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An introduction to the construction of subfusion frames

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Abstract: Fusion frames and subfusion frames are generalizations of frames in the Hilbert spaces. In this paper, we study subfusion frames and the relations between the fusion frames and subfusion frame operators. Also, we introduce new construction of subfusion frames. In particular, we study atomic resolution of the identity on the Hilbert spaces and derive new results.

Keywords: fusion frames, subfusion frames, resolution of the identity.

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1. Introduction

The concept of *Frames* for Hilbert spaces were initiated around 1952 by Duffin and Schaeffer, who studied some problems in the nonharmonic Fourier series [1]. Some years later, in [2–4], the authors introduced a new type of generalized frames as fusion frames, Bessel subfusion sequences and subfusion frames, and so established some results.

In the present paper, we introduce the new construction of subfusion frames and derive new results. In the reminder of this section, we briefly review the concept of frames, subfusion frames and their properties. In Section 2, we introduce the new construction of subfusion frames. In Section 3, we study the atomic resolution of the identity and obtain some results about them. Finally, Section 4 contains a discussion on alternate dual subfusion frames. Through this paper, H is used to denote a separable Hilbert space, I and J are countable index sets, and $\{V_i\}_{i\in I}$ is a sequence of closed subspaces of H.

Definition 1. [2] Let $\{V_i\}_{i\in I}$ be a family of closed subspaces of a Hilbert space H and $\{\alpha_i\}_{i\in I}$ be a family of weights, i.e., $\alpha_i > 0$ for all $i \in I$.

- $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ is called a *fusion frame*, if there exist positive constants C and D (*lower and upper fusion frame bounds*, respectively) such that

$$C\|f\|^2 \leq \sum_{i\in I} \alpha_i^2 \|\pi_{V_i}(f)\|^2 \leq D\|f\|^2, \quad for\ all\ f \in H, \tag{1}$$

  where π_{V_i} is the orthogonal projection onto the subspace V_i.
- If the right-hand side of the inequality in (1) is satisfied, then ν is called a *Bessel fusion sequence* for H with Bessel bound D.
- If $H = \oplus V_i$ and $\inf_{i\in I} \alpha_i > 0$, then ν is called an *orthogonal basis* of subspaces for H.
- The *fusion frame operator* $S_\nu : H \longrightarrow H$ is defined by $S_\nu(f) = \sum\limits_{i\in I} \alpha_i{}^2 \pi_{V_i} f$, for all $f \in H$. The operator S_ν is linear, bounded, positive, self-adjoint and invertible.
- The *synthesis operator* for $\{(V_i, \alpha_i)\}_{i\in I}$ is the operator $T_\nu : \oplus V_i \longrightarrow H$ defined by $T_\nu(\{f_i\}_{i\in I}) = \sum_{i\in I} \alpha_i f_i$, and the adjoint of the synthesis operator is called the *analysis operator*.

Remark 1. The analysis operator for $\{(V_i, \alpha_i)\}_{i\in I}$ is the operator $T_\nu^\star : H \longrightarrow \oplus V_i$ defined by $T_\nu^\star(f) = \{\alpha_i \pi_{V_i}(f)\}_{i\in I}$. So for every $f \in H$ we have

$$S_\nu f = T_\nu T_\nu^\star f = \sum_{i\in I} \alpha_i^2 \pi_{V_i} f \quad and \quad f = \sum_{i\in I} \alpha_i^2 S_\nu^{-1} \pi_{V_i} f = \sum_{i\in I} \alpha_i^2 \pi_{V_i} S_\nu^{-1} f. \tag{2}$$

Here we list, for the readers convenience, several results needed for our proofs.

Proposition 1. *[2] Let $\{(V_i, \alpha_i)\}_{i\in I}$ be a Bessel fusion sequence for H, W_i be a closed subspace of V_i and $\beta_i \leq \alpha_i$ for all $i \in I$. Then $\{(W_i, \beta_i)\}_{i\in I}$ is a Bessel fusion sequence for H.*

Theorem 1. *[5] If the operator $Sf = \sum_{i\in J} \alpha_i^2 \pi_{W_i} f$ of a collection of weighted subspaces $\{(W_i, \alpha_i)\}_{i\in J}$ is an invertible operator on H, then $\{(W_i, \alpha_i)\}_{i\in J}$ is a fusion frame for H.*

We are now ready to state a definition of subfusion frame and Bessel subfusion sequence as below.

Definition 2. [2] Let $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ be a fusion frame for H, W_i be a closed subspace of V_i and $\beta_i \leq \alpha_i$ for all $i \in I$. If $\omega = \{(W_i, \beta_i)\}_{i\in I}$ is a fusion frame for H, then ω is called a *subfusion frame of* ν. If ν and ω are Bessel fusion sequences for H, then ω is called a *Bessel subfusion sequence of* ν.

Remark 2. Let $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ and $\omega = \{(W_i, \beta_i)\}_{i\in I}$ be two Bessel fusion sequences for H, then the composed or mixed frame operator for them is defined by

$$S_{\omega\nu} f = \sum_i \alpha_i \beta_i \pi_{W_i} \pi_{V_i} f, \ \ \forall f \in H.$$

If $\omega = \{(W_i, \beta_i)\}_{i\in I}$ is a subfusion frame of $\nu = \{(V_i, \alpha_i)\}$ such that C is the lower bound of ω and D is the upper bound of ν, then $CI_d \leq S_{\omega\nu} \leq DI_d$, and $S_{\omega\nu}$ is invertible (see [3, Proposition 3.2]).

Concluding this section, let us recall the following results that will be needed in the sequel.

Definition 3. [4] A family of subspaces $\{W_i\}_{i\in J}$ is called *minimal*, if

$$W_i \cap \overline{span}_{j\in J, j\neq i}\{W_j\} = \{0\}$$

for each $i \in J$. Also, and a family of subspaces $\{W_i\}_{i\in J}$ of H is called *complete*, if $\overline{span}_{i\in J}\{W_i\} = H$.

Proposition 2. *[6] Let $F : H \to H$ be invertible on H. Suppose that $G : H \to H$ is a bounded operator and $\|Gf - Ff\| \leq \lambda \|f\|$, for all $f \in H$, where $\lambda \in [0, \frac{1}{\|F^{-1}\|}]$. Then G is invertible on H and $G^{-1} = \sum_{k=0}^{\infty} [F^{-1}(F - G)]^k F^{-1}$.*

2. Fusion frames and Subfusion frames

This section will be devoted to the fusion and subfusion frames. First, we remark that the following theorem was proved in [7, Theorem 4.3], but here we give another proof with extra information about the bounds.

Theorem 2. *Let $\{(W_i, \alpha_i)\}_{i\in J}$ be a fusion frame for H with fusion frame bounds C and D. Then the following hold.*

(i) If $I_d - \alpha_j^2 \pi_{W_j} S_W^{-1}$ is a bounded and invertible operator on H for some $j \in J$, then $\{(W_i, \alpha_i)\}_{i\in J, i\neq j}$ is a fusion frame with fusion frame bounds $\frac{C}{\|(I_d - \alpha_j^2 \pi_{W_j} S_W^{-1})^{-1}\|}$ and D.

(ii) If there is some $g \in W_j$ such that $g = \alpha_j^2 \pi_{W_j} S_W^{-1} g$, then $\{(W_i, \alpha_i)\}_{i\in J, i\neq j}$ is an incomplete set in H.

Proof. (i) Since S_W is invertible, by (2), $f = \sum_{i\in J} \alpha_i^2 \pi_{W_i} S_W^{-1} f$ for all $f \in H$. Now, if we put $T_j = I_d - \alpha_j^2 \pi_{W_j} S_W^{-1}$ and $S_{W_j} f = \sum_{i\in J, i\neq j} \alpha_i^2 \pi_{W_i} f$, then we have $T_j f = S_{W_j} S_W^{-1} f$. Since T_j and S_W are bounded and invertible operators on H, S_{W_j} is a positive and bounded invertible operator on H. So by Theorem 1, $\{(W_i, \alpha_i)\}_{i\in J, i\neq j}$ is a fusion frame with fusion frame bounds $\frac{C}{\|(I_d - \alpha_j^2 \pi_{W_j} S_W^{-1})^{-1}\|}$ and D.

(ii) If there is some $g \in W_j$ such that $g = \alpha_j^2 \pi_{W_j} S_W^{-1} g$, then T_j and S_{W_j} are not invertible operators on H. So $\{(W_i, \alpha_i)\}_{i\in J, i\neq j}$ is not a fusion frame. Therefore, by [4, Proposition 3.6], $\{(W_i, \alpha_i)\}_{i\in J, i\neq j}$ is an incomplete set in H. $\square$

Remark 3. Theorem 2 shows that if an element from a fusion frame is removed, the remaining set will be either a fusion frame or an incomplete set.

Using Theorem 2, one can easily obtain the following results. For more information see [7, Corollaries 4.4 , 4.5].

Corollary 3. *Let $\{(W_i, \alpha_i)\}_{i\in J}$ be a fusion frame for H and let $j \in J$. If $\alpha_j^2 < \frac{1}{\|S_W^{-1}\|}$, then $\{(W_i, \alpha_i)\}_{i\in J, i\neq j}$ is a fusion frame with same fusion frame bounds in Theorem 2.*

Corollary 4. *Let $\{(W_i, \alpha_i)\}_{i\in J}$ be a fusion frame for H, and $\alpha_j^2 < C$ for some $j \in J$. If V_j is a closed subspace of W_j, then for any $k > 0$, $\{(W_i, \alpha_i)\}_{i\in J, i\neq j} \cup \{(V_j, k)\}$ is a subfusion frame for $\{(W_i, \alpha_i)\}_{i\in J}$.*

Lemma 1. *Let $\{(W_i, \alpha_i)\}_{i\in J}$ be a Bessel fusion sequence for H with Bessel fusion bound D. Then for any $j \in J$, $\{(W_i, \alpha_i)\}_{i\in J} \cup \{(W_j^{\perp}, \alpha_j)\}$ is a fusion frame for H with fusion frame bound α_j^2 and $D + \alpha_j^2$. Moreover, if we put $S_{W_j} f = \sum_{i\in J} \alpha_i^2 \pi_{W_i} f + \alpha_j^2 \pi_{W_j^{\perp}} f$ for any $j \in J$, then $\alpha_j^2 \geq \frac{1}{\|S_{W_j}^{-1}\|}$.*

Now we study some new constructions of subfusion frames.

Theorem 5. *Let $\omega = \{(W_i, \alpha_i)\}_{i\in I}$ be a Bessel subfusion frame of $\nu = \{(V_i, \alpha_i)\}_{i\in I}$. Suppose that there exist $\lambda_1 < 0, \lambda_2 > -1$ such that $\|f - S_\omega f\| \leq \lambda_1 \|f\| + \lambda_2 \|S_\omega f\|$, for all $f \in H$. Then ω is a subfusion frame of ν.*

Proof. Given $f \in H$. We have $\|S_\omega f\| \geq \dfrac{1-\lambda_1}{1+\lambda_2} \|f\|$ since

$$\| f \| - \| S_\omega f \| \leq \|f - S_\omega f\| \leq \lambda_1 \|f\| + \lambda_2 \|S_\omega f\|. \tag{3}$$

But $\|S_\omega f\| \leq \sqrt{D} (\sum\limits_{i\in I} \alpha_i^2 \|\pi_{W_i} f\|^2)^{\frac{1}{2}}$, where D is the upper bound of ω. So

$$\frac{1}{D}\left(\frac{1-\lambda_1}{1+\lambda_2}\right)^2 \|f\|^2 \leq \sum_{i\in I} {\alpha_i}^2 \left\|\pi_{W_i} f\right\|^2 \leq \sum_{i\in I} {\alpha_i}^2 \left\|\pi_{V_i} f\right\|^2.$$

Therefore ω is a subfusion frame of ν. □

In the following theorem we give more characterizations of subfusion frame under the application of operators.

Theorem 6. *Let $\{(W_i, \beta_i)\}_{i\in I}$ be a subfusion frame of $\nu = \{(V_i, \alpha_i)\}_{i\in I}$. If T is an invertible operator on H, then $\{(TW_i, \beta_i)\}_{i\in I}$ is a subfusion frame of $\{(TV_i, \alpha_i)\}_{i\in I}$.*

Proof. Since $W_i \subset V_i$, we imply that $TW_i \subset TV_i$. Hence $\{(TW_i, \beta_i)\}_{i\in I}$ is a subfusion frame of $\{(TV_i, \alpha_i)\}_{i\in I}$. □

Corollary 7. *Let $\{(W_i, \beta_i)\}_{i\in I}$ be a subfusion frame of $\nu = \{(V_i, \alpha_i)\}_{i\in I}$. Suppose that T is an invertible operator on H which satisfying $T^*T(V_i) \subset W_i$ for all $i \in I$. Then $\{(TW_i, \beta_i)\}_{i\in I}$ is a subfusion frame of $\{(TV_i, \alpha_i)\}_{i\in I}$, $TS_\nu T^{-1}$ is a fusion frame operator for $\{(TV_i, \alpha_i)\}_{i\in I}$, and $TS_\omega T^{-1}$ is a fusion frame operator for $\{(TW_i, \beta_i)\}_{i\in I}$.*

Proof. Since T is invertible operator and $TW_i \subset TV_i$ for all $i \in I$, then $\{(TW_i, \beta_i)\}_{i\in I}$ is a subfusion frame of $\{(TV_i, \alpha_i)\}_{i\in I}$. Also, $T^*T(V_i) \subset W_i$ implies that $T^*T(W_i) \subset W_i$ and $T^*T(V_i) \subset V_i$. Now, by [**?** , Proposition 3.11], $TS_\nu T^{-1}$ is a fusion frame operator for $\{(TV_i, \alpha_i)\}_{i\in I}$ and $TS_\omega T^{-1}$ is a fusion frame operator for $\{(TW_i, \beta_i)\}_{i\in I}$. □

Next, we provide the condition under which a fusion frame can be a subfusion frame.

Proposition 3. *Let $\{(W_i, \alpha_i)\}_{i\in I}$ be a fusion frame for H with fusion frame bounds C, D, such that $\{W_i\}_{i\in I}$ is minimal. Then $C \leq \alpha_i^2 \leq D$. Moreover, for any weights β_i, $i \in I$ such that $C < \beta_i^2 < \alpha_i^2$, $\{(W_i, \beta_i)\}_{i\in I}$ is a subfusion frame for $\{(W_i, \alpha_i)\}_{i\in I}$.*

Proof. Suppose that $\{(W_i, \alpha_i)\}_{i\in I}$ is a fusion frame for H with fusion frame bounds C, D. So we have

$$C\|f\|^2 \leq \sum_{i\in J} \alpha_i^2 \|\pi_{W_i} f\|^2 \leq D\|f\|^2, f \in H.$$

Since $\{W_i\}_{i\in I}$ is minimal, for $0 \neq f \in W_i$, $\pi_{W_i} f = f$ and $\pi_{W_j} f = 0$ for $j \neq i$. Therefore, $C\|f\|^2 \leq \sum_{i\in I} \alpha_i^2 \|\pi_{W_i} f\|^2 = \alpha_i^2 \|f\|^2 \leq D\|f\|^2$, and so $C \leq \alpha_i^2 \leq D$. □

The next proposition immediately yield by definitions.

Proposition 4. *Let $\{(W_i, \alpha_i)\}_{i\in I}$ be a subfusion frame for $\{(V_i, \alpha_i)\}_{i\in I}$. If $W_i^{\perp}$ is the orthogonal complement of W_i in V_i, then*

$$S_\nu - S_\omega = S_{\omega^\perp}, \quad T_\nu^\star - T_\omega^\star = T_{\omega^\perp}^\star \quad \text{and} \quad \|T_\omega^\star\| \leq \|T_\nu^\star\|.$$

The next result was established in [2, Theorem 3.2], we provide here a short proof for it.

Proposition 5. *Let $\{(V_i, \alpha_i)\}_{i\in I}$ be a fusion frame with lower frame bound C, and $\{(W_i, \alpha_i)\}_{i\in I}$ be a subfusion frame for $\{(V_i, \alpha_i)\}_{i\in I}$. If $W_i^{\perp}$ is the orthogonal complement of W_i in V_i and $C > \|S_\omega\|$, then $\{(W_i^{\perp}, \alpha_i)\}_{i\in I}$ is a fusion frame.*

Proof. By Proposition 4, we know $S_\nu - S_\omega = S_{\nu^\perp}$, and so

$$\begin{aligned}
(C - \|S_\omega\|)\|f\| &\leq \|S_\nu f\| - \|S_W f\| \leq \|S_\nu f - S_\omega f\| \\
&= \|S_{\omega^\perp} f\| \leq \|S_\nu f\| + \|S_\omega f\| \\
&\leq \|S_\nu\|\|f\| + \|S_\omega\|\|f\|.
\end{aligned}$$

□

The following result is a direct consequence of the previous proposition.

Corollary 8. *Let $\{(V_i, \alpha_i)\}_{i\in I}$ be a fusion frame, and W_i be a closed subspace of V_i. Suppose that $\{(W_i^{\perp}, \alpha_i)\}_{i\in I}$ is a Bessel fusion sequence where $W_i^{\perp}$ is the orthogonal complement of W_i in V_i. If $\|S_{\omega^\perp}\|\|S_\nu^{-1}\| < 1$, then $\{(W_i, \alpha_i)\}_{i\in I}$ is a fusion frame, and $S_\omega^{-1} = \sum_{k=0}^{\infty} [S_\nu^{-1} S_{\omega^\perp}]^k S_\nu^{-1}$.*

Proof. From Proposition 4, we imply that $S_\nu - S_\omega = S_{\nu^\perp}$. So

$$\|(S_\nu - S_\omega) f\| = \|S_{\omega^\perp} f\| \leq \|S_{\omega^\perp}\|\|f\| < \frac{1}{\|S_\nu^{-1}\|}\|f\|.$$

Since S_ν is an invertible operator, by Proposition 2, S_ω is invertible on H. Therefore, by Theorem 1, $\{(W_i, \alpha_i)\}_{i\in I}$ is a fusion frame and $S_\omega^{-1} = \sum_{k=0}^{\infty} [S_\nu^{-1} S_{\omega^\perp}]^k S_\nu^{-1}$. □

3. Atomic resolution of the identity

In this section, we define atomic resolution of the identity on Hilbert space, specially derive some new results about them.

We recall that a family of bounded operators $\{T_i\}_{i\in I}$ on H is called an *atomic resolution* of the identity with respect to $\{\alpha_i\}_{i\in I}$ for H if there exist positive real numbers C and D such that for all $f \in H$,

$$C\|f\|^2 \leq \sum_{i\in I} \alpha_i^2 \|T_i(f)\|^2 \leq D\|f\|^2 \ \text{ and } \ f = \sum_{i\in I} T_i(f).$$

Lemma 2. *Let $\omega = \{(W_i, \beta_i)\}_{i\in I}$ be a subfusion of $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ such that C is the lower bound of ω and D is the upper bound of ν.*

(i) Suppose that $T_i : H \to H$ is given by $T_i = \alpha_i \beta_i \pi_{W_i} S_{\omega\nu}^{-1}$ $(i \in I)$. Then $\{T_i\}_{i\in I}$ is an atomic resolution of the identity with respect to $\{\alpha_i^{\frac{-1}{2}} \beta_i^{\frac{-1}{2}}\}_{i\in I}$ on H with bounds $\frac{1}{D}$ and $\frac{1}{C}$.

(ii) Suppose that $T_i : H \to H$ is given by $T_i = \alpha_i \beta_i S_{\omega\nu}^{-1} \pi_{W_i}$ $(i \in I)$. Then $\{T_i\}_{i\in I}$ is an atomic resolution of the identity with respect to $\{\alpha_i^{\frac{-1}{2}} \beta_i^{\frac{-1}{2}}\}_{i\in I}$ on H with bounds $\frac{C}{D^2}$ and $\frac{D}{C^2}$.

Proof. (i). We have $CI_d \leq S_{\omega v} \leq DI_d$, so for all $f \in H$,

$$\begin{aligned} \frac{1}{D}\|f\|^2 &\leq \langle S_{\omega v}^{-1} f, f\rangle = \langle S_{\omega v}^{-1} f, S_{\omega v}(S_{\omega v}^{-1} f)\rangle = \sum_{i\in I} \alpha_i \beta_i \|\pi_{W_i} S_{\omega v}^{-1} f\|^2 \\ &= \sum_{i\in I} \alpha_i^{-1}\beta_i^{-1}\|T_i f\|^2 \leq \frac{1}{C}\|f\|^2. \end{aligned}$$

Thus $f = \sum_{i\in I} \alpha_i\beta_i S_{\omega v}^{-1}\pi_{W_i} f = \sum_{i\in I} T_i f$ for all $f \in H$.

(ii). For any $f \in H$ we have

$$\|S_{\omega v}\|^{-1}\|\pi_{W_i} f\| \leq \|S_{\omega v}^{-1}\pi_{W_i} f\| \leq \|S_{\omega v}^{-1}\|\|\pi_{W_i} f\|,$$

then

$$\begin{aligned} \|S_{\omega v}^{-1}\|^{-2}\sum_{i\in I}\beta_i^2\|\pi_{W_i} f\|^2 &\leq \|S_{\omega v}^{-1}\|^{-2}\sum_{i\in I}\alpha_i\beta_i\|\pi_{W_i} f\|^2 \\ &\leq \sum_{i\in I}\alpha_i\beta_i\|S_{\omega v}^{-1}\pi_{W_i} f\|^2 \\ &\leq \sum_{i\in I}\alpha_i\beta_i\|S_{\omega v}^{-1}\|^2\|\pi_{W_i} f\|^2 \\ &\leq \sum_{i\in I}\alpha_i^2\|S_{\omega v}^{-1}\|^2\|\pi_{W_i} f\|^2. \end{aligned}$$

Thus

$$\|S_{\omega v}^{-1}\|^{-2}\sum_{i\in I}\beta_i^2\|\pi_{W_i} f\|^2 \leq \sum_{i\in I}\alpha_i\beta_i\|S_{\omega v}^{-1}\pi_{W_i} f\|^2 \leq \sum_{i\in I}\alpha_i^2\|S_{\omega v}^{-1}\|^2\|\pi_{W_i} f\|^2.$$

Therefore

$$\frac{C}{D^2}\|f\|^2 \leq \sum_{i\in I}\alpha_i^{-1}\beta_i^{-1}\|T_i f\|^2 \leq \frac{D}{C^2}\|f\|^2.$$

Also for all $f \in H$ we have,

$$f = S_{\omega v} S_{\omega v}^{-1} f = \sum_{i\in I}\alpha_i\beta_i\pi_{W_i} S_{\omega v}^{-1} f = \sum_{i\in I} T_i f.$$

□

Corollary 9. *Let $\omega = \{(W_i, \beta_i)\}_{i\in I}$ be a subfusion of $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ with the lower bound C and the upper bound D, and $\{f_j\}_{j\in J}$ be a frame with frame bounds A and B.*

(i) If $T_i = \alpha_i\beta_i\pi_{W_i} S_{\omega v}^{-1}$, then $\{\alpha_i^{\frac{-1}{2}}\beta_i^{\frac{-1}{2}} T_i^*(f_j)\}_{i\in I, j\in J}$ is a frame with bounds $\frac{A}{D}$ and $\frac{B}{C}$ for H. In particular, if $\{e_j\}_{\in J}$ is an orthonormal basis for H, then
$\{\alpha_i^{\frac{-1}{2}}\beta_i^{\frac{-1}{2}} T_i^*(e_j)\}_{i\in I, j\in J}$ is a frame with frame bounds $\frac{1}{D}$ and $\frac{1}{C}$ for H.

(ii) If $T_i = \alpha_i\beta_i S_{\omega v}^{-1}\pi_{W_i}$, then $\{\alpha_i^{\frac{-1}{2}}\beta_i^{\frac{-1}{2}} T_i^*(f_j)\}_{i\in I, j\in J}$ is a frame with bounds $\frac{AC}{D^2}$ and $\frac{BD}{C^2}$ for H. In particular, if $\{e_j\}_{\in J}$ is an orthonormal basis for H, then
$\{\alpha_i^{\frac{-1}{2}}\beta_i^{\frac{-1}{2}} T_i^*(e_j)\}_{i\in I, j\in J}$ is a frame with frame bounds $\frac{C}{D^2}$ and $\frac{D}{C^2}$ for H.

Proof. (i). Since $\{f_j\}_{j\in J}$ is a frame for H with frame bounds A and B, we obtain

$$A\sum_{i\in I}\alpha_i^{-1}\beta_i^{-1}\|T_i f\|^2 \leq \sum_{i\in I}\sum_{j\in J}|\langle\alpha_i^{\frac{-1}{2}}\beta_i^{\frac{-1}{2}} T_i f, f_j\rangle|^2 \leq B\sum_{i\in I}\alpha_i^{-1}\beta_i^{-1}\|T_i f\|^2,$$

for all $f \in H$. Hence

$$\frac{A}{D}\|f\|^2 \leq \sum_{i\in I}\sum_{j\in J}|\langle f, \alpha_i^{\frac{-1}{2}}\beta_i^{\frac{-1}{2}} T_i^* f_j\rangle|^2 \leq \frac{B}{C}\|f\|^2.$$

(ii). By a similar argument as in part (i), we obtain

$$\frac{AC}{D^2}\|f\|^2 \leq \sum_{i\in I}\sum_{j\in J}|\langle f, \alpha_i^{\frac{-1}{2}}\beta_i^{\frac{-1}{2}}T_i^* f_j\rangle|^2 \leq \frac{BD}{C^2}\|f\|^2.$$

□

4. Alternate dual

Let $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ be a fusion frame for a Hilbert space H, with fusion frame operator S_ν and let $\omega = \{(W_i, \beta_i)\}_{i\in I}$ be a Bessel fusion sequence for H. Then ω is called an *alternate dual* of ν if we have the following reconstruction formula,

$$f = \sum_{i\in I}\alpha_i\beta_i\pi_{W_i}S_\nu^{-1}\pi_{V_i}f, \quad for\ all\ f \in H.$$

Alternate dual frames are important in the literature of frame theory because of their important role in applications. In this section, we discuss some properties of alternate dual frames. In particular, we will show that if ω is a subfusion frame and an alternate dual of ν, such that $S_\nu = I$, then $S_\omega = S_{\omega\nu} = I$.

Proposition 6. *There exists a subfusion frame $\omega = \{(W_i, \alpha_i)\}_{i\in I}$ of $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ such that $S_\nu = I$ but $S_\omega \neq I$.*

Proof. We take $V_1 = V_2 = H, W_1 = \langle 0\rangle, W_2 = H$ and $\alpha_1 = \alpha_2 = \sqrt{\frac{1}{2}}$. Then $S_\nu f = f$ and $S_\omega f = \frac{1}{2}f$ for all $f \in H$. □

Lemma 3. *Let $\omega = \{(W_i, \alpha_i)\}_{i\in I}$ be a subfusion frame of $\nu = \{(V_i, \alpha_i)\}_{i\in I}$. If ω be an alternate dual of ν and $S_\nu = I$, then $S_\omega = I$.*

Proof. Since ω is an alternate dual of ν and $S_\nu = I$, then for all $f \in H$ we have,

$$f = \sum_{i\in I}\alpha_i^2\pi_{W_i}S_\nu^{-1}\pi_{V_i}f = \sum_{i\in I}\alpha_i^2\pi_{W_i}\pi_{V_i}f = \sum_{i\in I}\alpha_i^2\pi_{W_i}f = S_\omega f.$$

□

Using the argument similar to that above, we deduce the following results.

Proposition 7. *There exists a subfusion frame $\omega = \{(W_i, \beta_i)\}_{i\in I}$ of $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ such that $S_\nu = I$ but $S_{\omega\nu} \neq I$.*

Proof. Take $V_1 = V_2 = W_1 = W_2 = H$, $\alpha_1 = \alpha_2 = \sqrt{\frac{1}{2}}$, and $\beta_1 = \beta_2 = \sqrt{\frac{1}{3}}$. Then, for all $f \in H$, $S_\nu f = f$ and $S_{\omega\nu}f = 2\sqrt{\frac{1}{6}}f$. □

Lemma 4. *Let $\omega = \{(W_i, \beta_i)\}_{i\in I}$ be a subfusion frame of $\nu = \{(V_i, \alpha_i)\}_{i\in I}$. If ω is an alternate dual of ν and $S_\nu = I$, then $S_{\omega\nu} = I$.*

Proof. For all $f \in H$ we have

$$f = \sum_{i\in I}\alpha_i\beta_i\pi_{W_i}S_\nu^{-1}\pi_{V_i}f = \sum_{i\in I}\alpha_i\beta_i\pi_{W_i}\pi_{V_i}f = \sum_{i\in I}\alpha_i\beta_i\pi_{W_i}f = S_{\omega\nu}f.$$

□

5. Conclusion

In this paper, by presenting a much simpler proof for Theorem 4.3 of [7], we have provided conditions under which, by removing a member from a fusion frame, the remaining sequence is still a fusion frame. Then, we obtained the other interesting results from this theorem. Also, we have presented some new methods to get a subfusion frame from a fusion frame. Ultimately, after defining an alternate dual frame which is very important in practical topics, we have discussed some properties of a fusion frame operator.

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