

# Some new results of Ostrowski type inequalities using 4-step quadratic kernel and their applications

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**Abstract:** This work is a generalization of Ostrowski type integral inequalities using a special 4-step quadratic kernel. Some new and useful results are obtained. Applications to Quadrature Rules and special Probability distribution are also evaluated.

**Keywords:** Coefficient bounds; Fekete-Szego inequalities;  $p$ -valent functions

**MSC:** 42C15, 42C40, 47A05, 54A35.

## 1. Introduction

Mathematical inequalities constitute a highly versatile field with widespread applications in Mathematics, Statistics, and various other scientific domains. Notably, numerous specialized inequalities, such as the Cauchy-Schwartz inequality, Minkowski's inequality, Hermite-Hadamard inequality, and the Heisenberg Uncertainty Principle, have significantly influenced the trajectory of science. Owing to their extensive utility, various categories of inequalities have emerged, including integral, differential, and fractional inequalities.

One noteworthy subclass of integral inequalities is the Ostrowski type inequality, which finds broad and remarkable applications across multiple branches of mathematics, encompassing evaluation and analysis (for detailed references, see, for example, [1]-[9]). Ostrowski type inequalities play a pivotal role in diverse mathematical fields, such as error estimation in Numerical Integration and Probability Theory. In the realm of approximation theory, these inequalities serve to gauge the precision of polynomial or spline approximations to a given function within a specified interval.

With this context established, we now proceed to present our primary findings.

## 2. MAIN RESULTS

We develop a special 4-step quadratic kernel to produce new inequalities of Ostrowski type:

**Lemma 1.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f'$  is absolutely continuous on  $[a, b]$  for all  $x \in \left[a, \frac{a+b}{2}\right]$ . Then the following identity holds.

$$\begin{aligned} & \frac{1}{b-a} \int_a^b P(x, t) g''(t) dt \\ &= 2\tilde{a}g(x) - (\tilde{a}+b)g\left(\frac{\tilde{a}+x}{2}\right) + (\tilde{a}-b)g(\tilde{a}+b-x) \\ &+ \frac{1}{4}(8\tilde{a}^2 - 8\tilde{a}x + 4\tilde{a}b)g'(x) - \frac{1}{4}\left[(2\tilde{a}+b)^2 - (\tilde{a}^2 + 2\tilde{a}x + 2bx)\right] \\ &g'\left(\frac{\tilde{a}+x}{2}\right) + \frac{1}{4}g'(\tilde{a}+b-x)\left[(\tilde{a}+b)^2 - 4\tilde{a}(\tilde{a}-x) - 4bx\right] + \frac{1}{b-a} \int_a^b g(t) dt. \end{aligned} \quad (1)$$

**Proof.** Define  $P(x, t)$  as:

$$P(x, t) = \begin{cases} \frac{1}{2} (t - a)^2, & t \in (a, \frac{a+x}{2}] , \\ \frac{1}{2} \left(t - \frac{3a+b}{4}\right)^2, & t \in (\frac{a+x}{2}, x] , \\ \frac{1}{2} \left(t - \frac{a+b}{2}\right)^2, & t \in (x, a+b-x] , \\ \frac{1}{2} (t - b)^2, & t \in (a+b-x, b] \end{cases} \quad (2)$$

for all  $x \in [a, \frac{a+b}{2}]$ .

Consider

$$\begin{aligned} \frac{1}{b-a} \int_a^b P(x, t) g''(t) dt &= \frac{1}{b-a} \left[ \int_a^{\frac{a+x}{2}} \frac{1}{2} (t-a)^2 g''(t) dt + \int_{\frac{a+x}{2}}^x \frac{1}{2} \left(t - \frac{3a+b}{4}\right)^2 g''(t) dt \right. \\ &\quad \left. + \int_x^{a+b-x} \frac{1}{2} \left(t - \frac{a+b}{2}\right)^2 g''(t) dt + \int_{a+b-x}^b \frac{1}{2} (t-b)^2 g''(t) dt \right]. \end{aligned}$$

Integrating by parts we get the required result (1) which completes the proof.  $\square$

Let's produce new results by applying three different constraints on  $g''$  and  $g'''$ .

## 2.1. Case-I

**Theorem 1.** Let  $g: [a, b] \rightarrow \mathbb{R}$  is differentiable on  $(a, b)$ ,  $g'$  be absolutely continuous on  $[a, b]$  and  $\gamma \leq g''(t) \leq \Gamma, \forall t \in [a, b]$ , then  $\forall x \in [a, \frac{a+b}{2}]$ :

$$\begin{aligned} &\left| 2ag(x) - (a+b)g\left(\frac{a+x}{2}\right) + (a-b)g(a+b-x) \right. \\ &\quad \left. + \frac{1}{4}(8a^2 - 8ax + 4ab)g'(x) - \frac{1}{4}[(2a+b)^2 - (a^2 + 2ax + 2bx)] \right. \\ &\quad \left. g'\left(\frac{a+x}{2}\right) + \frac{1}{4}g'(a+b-x) \right. \\ &\quad \left. \times [(a+b)^2 - 4a(a-x) - 4bx] + \frac{g'(b) - g'(a)}{(b-a)^2} \times \frac{1}{48(b-a)} \left[ 9(x-a)^3 + \left(8x - \frac{3a+b}{2}\right)^3 - (x - (2a+b))^3 \right] \right. \\ &\quad \left. - \frac{1}{b-a} \int_a^b g(t) dt \right| \leq v(x)(b-a)(S - \gamma) \end{aligned} \quad (3)$$

and

$$\begin{aligned} &\left| 2ag(x) - (a+b)g\left(\frac{a+x}{2}\right) + (a-b)g(a+b-x) \right. \\ &\quad \left. + \frac{1}{4}(8a^2 - 8ax + 4ab)g'(x) - \frac{1}{4}[(2a+b)^2 - (a^2 + 2ax + 2bx)] \right. \\ &\quad \left. g'\left(\frac{a+x}{2}\right) + \frac{1}{4}g'(a+b-x) [(a+b)^2 - 4a(a-x) - 4bx] \right. \\ &\quad \left. + \frac{g'(b) - g'(a)}{(b-a)^2} \times \frac{1}{48(b-a)} \left[ 9(x-a)^3 + \left(8x - \frac{3a+b}{2}\right)^3 - (x - (2a+b))^3 \right] \right. \\ &\quad \left. - \frac{1}{b-a} \int_a^b g(t) dt \right| \leq v(x)(b-a)(\Gamma - S), \end{aligned} \quad (4)$$

where

$$S = \frac{g'(b) - g'(a)}{b - a} \quad (5)$$

and

$$v(x) = \frac{1}{48} \max \left\{ \left| 22a^3 + 16x^3 + 63ax^2 - 81a^2x + 21a^2b - 3ab^2 + 21bx^2 + 9b^2x - 60abx \right|, \left| -26a^3 - 16x^3 + 6b^3 + 45ax^2 - 57a^2x + 33a^2b - 33ab^2 + 39bx^2 + 24abx \right|, \left| 25a^3 + 16x^3 + 57ax^2 - 81a^2x + 24a^2b - 6ab^2 + 27bx^2 + 3b^2x - 60abx \right|, \left| -28a^3 + 16x^3 - 69ax^2 - 93a^2x - 15a^2b - 3ab^2 - 15bx^2 + 9b^2x + 48abx \right| \right\}. \quad (6)$$

**Proof.** We know that

$$\frac{1}{b-a} \int_a^b g''(t) dt = \frac{g'(b) - g'(a)}{b-a}.$$

Now consider

$$\begin{aligned} \frac{1}{b-a} \int_a^b P(x, t) dt &= \frac{1}{b-a} \left[ \int_a^{\frac{a+x}{2}} \frac{1}{2} (t-a)^2 dt + \int_{\frac{a+x}{2}}^x \frac{1}{2} \left( t - \frac{3a+b}{4} \right)^2 dt \right. \\ &\quad \left. + \int_x^{a+b-x} \frac{1}{2} \left( t - \frac{a+b}{2} \right)^2 dt + \int_{a+b-x}^b \frac{1}{2} (t-b)^2 dt \right] \end{aligned}$$

By integrating, we get

$$\begin{aligned} &\frac{1}{b-a} \int_a^b P(x, t) dt \\ &= \frac{1}{6(b-a)} \left[ \left( \frac{a+x}{2} - a \right)^3 + \left( x - \frac{3a+b}{2} \right)^3 + \left( \frac{x+2a+b}{2} \right)^3 \right. \\ &\quad \left. - \left( \frac{a+b-x}{2} - \frac{a+b}{2} \right)^3 - \left( x - \frac{a+b}{2} \right)^3 \right] - (a-x)^3 \\ &= \frac{1}{6(b-a)} \left[ \left( \frac{x-a}{2} \right)^3 + \left( \frac{2x-3a-b}{2} \right)^3 - \left( \frac{x-2a-b}{2} \right)^3 \right. \\ &\quad \left. + \left( \frac{a+b-2x}{2} \right)^3 - \left( \frac{2x-a-b}{2} \right)^3 - (a-x)^3 \right] \\ &= \frac{1}{48(b-a)} \left[ (x-a)^3 + (2x-3a-b)^3 - (x-2a-b)^3 \right. \\ &\quad \left. - (2x-a-b)^3 - 8(a-x)^3 \right] \\ &= \frac{1}{48(b-a)} \left[ 9(x-a)^3 + 8 \left( x - \frac{3a+b}{2} \right)^3 - (x-(2a+b))^3 \right]. \end{aligned} \quad (7)$$

Now we proceed by considering

$$\begin{aligned} &\frac{1}{b-a} \int_a^b P(x, t) g''(t) dt - \frac{1}{(b-a)^2} \int_a^b P(x, t) dt \int_a^b g''(t) dt \\ &= 2a g'(x) - (a+b) g' \left( \frac{a+x}{2} \right) + (a-b) g'(a+b-x) \\ &\quad + \frac{1}{4} (8a^2 - 8ax + 4ab) g'(x) - \frac{1}{4} [(2a+b)^2 - (a^2 + 2ax + 2bx)] \end{aligned}$$

$$\begin{aligned} & \check{g}' \left( \frac{\check{a} + x}{2} \right) + \frac{1}{4} \check{g}' (\check{a} + b - x) \left[ (\check{a} + b)^2 - 4\check{a}(\check{a} - x) - 4bx \right] \\ & + \frac{1}{b - \check{a}} \int_{\check{a}}^b \check{g}(\check{t}) - \frac{\check{g}'(b) - \check{g}'(\check{a})}{b - \check{a}} \\ & \times \frac{1}{48(b - \check{a})} \left[ 9(x - \check{a})^3 + 8 \left( x - \frac{3\check{a} + b}{2} \right)^3 - (x - (2\check{a} + b))^3 \right]. \end{aligned}$$

Define

$$R_n(x) = \frac{1}{b - \check{a}} \int_{\check{a}}^b P(x, \check{t}) \check{g}''(\check{t}) d\check{t} - \frac{1}{(b - \check{a})^2} \int_{\check{a}}^b P(x, \check{t}) d\check{t} \int_{\check{a}}^b \check{g}''(\check{t}) d\check{t}.$$

Let  $C \in R$  be a constant of arbitrary nature, then we have

$$\begin{aligned} R_n(x) & \\ & = \frac{1}{b - \check{a}} \int_{\check{a}}^b (\check{g}''(\check{t}) - C) \left[ P(x, \check{t}) - \frac{1}{b - \check{a}} \int_{\check{a}}^b P(x, s) ds \right] d\check{t} \end{aligned} \quad (8)$$

which leads to the following relation:

$$\begin{aligned} |R_n(x)| & \\ & \leq \frac{1}{b - \check{a}} \max_{\check{t} \in [\check{a}, b]} \left| P(x, \check{t}) - \frac{1}{b - \check{a}} \int_{\check{a}}^b P(x, s) ds \right| \\ & \times \int_{\check{a}}^b |\check{g}''(\check{t}) - C| d\check{t}. \end{aligned} \quad (9)$$

We also have

$$\begin{aligned} & \max \left| P(x, \check{t}) - \frac{1}{b - \check{a}} \int_{\check{a}}^b P(x, s) ds \right| \\ & = \max \left\{ \left| \frac{1}{2} \left( \frac{x - \check{a}}{2} \right)^2 - \frac{\lambda(x)}{b - \check{a}} \right|, \left| \frac{1}{2} \left( x - \frac{3\check{a} + b}{2} \right)^2 - \frac{\lambda(x)}{b - \check{a}} \right|, \right. \\ & \left. \left| \frac{1}{2} \left( x - \frac{\check{a} + b}{2} \right)^2 - \frac{\lambda(x)}{b - \check{a}} \right|, \left| \frac{\lambda(x)}{b - \check{a}} \right| \right\} \end{aligned}$$

where

$$\begin{aligned} \lambda(x) & \\ & = \frac{1}{48(b - \check{a})} \left[ 9(x - \check{a})^3 + 8 \left( x - \frac{3\check{a} + b}{2} \right)^3 - (x - (2\check{a} + b))^3 \right]. \end{aligned}$$

Consider

$$\begin{aligned}
 & \left| \frac{1}{2} \left( \frac{x - \check{a}}{2} \right)^2 - \frac{\lambda(x)}{b - \check{a}} \right| \\
 &= \left| \frac{x^2 + \check{a}^2 - 2\check{a}x}{8} - \frac{1}{48(b - \check{a})} \right. \\
 & \quad \left. \left[ 9(x - \check{a})^3 + 8 \left( x - \frac{3\check{a} + b}{2} \right)^3 - (x - (2\check{a} + b))^3 \right] \right| \\
 &= \frac{1}{48(b - \check{a})} \left| 22\check{a}^3 + 16x^3 + 63\check{a}x^2 - 81\check{a}^2x + 21\check{a}^2b - 3\check{a}b^2 \right. \\
 & \quad \left. + 21bx^2 + 9b^2x - 60\check{a}bx \right|,
 \end{aligned} \tag{10}$$

similarly, we obtain

$$\begin{aligned}
 & \left| \frac{1}{2} \left( x - \frac{3\check{a} + b}{2} \right)^2 - \frac{\lambda(x)}{b - \check{a}} \right| \\
 &= \left| \frac{4x^2 + \check{a}^2 + b^2 - 4\check{a}x + 4bx - 2\check{a}b}{8} - \frac{1}{48(b - \check{a})} \right. \\
 & \quad \left. \left[ 9(x - \check{a})^3 + 8 \left( x - \frac{3\check{a} + b}{2} \right)^3 - (x - (2\check{a} + b))^3 \right] \right| \\
 &= \frac{1}{48(b - \check{a})} \left| -26\check{a}^3 - 16x^3 + 6b^3 + 45\check{a}x^2 - 57\check{a}^2x + 33\check{a}^2b \right. \\
 & \quad \left. - 33\check{a}b^2 + 39bx^2 + 24\check{a}bx \right|
 \end{aligned} \tag{11}$$

and

$$\begin{aligned}
 & \left| \frac{1}{2} \left( x - \frac{\check{a} - b}{2} \right)^2 - \frac{\lambda(x)}{b - \check{a}} \right| \\
 &= \left| \frac{4x^2 + \check{a}^2 + b^2 - 4\check{a}x + 4bx - 2\check{a}b}{8} - \frac{1}{48(b - \check{a})} \right. \\
 & \quad \left. \left[ 9(x - \check{a})^3 + 8 \left( x - \frac{3\check{a} + b}{2} \right)^3 - (x - (2\check{a} + b))^3 \right] \right| \\
 &= \frac{1}{48(b - \check{a})} \left| 25\check{a}^3 + 16x^3 + 57\check{a}x^2 - 81\check{a}^2x + 24\check{a}^2b - 6\check{a}b^2 \right. \\
 & \quad \left. + 27bx^2 + 3b^2x - 60\check{a}bx \right|.
 \end{aligned} \tag{12}$$

We can also conclude:

$$\begin{aligned}
 & \left| \frac{\lambda(x)}{b - \check{a}} \right| \\
 &= \frac{1}{48(b - \check{a})} \left| -28\check{a}^3 + 16x^3 - 69\check{a}x^2 - 93\check{a}^2x - 15\check{a}^2b \right. \\
 & \quad \left. - 3\check{a}b^2 - 15bx^2 + 9b^2x + 48\check{a}bx \right|.
 \end{aligned} \tag{13}$$

With the combination of (10) till (13) we get (6).

Furthermore,

$$\int_{\check{a}}^b |\check{g}''(\check{t}) - \gamma| d\check{t} = (S - \gamma)(b - \check{a}) \tag{14}$$

$$\int_{\check{a}}^b |\check{g}''(\check{t}) - \Gamma| d\check{t} = (\Gamma - S)(b - \check{a}). \tag{15}$$

We have obtained (3) and (4) by combining (7) till (15).  $\square$

## 2.2. Case-II

**Theorem 2.** Let  $g: [a, b] \rightarrow \mathbb{R}$  be a three times differentiable function on  $(a, b)$ . If  $g''' \in L^2[a, b]$ , then  $\forall x \in \left[a, \frac{a+b}{2}\right]$ :

$$\begin{aligned} & \left| 2ag(x) - (a+b)g\left(\frac{a+x}{2}\right) + (a-b)g(a+b-x) \right. \\ & + \frac{1}{4}(8a^2 - 8ax + 4ab)g'(x) - \frac{1}{4}[(2a+b)^2 - \\ & (a^2 + 2ax + 2bx)] \\ & \times g'\left(\frac{a+x}{2}\right) + \frac{1}{4}g'(a+b-x)[(a+b)^2 - 4a(a-x) \\ & 4bx + \frac{g'(b) - g'(a)}{(b-a)^2} \\ & \times \frac{1}{48(b-a)} \left[ 9(x-a)^3 + \left(8x - \frac{3a+b}{2}\right)^3 \right. \\ & \left. \left. - (x - (2a+b))^3 \right] - \frac{1}{b-a} \int_a^b g(t) dt \right| \\ & \leq \frac{1}{\pi} \|g'''\|_2 \left\{ \frac{33}{160}(x-a)^5 + 32\left(x - \frac{3a+b}{2}\right)^5 \right. \\ & \left. - (x - (2a+b))^5 + 64\left(x - \frac{a+b}{2}\right)^5 \right\} \times \frac{1}{b-a} \left\{ 9(x-a)^3 \right. \\ & \left. + \left(8x - \frac{3a+b}{2}\right)^3 - (x - (2a+b))^3 \right\} \\ & \left. - \frac{1}{b-a} \left\{ 9(x-a)^3 + \left(8x - \frac{3a+b}{2}\right)^3 - (x - (2a+b))^3 \right\} \right\}^{\frac{1}{2}}. \end{aligned} \quad (16)$$

**Proof.** Let  $R_n(x)$  be given as

$$\begin{aligned} R_n(x) &= \frac{1}{b-a} \int_a^b P(x, t) g''(t) dt - \frac{1}{(b-a)^2} \int_a^b P(x, t) dt \int_a^b g''(t) dt \\ &= \left[ 2ag(x) - (a+b)g\left(\frac{a+x}{2}\right) + (a-b) \right. \\ & g(a+b-x) + \frac{1}{4}(8a^2 - 8ax + 4ab)g'(x) - \frac{1}{4}[(2a+b)^2 \\ & - (a^2 + 2ax + 2bx)]g'\left(\frac{a+x}{2}\right) + \frac{1}{4}g'(a+b-x)[(a+b)^2 \\ & - 4a(a-x) - 4bx] + \frac{g'(b) - g'(a)}{(b-a)^2} \\ & \times \frac{1}{48(b-a)} \left[ 9(x-a)^3 + \left(8x - \frac{3a+b}{2}\right)^3 \right. \\ & \left. - (x - (2a+b))^3 \right] - \frac{1}{b-a} \int_a^b g(t) dt \\ & \left. - (x - (2a+b))^3 \right] - \frac{1}{b-a} \int_a^b g(t) dt. \end{aligned}$$

By choosing  $C = \check{g}''\left(\frac{\check{a}+\flat}{2}\right)$  and using Cauchy's Inequality, we get

$$\begin{aligned} & |R_n(x)| \\ & \leq \frac{1}{\flat-\check{a}} \int_{\check{a}}^{\flat} \left| \check{g}''(\check{t}) - \check{g}''\left(\frac{\check{a}+\flat}{2}\right) \right| |p(x, \check{t}) \\ & \quad - \frac{1}{\flat-\check{a}} \int_{\check{a}}^{\flat} p(x, s) ds| d\check{t} \\ & \leq \frac{1}{\flat-\check{a}} \left[ \int_{\check{a}}^{\flat} \left( \check{g}''(\check{t}) - \check{g}''\left(\frac{\check{a}+\flat}{2}\right) \right)^2 d\check{t} \right]^{\frac{1}{2}} \\ & \quad \times \left[ \int_{\check{a}}^{\flat} \left( P(x, \check{t}) - \frac{1}{\flat-\check{a}} \int_{\check{a}}^{\flat} P(x, s) ds \right)^2 d\check{t} \right]^{\frac{1}{2}}. \end{aligned}$$

In the same way, using Diaz-Metcalf inequality, we obtain

$$\int_{\check{a}}^{\flat} \left( \check{g}''(\check{t}) - \check{g}''\left(\frac{\check{a}+\flat}{2}\right) \right)^2 d\check{t} \leq \frac{(\flat-\check{a})^2}{\pi^2} \|\check{g}'''\|_2^2 \quad (17)$$

and

$$\begin{aligned} & \int_{\check{a}}^{\flat} \left( p(x, \check{t}) - \frac{1}{\flat-\check{a}} \int_{\check{a}}^{\flat} p(x, s) ds \right)^2 d\check{t} \\ & = \int_{\check{a}}^{\flat} P(x, \check{t})^2 d\check{t} - \frac{1}{\flat-\check{a}} \left\{ 9(x-\check{a})^3 + \left( 8x - \frac{3\check{a}+\flat}{2} \right)^3 - (x-(2\check{a}+\flat))^3 \right\} \\ & = \frac{1}{160} \left[ \left\{ 33(x-\check{a})^5 + 32 \left( x - \frac{3\check{a}+\flat}{2} \right)^5 - (x-(2\check{a}+\flat))^5 \right. \right. \\ & \quad \left. \left. - 64 \left( x - \frac{\check{a}+\flat}{2} \right)^5 \right\} \right. \\ & \quad \left. - \frac{1}{\flat-\check{a}} \left\{ 9(x-\check{a})^3 + \left( 8x - \frac{3\check{a}+\flat}{2} \right)^3 - (x-(2\check{a}+\flat))^3 \right\} \right] \quad (18) \end{aligned}$$

Therefore, using the above relations (17)-(18), we obtain (16) which completes proof.  $\square$

### 2.3. Case-III

**Theorem 3.** Let  $\check{g} : [\check{a}; \flat] \rightarrow \mathbb{R}$  be a function which is absolutely continuous on  $(\check{a}, \flat)$  such that  $\check{g}'' \in L^2[\check{a}, \flat]$ . Then  $\forall x \in \left[\check{a}, \frac{\check{a}+\flat}{2}\right]$ :

$$\begin{aligned}
 & \left| 2\check{a}\check{g}(x) - (\check{a} + \flat)\check{g}\left(\frac{\check{a}+x}{2}\right) + (\check{a} - \flat)\check{g}(\check{a} + \flat - x) \right. \\
 & + \frac{1}{4}(8\check{a}^2 - 8\check{a}x + 4\check{a}\flat)\check{g}'(x) - \frac{1}{4}[(2\check{a} + \flat)^2 - (\check{a}^2 + 2\check{a}x + 2\flat x)] \\
 & \check{g}'\left(\frac{\check{a}+x}{2}\right) + \frac{1}{4}\check{g}'(\check{a} + \flat - x)[(\check{a} + \flat)^2 - 4\check{a}(\check{a} - x) - 4\flat x] \\
 & \left. + \frac{\check{g}'(\flat) - \check{g}'(\check{a})}{(\flat - \check{a})^2} \times \frac{1}{48(\flat - \check{a})} \right. \\
 & \times \left[ 9(x - \check{a})^3 + \left(8x - \frac{3\check{a} + \flat}{2}\right)^3 - (x - (2\check{a} + \flat))^3 \right] - \frac{1}{\flat - \check{a}} \int_{\check{a}}^{\flat} \check{g}(\check{t}) d\check{t} \Bigg| \\
 & \leq \frac{\sqrt{\sigma(\check{g}'')}}{\flat - \check{a}} \\
 & \times \left[ \left\{ \frac{33}{160}(x - \check{a})^5 + 32\left(x - \frac{3\check{a} + \flat}{2}\right)^5 - (x - (2\check{a} + \flat))^5 \right. \right. \\
 & \left. \left. 64\left(x - \frac{\check{a} + \flat}{2}\right)^5 \right\} \times \frac{1}{\flat - \check{a}} \left\{ 9(x - \check{a})^3 + \left(8x - \frac{3\check{a} + \flat}{2}\right)^3 \right. \right. \\
 & \left. \left. - (x - (2\check{a} + \flat))^3 \right\}^3 \right. \\
 & \left. - \frac{1}{\flat - \check{a}} \left\{ 9(x - \check{a})^3 + \left(8x - \frac{3\check{a} + \flat}{2}\right)^3 - (x - (2\check{a} + \flat))^3 \right\} \right]^{\frac{1}{2}}
 \end{aligned} \tag{19}$$

where

$$\begin{aligned}
 \sigma(\check{g}'') &= \|\check{g}''\|_2^2 - \frac{(\check{g}'(\flat) - \check{g}'(\check{a}))^2}{\flat - \check{a}} \\
 &= \|\check{g}''\|_2^2 - S^2(\flat - \check{a}),
 \end{aligned}$$

where

$$S = \frac{\check{g}'(\flat) - \check{g}'(\check{a})}{\flat - \check{a}}.$$

**Proof.** Let  $R_n(x)$  be given as in (??), By choosing  $C = \frac{1}{\flat - \check{a}} \int_{\check{a}}^{\flat} \check{g}''(s) ds$  in (8) and using Cauchy's inequality and (18), we get

$$\begin{aligned}
 & |R_n(x)| \\
 & \leq \frac{1}{\flat - \check{a}} \int_{\check{a}}^{\flat} \left| \check{g}''(\check{t}) - \check{g}''\left(\frac{\check{a} + \flat}{2}\right) \right| \left| P(x, \check{t}) - \frac{1}{\flat - \check{a}} \int_{\check{a}}^{\flat} P(x, s) ds \right| d\check{t} \\
 & \leq \frac{1}{\flat - \check{a}} \left[ \int_{\check{a}}^{\flat} \left( \check{g}''(\check{t}) - \check{g}''\left(\frac{\check{a} + \flat}{2}\right) \right)^2 d\check{t} \right]^{\frac{1}{2}} \\
 & \times \left[ \int_{\check{a}}^{\flat} \left( P(x, \check{t}) - \frac{1}{\flat - \check{a}} \int_{\check{a}}^{\flat} P(x, s) ds \right)^2 d\check{t} \right]^{\frac{1}{2}}.
 \end{aligned}$$



$$\begin{aligned}
&= \frac{\sqrt{\sigma(\check{g}'')}}{b-\check{a}} \left[ \left\{ \frac{33}{160} (x-\check{a})^5 + 32 \left( x - \frac{3\check{a}+b}{2} \right)^5 \right. \right. \\
&\quad \left. \left. - (x-(2\check{a}+b))^5 \right\} \times \frac{1}{b-\check{a}} \left\{ 9(x-\check{a})^3 \right. \right. \\
&\quad \left. \left. + \left( 8x - \frac{3\check{a}+b}{2} \right)^3 - (x-(2\check{a}+b))^3 \right\} \right. \\
&\quad \left. - \frac{1}{b-\check{a}} \left\{ 9(x-\check{a})^3 + \left( 8x - \frac{3\check{a}+b}{2} \right)^3 - (x-(2\check{a}+b))^3 \right\} \right]^{\frac{1}{2}}.
\end{aligned}$$

which completes the proof.  $\square$

### 3. An application to Composite Quadrature Rules

Let  $I_n : \check{a} = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$  be a distribution of the interval  $[\check{a}, b]$  and  $\check{\varsigma} = (\check{\varsigma}_0, \check{\varsigma}_1, \dots, \check{\varsigma}_{n-1})$  a sequence of intermediate points  $\check{\varsigma}_i \in [x_i, x_{i+1}]$  and  $h = x_{i+1} - x_i$  ( $i = 0, 1, \dots, n-1$ ).

Then:

$$\begin{aligned}
&\check{a}_n(\check{g}, \check{g}', \check{\varsigma}, I_n) \\
&= \left| 2x_i \check{g}(\check{\varsigma}_i) - (x_i + x_{i+1}) \check{g}\left(\frac{x_i + \check{\varsigma}_i}{2}\right) + h_i \check{g}(x_i + x_{i+1} - \check{\varsigma}_i) \right. \\
&\quad \left. + \frac{1}{4} (8x_i^2 - 8x_i \check{\varsigma}_i + 4x_i x_{i+1}) \check{g}'(\check{\varsigma}_i) - \frac{1}{4} [(2x_i + x_{i+1})^2 \right. \\
&\quad \left. - (x_i^2 + 2x_i \check{\varsigma}_i + 2x_{i+1} \check{\varsigma}_i)] \right. \\
&\quad \check{g}'\left(\frac{x_i + \check{\varsigma}_i}{2}\right) + \frac{1}{4} \check{g}'(x_i + x_{i+1} - \check{\varsigma}_i) [(x_i + x_{i+1})^2 - 4x_i(x_i - \check{\varsigma}_i) \\
&\quad \left. - 4x_{i+1} \check{\varsigma}_i] + \frac{\check{g}'(x_{i+1}) - \check{g}'(x_i)}{h_i^2} \right. \\
&\quad \left. \times \frac{1}{48h_i} \left[ 9(\check{\varsigma}_i - x_i)^3 + \left( 8\check{\varsigma}_i - \frac{3x_i + x_{i+1}}{2} \right)^3 \right. \right. \\
&\quad \left. \left. - (\check{\varsigma}_i - (2x_i + x_{i+1}))^3 \right] \right|
\end{aligned} \tag{20}$$

for all  $\check{\varsigma}_i \in \left[ x_i, \frac{x_i + x_{i+1}}{2} \right]$ .

**Theorem 4.** Let  $\check{g} : [\check{a}; b] \rightarrow R$  be such that  $\check{g}'$  is absolutely continuous function. If  $\check{g}'' \in L^1[\check{a}, b]$  and  $\gamma \leq \check{g}''(x) \leq \Gamma$ , then :

$$\begin{aligned}
&\int_{\check{a}}^b \check{g}(\check{t}) d\check{t} = \check{a}_n(\check{g}, \check{g}', \check{\varsigma}, I_n) + R_n^1(\check{g}, \check{g}', \check{\varsigma}, I_n), \\
&\int_{\check{a}}^b \check{g}(\check{t}) d\check{t} = \check{a}_n(\check{g}, \check{g}', \check{\varsigma}, I_n) + R_n^2(\check{g}, \check{g}', \check{\varsigma}, I_n).
\end{aligned}$$

Where  $\check{a}_n(\check{g}, \check{g}', \check{\varsigma}, I_n)$  is given in (20) and remainder satisfies:

$$\left| R_n^1(\check{g}, \check{g}', \check{\varsigma}, I_n) \right| \leq (S - \gamma) \sum_{i=0}^{n-1} h_i^2 v(\check{\varsigma}_i) \tag{21}$$

and

$$\left| R_n^2(\check{g}, \check{g}', \check{\varsigma}, I_n) \right| \leq (\Gamma - S) \sum_{i=0}^{n-1} h_i^2 v(\check{\varsigma}_i). \tag{22}$$

**Theorem 5.** Let  $\check{g} : [\check{a}; b] \rightarrow R$  be a mapping which is three times differentiable on  $(\check{a}, b)$  with  $\check{g}''' \in L^2[\check{a}, b]$ . Then:

$$\int_{\check{a}}^b \check{g}(\check{t}) d\check{t} = \check{a}_n(\check{g}, \check{g}', \check{g}'', I_n) + R_n^3(\check{g}, \check{g}', \check{g}'', I_n)$$

where  $\check{a}_n(\check{g}, \check{g}', \check{g}'', I_n)$  is defined by formula (20) and the remainder  $R_n^3(\check{g}, \check{g}', \check{g}'', I_n)$  satisfies the estimation

$$\begin{aligned} & \left| R_n^3(\check{g}, \check{g}', \check{g}'', I_n) \right| \\ & \leq \frac{1}{\pi} \|\check{g}'''\|_2 \left\{ \left[ \frac{33}{160} (\check{s}_i - x_i)^5 + 32 \left( \check{s}_i - \frac{3x_i + x_{i+1}}{2} \right)^5 \right. \right. \\ & \quad \left. \left. - (\check{s}_i - (2x_i + x_{i+1}))^5 - 64 \left( \check{s}_i - \frac{x_i + x_{i+1}}{2} \right)^5 \right] \right\} \\ & \quad \times \frac{1}{h_i} \left\{ 9(\check{s}_i - x_i)^3 + \left( 8\check{s}_i - \frac{3x_i + x_{i+1}}{2} \right)^3 - (\check{s}_i - (2x_i + x_{i+1}))^3 \right\} \\ & \quad \left[ -\frac{1}{h_i} 9(\check{s}_i - x_i)^3 + \left( 8\check{s}_i - \frac{3x_i + x_{i+1}}{2} \right)^3 \right. \\ & \quad \left. - (\check{s}_i - (2x_i + x_{i+1}))^3 \right] \frac{1}{2}. \end{aligned} \quad (23)$$

#### 4. An Application to Cumulative Distribution Function

As we know that " If  $X$  is a random variable which has values in the interval  $[\check{a}, b]$  with the probability density function  $\check{g} : [\check{a}, b] \rightarrow [0, 1]$  and cumulative distribution function

$$\check{g}(x) = \Pr(X \leq x) = \int_{\check{a}}^x \check{g}(\check{t}) d\check{t}, \quad (24)$$

$$\check{g}(b) = \Pr(X \leq b) = \int_{\check{a}}^b \check{g}(u) du = 1''. \quad (25)$$

**Theorem 6.** Using the conditions of Theorem 2, we obtain following inequality which holds

$$\begin{aligned} & \left| \frac{b - E(X)}{b - \check{a}} - 2\check{a}\check{g}(x) - (\check{a} + b)\check{g}\left(\frac{\check{a} + x}{2}\right) + (\check{a} - b)\check{g}(\check{a} + b - x) \right. \\ & \quad + \frac{1}{4} (8\check{a}^2 - 8\check{a}x + 4\check{a}b) \check{g}'(x) - \frac{1}{4} \left[ (2\check{a} + b)^2 - (\check{a}^2 + 2\check{a}x + 2bx) \right] \\ & \quad \check{g}'\left(\frac{\check{a} + x}{2}\right) + \frac{1}{4} \check{g}'(\check{a} + b - x) \\ & \quad \times \left[ (\check{a} + b)^2 - 4\check{a}(\check{a} - x) - 4bx \right] + \frac{\check{g}'(b) - \check{g}'(\check{a})}{(b - \check{a})^2} \\ & \quad \times \frac{1}{48(b - \check{a})} \left[ 9(x - \check{a})^3 + \left( 8x - \frac{3\check{a} + b}{2} \right)^3 - (x - (2\check{a} + b))^3 \right] \left. \right| \\ & \leq v(x)(b - \check{a})(S - \gamma) \end{aligned} \quad (26)$$

$$\begin{aligned}
& \left| \frac{b-E(X)}{b-a} - 2a g(x) - (a+b) g\left(\frac{a+x}{2}\right) + (a-b) g(a+b-x) \right. \\
& + \frac{1}{4} (8a^2 - 8ax + 4ab) g'(x) - \frac{1}{4} [(2a+b)^2 - (a^2 + 2ax + 2bx)] \\
& g'\left(\frac{a+x}{2}\right) + \frac{1}{4} g'(a+b-x) \\
& \times [(a+b)^2 - 4a(a-x) - 4bx] + \frac{g'(b) - g'(a)}{(b-a)^2} \\
& \times \frac{1}{48(b-a)} \left[ 9(x-a)^3 + \left(8x - \frac{3a+b}{2}\right)^3 - (x-(2a+b))^3 \right] \Bigg| \\
& \leq v(x)(b-a)(\Gamma-S)
\end{aligned} \tag{27}$$

for all  $x$ . Where  $E(X)$  is the expectation of  $X$ .

**Theorem 7.** With the conditions of Theorem 5, we have the following inequality which holds

$$\begin{aligned}
& \left| \frac{b-E(X)}{b-a} - 2a g(x) - (a+b) g\left(\frac{a+x}{2}\right) \right. \\
& + (a-b) g(a+b-x) + \frac{1}{4} (8a^2 - 8ax + 4ab) g'(x) - \frac{1}{4} [(2a+b)^2 \\
& - (a^2 + 2ax + 2bx)] g'\left(\frac{a+x}{2}\right) + \frac{1}{4} g'(a+b-x) [(a+b)^2 \\
& - 4a(a-x) - 4bx] + \frac{g'(b) - g'(a)}{(b-a)^2} \\
& \times \frac{1}{48(b-a)} \left[ 9(x-a)^3 + \left(8x - \frac{3a+b}{2}\right)^3 - (x-(2a+b))^3 \right] \Bigg| \\
& \leq \frac{1}{\pi} \|g'''\|_2 \left[ \left\{ \frac{33}{160} (x-a)^5 + 32 \left(x - \frac{3a+b}{2}\right)^5 - (x-(2a+b))^5 \right. \right. \\
& \left. \left. 64 \left(x - \frac{a+b}{2}\right)^5 \right\} \times \frac{1}{b-a} \left\{ 9(x-a)^3 + \left(8x - \frac{3a+b}{2}\right)^3 \right. \right. \\
& \left. \left. - (x-(2a+b))^3 \right\} \right. \\
& \left. - \frac{1}{b-a} \left\{ 9(x-a)^3 + \left(8x - \frac{3a+b}{2}\right)^3 - (x-(2a+b))^3 \right\} \right]^{\frac{1}{2}}.
\end{aligned}$$

for all  $x \in \left[a, \frac{a+b}{2}\right]$ , where  $E(X)$  is the expectation of  $X$ .

**Theorem 8.** By using same conditions as of Theorem 9, we obtain the inequality:

$$\begin{aligned}
& \left| \frac{b-E(X)}{b-a} - 2a g(x) - (a+b) g\left(\frac{a+x}{2}\right) + (a-b) \right. \\
& g(a+b-x) + \frac{1}{4} (8a^2 - 8ax + 4ab) g'(x) - \frac{1}{4} [(2a+b)^2 \\
& - (a^2 + 2ax + 2bx)] g'\left(\frac{a+x}{2}\right) + \frac{1}{4} g'(a+b-x) [(a+b)^2 \\
& - 4a(a-x) - 4bx] + \frac{g'(b) - g'(a)}{(b-a)^2} \\
& \times \frac{1}{48(b-a)} \left[ 9(x-a)^3 + \left(8x - \frac{3a+b}{2}\right)^3 - (x-(2a+b))^3 \right] \Bigg|
\end{aligned} \tag{28}$$

$$\begin{aligned} &\leq \frac{\sqrt{\sigma(\check{g}'')}}{b-\check{a}} \left[ \left\{ \frac{33}{160} (x-\check{a})^5 + 32 \left( x - \frac{3\check{a}+b}{2} \right)^5 \right. \right. \\ &\quad \left. \left. - (x-(2\check{a}+b))^5 \right\} 64 \left( x - \frac{\check{a}+b}{2} \right)^5 \right] \times \frac{1}{b-\check{a}} \left\{ 9(x-\check{a})^3 \right. \\ &\quad \left. + \left( 8x - \frac{3\check{a}+b}{2} \right)^3 - (x-(2\check{a}+b))^3 \right\}^3 \\ &\quad \left. - \frac{1}{b-\check{a}} \left\{ 9(x-\check{a})^3 + \left( 8x - \frac{3\check{a}+b}{2} \right)^3 - (x-(2\check{a}+b))^3 \right\} \right]^{\frac{1}{2}}. \end{aligned}$$

**Proof.** Keeping in mind, the conditions of Theorem 19, if we put  $x = \frac{3\check{a}+b}{4}$  in (16), then:

$$\begin{aligned} &\left| \frac{b-E(X)}{b-\check{a}} - 2\check{a}\check{g}'\left(\frac{3\check{a}+b}{4}\right) - (\check{a}+b)\check{g}'\left(\frac{7\check{a}+b}{4}\right) + (\check{a}-b) \right. \\ &\quad \left. \check{g}'\left(\frac{\check{a}+3b}{4}\right) + \frac{1}{2}\check{a}(\check{a}+b)\check{g}'\left(\frac{3\check{a}+b}{4}\right) - \frac{1}{8}[(2\check{a}+b)^2 - \check{a}^2] \right. \\ &\quad \left. \check{g}'\left(\frac{7\check{a}+b}{8}\right) - \frac{1}{2}b(b-\check{a}) + \frac{\check{g}'(b) - \check{g}'(\check{a})}{(b-\check{a})^2} \right| \\ &\quad \times \frac{1}{3072(b-\check{a})} [9(b-\check{a})^3 + 216(3\check{a}+b)^3 - (5\check{a}+3b)^3] \\ &\leq \frac{\sqrt{\sigma(\check{g}'')}}{b-\check{a}} \left[ \frac{33}{160} \left( \frac{b-\check{a}}{4} \right)^5 - (3\check{a}+b)^5 - (5\check{a}+3b)^5 \right] \\ &\quad \times \frac{1}{b-\check{a}} [9(b-\check{a})^3 + 216(3\check{a}+b)^3 - (5\check{a}+3b)^3]^{\frac{1}{2}}. \end{aligned}$$

Hence the required inequality.  $\square$

## 5. Conclusion

This work explored the Ostrowski-type Inequalities by using the innovative 4-step Quadratic kernel has yielded profound insights into the realm of Mathematics, precisely to Statistics and Numerical Analysis. Through a tedious journey of theoretical development and analysis, this work has successfully established the effectiveness and versatility of 4-step Quadratic kernel. Some more developments have been done by several authors, [8]-[16].

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## References

- [1] Cerone, P. (2002). A new Ostrowski type inequality involving integral Means over end intervals. *Tamkang Journal of Mathematics*, 33(2), 109-118.
- [2] Dragömer, S. S., & Wang, S. (1997). An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules. *Computers and Mathematics with Applications*, 33(1), 16-20.
- [3] Dragömer, S. S., & Wang, S. (1998). Applications of Ostrowski inequality to the estimation of error bounds for some special means and some numerical quadrature rules. *Applied Mathematics Letters*, 11, 105-109.
- [4] Alomari, M. W. (2012). A companion of Dragomir's generalization of Ostrowski's inequality and applications in Numerical Integration. *Ukrainian Mathematical Journal*, 64(4), 491-510.
- [5] Qayyum, A., Shoaib, M., & Erden, S. (2019). Generalized fractional Ostrowski type inequality for higher order derivatives. *New Trends in Mathematical Sciences*, 4(2), 111-124.

- [6] Fahad, S., Mustafa, M. A., Ullah, Z., Hussain, T., & Qayyum, A. (2022). Weighted Ostrowski's Type Integral Inequalities for Mapping Whose First Derivative Is Bounded. *International Journal of Analysis and Applications*, 20, 16-16.
- [7] Kashif, A. R., Khan, T. S., Qayyum, A., & Faye, I. (2018). A comparison and error analysis of error bounds. *International journal of analysis and applications*, 16(5), 751-762.
- [8] Guessab, A., & Schmeisser, G. (2002). Sharp integral inequalities of the Hermite-Hadamard type. *Journal of Approximation Theory*, 115(2), 260-288.
- [9] Qayyum, A., Shoaib, M., & Faye, I. (2015). On New Weighted Ostrowski Type Inequalities Involving Integral Means Over End Intervals and Application. *Turkish Journal of Analysis and Number Theory*, 3(2), 61-67.
- [10] Amjad, J., Qayyum, A., Arslan, M., & Fahad, S. (2022). Some new generalized Ostrowski type inequalities with new error bounds. *Innovative Journal of Mathematics*, 1(2), 30-43.
- [11] Qayyum, A. (2008). A weighted Ostrowski-Grüss type inequality for twice differentiable mappings and applications. *International Journal of Computer Mathematics*, 1, 63-71.
- [12] Qayyum, A., Shoaib, M., Matouk, A. E., & Latif, M. A. (2014). On New Generalized Ostrowski Type Integral inequalities. *Abstract and Applied Analysis*, 2014, 1-8.
- [13] Qayyum, A., Shoaib, M., & Latif, M. A. (2014). A generalized inequality of ostrowski type for twice differentiable bounded mappings and applications. *Applied Mathematical Sciences*, 8(38), 1889-1901.
- [14] Budak, H., Sarikaya, M. Z. & Qayyum, A. (2021). New refinements and applications of Ostrowski type inequalities for mappings whose nth derivatives are of bounded variation. *TWMS Journal of Applied and Engineering Mathematics*, 11(2), 424-435.
- [15] Nasir, J., Qaisar, S., Butt, S. I., & Qayyum, A. (2022). Some Ostrowski type inequalities for mappings whose second derivatives are preinvex functions via fractional integral operator. *AIMS Mathematics*, (3), 3303-3320.
- [16] Iftikhar, M., Qayyum, A., Fahad, S., & Arslan, M. (2021). A new version of Ostrowski type integral inequalities for different differentiable mapping. *Open Journal of Mathematical Sciences*, 5(1), 353-359.



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