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The local fractional natural transform and its applications to differential equations on Cantor sets

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Abstract: The work that we have done in this paper is the coupling method between the local fractional derivative and the Natural transform (we can call it the local fractional Natural transform), where we have provided some essential results and properties. We have applied this method to some linear local fractional differential equations on Cantor sets to get nondifferentiable solutions. The results show this transform's effectiveness when we combine it with this operator.

Keywords: local fractional calculus; local fractional Laplace transform; natural transform method; local fractional differential equations.

MSC: 44A05, 26A33, 44A20, 34K37

1. Introduction

Transformations defined by integrals play an essential role in facilitating the solution of linear differential equations, as they allow us to reduce the resolution of linear differential equations with constant coefficients to the solution of algebraic equations. Engineers also widely use them to solve differential equations and determine the transfer function of a linear system. Among the most famous transformations, we find the Laplace transform method [1], the Fourier transform method [2], the Hankel transform Method [3], the Mellin transform method [4], and other transformations have appeared in the recent period, we cite for example, the Sumudu transform method [5], the Natural transform method [6], the Ezaki transform method [7], the Aboodh transform method [8], the ZZ-transform method [9], the Shehu transform method [10] and others.

Our work in this paper is based on the Natural transform method, which was developed in 2008 by Z.H. Khan and W.A. Khan [6], and has been used by many researchers in the resolution of differential equations of integer order [11–16]. Differential equations of fractional order [17–22], and we will extend it to solve linear local fractional differential equations. We support this work with illustrative examples showing how to apply this transformation using local fractional derivatives.

The present paper has been organized as follows: Section 2 presents some basic definitions and properties of the local fractional calculus and local fractional Laplace transform method. In section 3, we present some significant results. In section 4, we apply the local fractional Natural transform method (LFNTM) to solve the proposed example. We finish our paper with a conclusion that sums up our work.

2. Some basic theory of local fractional calculus

In this section, we present the basic theory of local fractional calculus as local fractional derivative, local fractional integral and and local fractional Laplace transform method.

Definition 1. [23–25] If there exists the relation

$$|f(x) - f(x_0)| < \epsilon^{\alpha}, \ \ 0 < \alpha \leqslant 1, \tag{1}$$

with $|x-x_0|<\delta$, for $\epsilon,\delta\in\mathbb{R}_+^*$. Now $f(x)$ is called local fractional continuous, denoted by $f(x)\in C_\alpha(a,b)$ on the interval (a,b).

Definition 2. [23–25] Setting $f(x)\in C_\alpha(a,b)$, the local fractional derivative of $f(x)$ of order α at $x=x_0$ is defined as

$$f^{(\alpha)}(x)=\left.\frac{d^\alpha f}{dx^\alpha}\right|_{x=x_0}=\lim_{x\to x_0}\frac{\Delta^\alpha(f(x)-f(x_0))}{(x-x_0)^\alpha}, \tag{2}$$

where

$$\Delta^\alpha(f(x)-f(x_0))\cong\Gamma(1+\alpha)\left[(f(x)-f(x_0))\right]. \tag{3}$$

The local fractional partial differential operator of order α was given by

$$\frac{\partial^\alpha u(x_0,t)}{\partial t^\alpha}=\lim_{x\to x_0}\frac{\Delta^\alpha(u(x_0,t)-u(x_0,t_0))}{(t-t_0)^\alpha}, \tag{4}$$

where

$$\Delta^\alpha(u(x_0,t)-u(x_0,t_0))\cong\Gamma(1+\alpha)\left[u(x_0,t)-u(x_0,t_0))\right]. \tag{5}$$

Definition 3. [23–25] The local fractional integral of $f(x)$ of order α in the interval $[a,b]$ is defined as

$$\begin{aligned} {}_aI_b^{(\alpha)}f(x) &= \frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)(dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\longrightarrow 0}\sum_{j=0}^{N-1}f(t_j)(\Delta t_j)^\alpha, \end{aligned} \tag{6}$$

where $\Delta t_j=t_{j+1}-t_j$, $\Delta t=\max\{\Delta t_0,\ \Delta t_1,\ \Delta t_2,\cdots\}$ and $[t_j,t_{j+1}]$, $t_0=a$, $t_N=b$, is a partition of the interval $[a,b]$.

Theorem 1. *[23–25]) (local fractional Taylor's theorem). Suppose that $f^{(k+1)\alpha}(x)\in C_\alpha(a,b)$, for $k=0,1,....,n$ and $0<\alpha\leqslant 1$. Then, one has*

$$f(x)=\sum_{k=0}^{n}\frac{f^{k\alpha}(x_0)}{\Gamma(1+k\alpha)}(x-x_0)^{k\alpha}+\frac{f^{(n+1)\alpha}(\xi)}{\Gamma(1+(n+1)\alpha)}(x-x_0)^{(n+1)\alpha}, \tag{7}$$

with $a<x_0<\xi<x<b$, $\forall x\in(a,b)$, where

$$f^{(k+1)\alpha}(x)=\overbrace{D_x^{(\alpha)}\cdots D_x^{(\alpha)}}^{(k+1)\ times}f(x). \tag{8}$$

Proof. see [24,25]. □

Theorem 2. *Local fractional Maclaurin series of the Mittag-Leffler functions on Cantor sets is given by [23–25]*

$$E_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(1+k\alpha)},\ x\in\mathbb{R},\ \ 0<\alpha\leqslant 1, \tag{9}$$

and local fractional Maclaurin series of the Mittag-Leffler functions on Cantor sets with the parameter η reads as follows

$$E_\alpha(\eta^\alpha x^\alpha)=\sum_{k=0}^{\infty}\frac{\eta^{k\alpha}x^{k\alpha}}{\Gamma(1+k\alpha)},\ x\in\mathbb{R},\ \ 0<\alpha\leqslant 1. \tag{10}$$

Definition 4. [23,26,27] The local fractional Laplace transform of $f(x)$ of order α is defined as

$$L_\alpha \{f(x)\} = F_\alpha(s) = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(x) (dx)^\alpha. \tag{11}$$

If $LFL_\alpha \{f(x)\} = F_\alpha(s)$, the inverse formula of (11) is defined as

$$f(x) = L_\alpha^{-1} \{F_\alpha(s)\} = \frac{1}{(2\pi)^\alpha} \int_{\beta-i\infty}^{\beta+i\infty} E_\alpha(s^\alpha x^\alpha) F_\alpha(s) (ds)^\alpha, \tag{12}$$

where $f(x)$ is local fractional continuous, $s^\alpha = \beta^\alpha + i^\alpha \infty^\alpha$, and $\mathrm{Re}(s) = \beta > 0$.

Theorem 3. *[24] If $L_\alpha \{f(x)\} = F_\alpha(s)$, then one has*

$$L_\alpha \left\{ f^{(\alpha)}(x) \right\} = s^\alpha L_\alpha \{f(x)\} - f(0). \tag{13}$$

Proof. see [24] □

Theorem 4. *[24] If $L_\alpha \{f(x)\} = F_\alpha(s)$, then one has*

$$L_\alpha \{ {}_0I_x^\alpha f(x) \} = \frac{1}{s^\alpha} L_\alpha \{f(x)\}. \tag{14}$$

Proof. see [24] □

Theorem 5. *[24] If $L_\alpha \{f(x)\} = F_\alpha(s)$ and $L_\alpha \{g(x)\} = G_\alpha(s)$, then one has*

$$L_\alpha \{f(x) * g(x)\} = F_\alpha(z) G_\alpha(z), \tag{15}$$

where

$$f(x) * g(x) = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty f(t) g(x-t) (dt)^\alpha. \tag{16}$$

Proof. see [24] □

Theorem 6. *[28] Suppose that $f(x) \in C_\alpha[a,b]$, then there is a function*

$$\Pi(x) = {}_aI_x^{(\alpha)} f(x),$$

the function has its derivative with respect to $(dx)^\alpha$,

$$\frac{d^\alpha \Pi(x)}{(dx)^\alpha} = f(x),\ a \leqslant x \leqslant b.$$

Proof. see [28] □

3. Main Result

In this section, we derive the local fractional Natural transform method ($LFNT$) and some properties are discussed.

If there is a new transform operator $LFN_\alpha : f(x) \longrightarrow R_\alpha(s,u)$, namely,

$$LFN_\alpha \{f(x)\} = LFN_\alpha \left\{ \sum_{k=0}^\infty a_k x^{k\alpha} \right\} = \sum_{k=0}^\infty \Gamma(1+k\alpha)\, a_k \frac{u^{k\alpha}}{s^{(k+1)\alpha}}. \tag{17}$$

If $f(x) = E_\alpha(i^\alpha x^\alpha)$, we obtain

$$
\begin{aligned}
LFN_\alpha\left\{E_\alpha(i^\alpha x^\alpha)\right\} &= LFN_\alpha\left\{\sum_{k=0}^{\infty}\frac{i^{k\alpha}x^{k\alpha}}{\Gamma(1+k\alpha)}\right\}\\
&= \sum_{k=0}^{\infty}\frac{i^{k\alpha}u^{k\alpha}}{s^{(k+1)\alpha}},
\end{aligned}
\tag{18}
$$

and if $f(x)=\frac{x^\alpha}{\Gamma(1+\alpha)}$, we get

$$
LFN_\alpha\left\{\frac{x^\alpha}{\Gamma(1+\alpha)}\right\}=\frac{u^\alpha}{s^{2\alpha}}. \tag{19}
$$

These results can be generalized by providing the following definition.

Definition 5. The local fractional Natural transform of $f(x)$ of order α is defined as

$$
LFN_\alpha\left\{f(x)\right\} = R_\alpha(s,u)=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty E_\alpha(-s^\alpha x^\alpha)f(ux)(dx)^\alpha,\ 0<\alpha\leqslant 1. \tag{20}
$$

The inverse transformation can be obtained as follows

$$
LFN_\alpha^{-1}\left\{ R_\alpha(s,u)\right\}=f(x). \tag{21}
$$

Theorem 7. *(linearity). If $LFN_\alpha\left\{f(x)\right\} = R_\alpha(s,u)$, and $LFN_\alpha\left\{g(x)\right\} = T_\alpha(s,u)$ then one has*

$$
LFN_\alpha\left\{f(x)+g(x)\right\}=R_\alpha(s,u)+T_\alpha(s,u). \tag{22}
$$

Proof. Using the formula (20) of the definition, we can easily prove the theorem. □

Theorem 8. *(local fractional Natural-Laplace and Laplace-Natural duality). If $L_\alpha\left\{f(x)\right\} = F_\alpha(s)$ and $LFN_\alpha\left\{g(x)\right\} = R_\alpha(s,u)$, then one has*

$$
LFN_\alpha\left\{f(x)\right\}=\frac{1}{u^\alpha}F_\alpha\left(\frac{s}{u}\right). \tag{23}
$$

$$
L_\alpha\left\{f(x)\right\}=u^\alpha R_\alpha(su,u). \tag{24}
$$

Proof. We show formula (23) and using the formula (20) gives

$$
LFN_\alpha\left\{f(x)\right\}=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty E_\alpha(-s^\alpha x^\alpha)\frac{f(ux)}{u^\alpha}(udx)^\alpha.
$$

By substituting $v=ux$ and $dv=udx$ in the previous formula, we get

$$
\begin{aligned}
LFN_\alpha\left\{f(x)\right\} &= \frac{1}{u^\alpha}\frac{1}{\Gamma(1+\alpha)}\int_0^\infty E_\alpha(-\left(\frac{s}{u}\right)^\alpha v^\alpha)f(v)(dv)^\alpha\\
&= \frac{1}{u^\alpha}\frac{1}{\Gamma(1+\alpha)}\int_0^\infty E_\alpha(-\left(\frac{s}{u}\right)^\alpha x^\alpha)f(x)(dx)^\alpha\\
&= \frac{1}{u^\alpha}F_\alpha\left(\frac{s}{u}\right).
\end{aligned}
$$

Proof of the formula (24). We have

$$
L_\alpha\left\{f(x)\right\}=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty E_\alpha(-s^\alpha x^\alpha)f(x)(dx)^\alpha.
$$

By substituting $x=uv$ and $dx=udv$, we get

$$\begin{aligned} L_\alpha \{f(x)\} &= \frac{1}{\Gamma(1+\alpha)} \int_0^\infty E_\alpha(-s^\alpha u^\alpha v^\alpha) f(uv)(udv)^\alpha \\ &= u^\alpha \frac{1}{\Gamma(1+\alpha)} \int_0^\infty E_\alpha(-(su)^\alpha v^\alpha) f(uv)(dv)^\alpha. \\ &= u^\alpha R_\alpha(su,u). \end{aligned}$$

This and the proof. □

Theorem 9. *(local fractional Natural transform of local fractional derivative). If $LFN_\alpha \{f(x)\} = R_\alpha(s,u)$, then one has*

$$LFN_\alpha \left\{D_{0+}^\alpha f(x)\right\} = \frac{s^\alpha}{u^\alpha} R_\alpha(s,u) - \frac{1}{u^\alpha} f(0),\ 0 < \alpha \leqslant 1, \tag{25}$$

and

$$LFN_\alpha \left\{D_{0+}^{n\alpha} f(x)\right\} = \frac{s^{n\alpha}}{u^{n\alpha}} R_\alpha(s,u) - \sum_{k=0}^{n-1} \frac{s^{(n-k-1)\alpha}}{u^{(n-k)\alpha}} f^{(k\alpha)}(0),\ 0 < \alpha \leqslant 1. \tag{26}$$

Proof. The idea of the proof of formula (25) is found in [29].
To demonstrate the validity of the formula (26), we use the mathematical induction.
If $n = 1$ and according to formula (26), we obtain

$$LFN_\alpha \left\{D_{0+}^\alpha f(x)\right\} = \frac{s^\alpha}{u^\alpha} R_\alpha(s,u) - \frac{1}{u^\alpha} f(0),$$

so, according to the (25), we note that the formula holds when $n = 1$.
Assume inductively that the formula holds for n, so that

$$LFN_\alpha \left\{D_{0+}^{n\alpha} f(x)\right\} = \frac{s^{n\alpha}}{u^{n\alpha}} R_\alpha(s,u) - \sum_{k=0}^{n-1} \frac{s^{(n-k-1)\alpha}}{u^{(n-k)\alpha}} f^{(k\alpha)}(0), \tag{27}$$

and show that it stays true at rank $n+1$. Let $D_{0+}^{n\alpha} f(x) = g(x)$and according to (25) and (27), we have

$$\begin{aligned} LFN_\alpha \left[D_{0+}^{(n+1)\alpha} f(x)\right] &= LFN_\alpha \left[D_{0+}^\alpha g(x)\right] = \frac{s^\alpha}{u^\alpha} G_\alpha(s,u) - \frac{1}{u^\alpha} g(0) \\ &= \frac{s^\alpha}{u^\alpha} \left[\frac{s^{n\alpha}}{u^{n\alpha}} R_\alpha(s,u) - \sum_{k=0}^{n-1} \frac{s^{(n-k-1)\alpha}}{u^{(n-k)\alpha}} f^{(k\alpha)}(0)\right] - \frac{1}{u^\alpha} g(0) \\ &= \frac{s^{(n+1)\alpha}}{u^{(n+1)\alpha}} R_\alpha(s,u) - \sum_{k=0}^{n-1} \frac{s^{(n-k)\alpha}}{u^{(n-k+1)\alpha}} f^{(k\alpha)}(0) - \frac{1}{u^\alpha} D_{0+}^{n\alpha} f(0) \\ &= \frac{s^{(n+1)\alpha}}{u^{(n+1)\alpha}} R_\alpha(s,u) - \sum_{k=0}^{n} \frac{s^{(n-k)\alpha}}{u^{(n-k+1)\alpha}} f^{(k\alpha)}(0). \end{aligned}$$

Thus by the principle of mathematical induction, the formula (26) holds for all $n \geqslant 1$.
□

Theorem 10. *(Local fractional Natural transform of local fractional integral). If $LFN_\alpha \{f(x)\} = R_\alpha(s,u)$, then one has*

$$LFN_\alpha \left\{{}_0 I_x^{(\alpha)} f(x)\right\} = \frac{u^\alpha}{s^\alpha} R_\alpha(s,u). \tag{28}$$

Proof. Let $h(x) = {}_0 I_x^{(\alpha)} f(x)$. According to the (theorem 3.2.9 in [28]), we get

$$D_{0+}^\alpha h(x) = f(x), \tag{29}$$

and $h(0)=0$.
Taking the local fractional Natural transform on both sides of (29), we have

$$LFN_\alpha\left\{D_{0+}^\alpha h(x)\right\}=LFN_\alpha^+\left\{f(x)\right\}.$$

Which give

$$\frac{s^\alpha}{u^\alpha}LFN_\alpha^+\left\{h(x)\right\}=R_\alpha(s,u),$$

because $h(0)=0$, and $LFN_\alpha\left\{f(x)\right\}=R_\alpha(s,u)$. Thus, we get

$$LFN_\alpha\left\{{}_0I_x^{(\alpha)}f(x)\right\}=\frac{u^\alpha}{s^\alpha}R_\alpha(s,u).$$

□

Theorem 11. *(local fractional convolution). If* $LFN_\alpha\left\{f(x)\right\}=R_\alpha(s,u)$ *and* $LFN_\alpha\left\{g(x)\right\}=T_\alpha(s,u)$*, then one has*

$$LFN_\alpha\left\{f(x)*g(x)\right\}=u^\alpha R_\alpha(s,u)T_\alpha(s,u),$$

where

$$f(x)*g(x)=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty f(\theta)g(r-\theta)(dr)^\alpha.$$

Proof. Using the two formulas (15) and (23) give

$$\begin{aligned}LFN_\alpha\left\{f(x)*g(x)\right\}&=\frac{1}{u^\alpha}L_\alpha\left\{f(x)*g(x)\right\}\\&=\frac{1}{u^\alpha}L_\alpha\left\{f(x)\right\}L_\alpha\left\{g(x)\right\}\\&=\frac{1}{u^\alpha}u^\alpha LFN_\alpha^+\left\{f(x)\right\}u^\alpha LFN_\alpha^+\left\{g(x)\right\}\\&=u^\alpha R_\alpha(s,u)T_\alpha(s,u).\end{aligned}$$

□

Table1: Local fractional Natural transform of some special functions

$f(x)$	$LFN_\alpha\left\{f(x)\right\}$	Remarks
c	$\frac{c}{s^\alpha}$	c is a constant
$\frac{x^\alpha}{\Gamma(1+\alpha)}$	$\frac{u^\alpha}{s^{2\alpha}}$	
$E_\alpha\left(ax^\alpha\right)$	$\frac{1}{s^\alpha-au^\alpha}$	$E_\alpha\left(ax^\alpha\right)=\sum_{k=0}^{\infty}\frac{a^{k\alpha}x^{k\alpha}}{\Gamma(1+k\alpha)}$
$x^\alpha E_\alpha\left(ax^\alpha\right)$	$\frac{u^\alpha}{(s^\alpha-au^\alpha)^2}$	
$\sin_\alpha(ax^\alpha)$	$\frac{au^\alpha}{s^{2\alpha}+au^{2\alpha}}$	$\sin_\alpha(x^\alpha)=\sum_{m=0}^{+\infty}(-1)^m\frac{x^{(2m+1)\alpha}}{\Gamma(1+(2m+1)\alpha)}$
$\cos_\alpha\left(ax^\alpha\right)$	$\frac{s^\alpha}{s^{2\alpha}+au^{2\alpha}}$	$\cos_\alpha\left(x^\alpha\right)=\sum_{m=0}^{+\infty}(-1)^m\frac{x^{2m\alpha}}{\Gamma(1+2m\alpha)}$
$\sinh_\alpha(ax^\alpha)$	$\frac{au^\alpha}{s^{2\alpha}-a^2u^{2\alpha}}$	$\sinh_\alpha(ax^\alpha)=\sum_{m=0}^{+\infty}\frac{x^{(2m+1)\alpha}}{\Gamma(1+(2m+1)\alpha)}$
$\cosh_\alpha\left(ax^\alpha\right)$	$\frac{s^\alpha}{s^{2\alpha}-a^2u^{2\alpha}}$	$\cosh_\alpha\left(ax^\alpha\right)=\sum_{m=0}^{+\infty}\frac{x^{2m\alpha}}{\Gamma(1+2m\alpha)}$
$E_\alpha(bx^\alpha)\sin_\alpha(ax^\alpha)$	$\frac{au^\alpha}{(s^\alpha-bu^\alpha)^2+a^2u^{2\alpha}}$	
$E_\alpha(bx^\alpha)\cos_\alpha(ax^\alpha)$	$\frac{s^\alpha-bu^\alpha}{(s^\alpha-bu^\alpha)^2+a^2u^{2\alpha}}$	
$E_\alpha(bx^\alpha)\sinh_\alpha(ax^\alpha)$	$\frac{au^\alpha}{(s^\alpha-bu^\alpha)^2-a^2u^{2\alpha}}$	
$E_\alpha(bx^\alpha)\cosh_\alpha\left(ax^\alpha\right)$	$\frac{s^\alpha-bu^\alpha}{(s^\alpha-bu^\alpha)^2-a^2u^{2\alpha}}$	
$\frac{E_\alpha(bx^\alpha)-E_\alpha(ax^\alpha)}{b-a};a\neq b$	$\frac{u^\alpha}{(s^\alpha-bu^\alpha)(s^\alpha-au^\alpha)}$	

Remark 1. The results recorded in this table are the result of using the formula (17) and the Mc-Laurin's series.

4. Applications to local fractional differential equations

In this section, we will apply the local fractional Natural transform $(LFNT)$ to some suggested local fractional differential equations.

Example 1. Let's take the following initial value problem with local fractional derivative

$$\begin{cases} \frac{d^{\alpha}y(x)}{dx^{\alpha}} + y(x) = 0, \\ y(0) = 1, \end{cases} \tag{30}$$

where $0 < \alpha \leqslant 1$.

Taking local fractional Natural transform on both sides of given equation, we have

$$\frac{s^{\alpha}}{u^{\alpha}} LFN_{\alpha}\{y(x)\} - \frac{1}{u^{\alpha}} y(0) + LFN_{\alpha}\{y(x)\} = 0. \tag{31}$$

Then

$$\left(\frac{s^{\alpha}}{u^{\alpha}} + 1\right) LFN_{\alpha}\{y(x)\} = \frac{1}{u^{\alpha}}. \tag{32}$$

Which give

$$LFN_{\alpha}\{y(x)\} = \frac{1}{s^{\alpha} + u^{\alpha}}. \tag{33}$$

By applying the inverse transformation on both sides of equation (33) and using the previous table, we get

$$y(x) = E_{\alpha}(-x^{\alpha}), \tag{34}$$

which is the exact solution of the problem (30).

Example 2. Next, we consider the following local fractional differential equation

$$\frac{d^{\alpha}y(x)}{dx^{\alpha}} + 2y(x) = 4, \tag{35}$$

where $0 < \alpha \leqslant 1$, and subject to the initial condition

$$y(0) = 3. \tag{36}$$

Taking the local fractional Natural transform on both sides of quation (35), we have

$$\frac{s^{\alpha}}{u^{\alpha}} LFN_{\alpha}\{y(x)\} - \frac{1}{u^{\alpha}} y(0) + 2LFN_{\alpha}\{y(x)\} = LFN_{\alpha}\{4\}. \tag{37}$$

By following the same steps as the previous example, we get

$$LFN_{\alpha}\{y(x)\} = \frac{1}{s^{\alpha} + 2u^{\alpha}} + \frac{2}{s^{\alpha}}. \tag{38}$$

Take the inverse transformation on both sides of equation (38),we get

$$y(x) = E_{\alpha}(-2x^{\alpha}) + 2. \tag{39}$$

Result (35) represents the exact solution to the equation (35).

Example 3. Finally, we consider the following local fractional differential equation

$$\frac{d^{2\alpha}y(x)}{dx^{2\alpha}} + y(x) = -\frac{x^{\alpha}}{\Gamma(1+\alpha)}, \tag{40}$$

where $0 < \alpha \leqslant 1$, with the initial conditions

$$y(0) = 0, \; y^{(\alpha)}(0) = 0. \tag{41}$$

Taiking local fractional Natural transform on both sides of equation (40), we have

$$\frac{s^{2\alpha}}{u^{2\alpha}} LFN_\alpha \{y(x)\} + LFN_\alpha \{y(x)\} = -LFN_\alpha \left\{ \frac{x^\alpha}{\Gamma(1+\alpha)} \right\}. \quad (42)$$

By following the steps of the first example, we get

$$LFN_\alpha \{y(x)\} = \frac{u^\alpha}{s^{2\alpha} + u^{2\alpha}} - \frac{u^\alpha}{s^{2\alpha}}. \quad (43)$$

By taking the inverse transformation of the last formula, we obtain the exact solution to the equation (40), which is

$$y(x) = \sin_\alpha(x^\alpha) - \frac{x^\alpha}{\Gamma(1+\alpha)}. \quad (44)$$

5. Conclusion

In this work, we have combined the Natural transform with the local fractional derivative, and we present some significant results of this combination. This method was applied to some linear local fractional differential equations, so the solutions were accurate and of the type of non-differentiable functions. From these results, this method is efficient in solving linear local fractional differential equations of this type.

Conflicts of Interest: The author declares no conflict of interest.

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