambda \in \mathbb{C}.$$

□

Proposition 8. *Let $B \in J(E)$ where E is a complex Hilbert space of* dim ≥ 2. *Then*

(i) $0 \in W_0(B) \Rightarrow \|B\|^2 + |\lambda|^2 \le \|B + \lambda I\|^2, \forall \lambda \in \mathbb{C}$
(ii) $\|B\| \le \|B + \lambda I\| \Rightarrow 0 \in W_0(B)$
(iii) $\forall B \in B(H)$, *there exists a unique λ_0 such that* $\|B - \lambda_0 I\| \le \|A - \lambda I\|, \forall\, \lambda \in \mathbb{C}$

Proof. (i) Assume that $0 \in W_0(B)$.Let $\{y_k\}$ be a normed sequence of E such that

$$\lim_k (\|B^*B\| - B^*B)y_k) = 0, \lim_k (By_k, y_k) = 0.$$

Then

$$\lim_k \|(B+\lambda)y_k\|^2 = \|B^*B\| + |\lambda|^2.$$

Therefore

$$\|B+\lambda\|^2 \ge \|B\|^2 + |\lambda|^2.$$

(ii) Suppose that $0 \in W_0(B)$. We prove that there exists $\lambda \in \mathbb{C}$ such that $\|B+\lambda\| \le \|B\|$. By transformation $A\exp(i\theta)$ of A, we can suppose that $d(0, W_0(B)) = n$, where $n > 0$ and $n \in W_0(B)$. Let

$$G_n = \{y \in E : \|y\| = 1, Re(By, y) \le \frac{n}{2}\}, H_n = \{y \in E : \|y\| = 1, y \in G_n\}.$$

We have

$$\sup\{\|By\|, y \in G_n\} = \beta \le \|B\|.$$

Indeed, assume that $\beta\|B\|$. Let y_k be a sequence such that $\|y_k\| = 1$ and $\lim_k \|By_n\| = \|B\|$. It remains to extract a subsequence, which gives an element $W_0(B), \phi = \lim_k (By_k, y_k)$ be such that

$$\lambda \le \frac{1}{2}(\|B\| - \beta),$$

we have for all $y \in G_n$.

$$\|(B+\lambda)y\| \le \beta + |\lambda| \le \frac{1}{2}(\beta + \|B\|) \le \|B\|.$$

If λ is a negative real number satisfying

$$|\lambda| \leq \frac{1}{2}(\|B\| - \beta), |\lambda| \in [0, n],$$

then

$$\|B + \lambda\| = \sup\{\|(B + \lambda)y\|, y \in G_n \bigcup G_n\} \leq \|B\|.$$

We have then established from that

$$\sup\{\|BY - YB\|, \|y\| = 1\} = 2\inf\{\|B + \lambda\|, \lambda \in \mathbb{C}\}.$$

(iii)Since $\|B - \lambda\| \geq |\lambda| - \|B\|$, hence if $|\lambda| \geq 2\|B\|$, then $\|B - \lambda\| > \|B\|$ and,

$$\inf\{\|B - \lambda\|, \lambda \in \mathbb{C}\} = \inf\{\|B - \lambda\|, |\lambda| \leq 2\|B\|\}.$$

□

4. Conclusion

In this paper, we have determined the upper and lower norm estimates of derivations implemented by self-adjoint operators. We recommend more studies on their orthogonality of such derivations on self-adjoint operators.

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