Open Journal of
Mathematical Analysis

PSR Press

Article

Floquet exponent of solution to homogeneous growth-fragmentation equation

MEAS Len[1,*]

[1] Department of Mathematics, Royal University of Phnom Penh, Phnom Penh, Cambodia.
* Correspondence: meas.len@rupp.edu.kh

Communicated by: Absar Ul Haq
Received: 22 September 2023; Accepted: 2 Decemebr 2023; Published: 29 December 2023.

Abstract: In this work, we establish the existence and uniqueness of solution of Floquet eigenvalue and its adjoint to homogeneous growth-fragmentation equation with positive and periodic coefficients. We study the Floquet exponent, which measures the growth rate of a population. Finally, we establish the long term behavior of solution to the homogeneous growth-fragmentation equation by entropy method [1–3].

Keywords: Homogeneous Growth-Fragmentation Equation; Floquet theory; Entropy method

MSC: 42C15; 42C40; 47A05; 54A35.

1. Introduction

The growth-fragmentation model finds application in diverse contexts such as cell division, polymerization, neuroscience, prion proliferation, and telecommunications. This model encapsulates a critical biological phenomenon: the competition between growth and fragmentation, which exhibit opposing dynamics. Growth tends to increase the population size, whereas fragmentation reduces it.

In this study, we examine the homogeneous growth-fragmentation equation, a partial differential equation (PDE) modeling the dynamics of a cell population. This PDE describes cells of size $y > 0$ dividing into two parts, each of size $0 < x < y$, with equal probability. The equation is expressed as follows: for all $t, x > 0$,

$$\begin{aligned} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + \beta(t,x) n(t,x) & \\ = 2\int_x^\infty \beta(t,y) n(t,y) \frac{dy}{y}, & \\ n(t, x=0) = 0, \quad t > 0, & \\ n(t=0, x) = n^0(x), \quad x \geq 0. & \end{aligned} \tag{1}$$

Here, $\beta(t,y)$ represents the division rate of cells of size y at time t, and $\frac{1}{y}$ denotes the uniform probability of division into cells of size $x < y$. We assume β is T-periodic, positive, and bounded, satisfying

$$1 < \inf_{t \in (0,T)} \int_0^\infty \beta(t,y) \frac{dy}{y} e^{-\int_0^x \beta(t-x+z,z)dz} dx$$

and

$$\sup_{t \in (0,T)} \int_0^\infty \beta(t,y) \frac{dy}{y} e^{-\int_0^x \beta(t-x+z,z)dz} dx < \infty.$$

The mathematical inquiry central to this biological phenomenon involves the existence and uniqueness of the solution to the corresponding eigenvalue problem and its adjoint, as well as the exponential decay of the solution for the growth-fragmentation equation, as studied in various works [4–7].

Our investigation is motivated by the need to model cell division in cancer treatments, such as resonance and chrono-therapy, which are based on circadian rhythms [8].

The principal result of this research is the demonstration of the existence and uniqueness of $(\lambda_{\text{per}}, N, \phi)$ for the associated Floquet eigenvalue problem to equation (1), presented as

$$\begin{aligned} &\frac{\partial}{\partial t}N(t,x) + \frac{\partial}{\partial x}N(t,x) + (\lambda_{\text{per}} + \beta(t,x))N(t,x) \\ &\qquad = 2\int_x^{\infty} \beta(t,y)N(t,y)\frac{dy}{y}, \\ &N(t,x=0) = 0, \\ &\quad N(t,x) > 0, \quad T\text{-periodic}, \quad \int_0^T \int_0^{\infty} N(t,x)dxdt = 1. \end{aligned} \tag{2}$$

The adjoint eigenvalue problem is formulated as

$$\begin{aligned} &-\frac{\partial}{\partial t}\phi(t,x) - \frac{\partial}{\partial x}\phi(t,x) + (\lambda_{\text{per}} + \beta(t,x))\phi(t,x) \\ &\qquad = 2\frac{\beta(t,x)}{x}\int_0^{x} \phi(t,y)dy, \\ &\quad \phi(t,x) > 0, \quad T\text{-periodic}, \quad \int_0^{\infty} N(t,x)\phi(t,x)dx = 1. \end{aligned} \tag{3}$$

Theorem 1. *Given the assumptions on β, there exists a unique Floquet exponent $\lambda_{per} > 0$ and functions $N, \phi \in C(\mathbb{R}_+, L^1(\mathbb{R}_+; \phi(.,x)dx))$ for the Floquet eigenvalue problems (2) and (3).*

Furthermore, the long-term behavior is established using two main methodologies: the existence and uniqueness of a positive dominant Floquet eigenvalue associated with a positive eigenvector, and the General Relative Entropy method [1–3].

Theorem 2. *Under the assumptions on β and with $n^0 \in L^1(\mathbb{R}_+, \phi(0,x)dx)$, it holds true that*

$$\int_0^{\infty} |n(t,x)e^{-\lambda_{per}t} - \rho N(t,x)|\phi(t,x)dx \xrightarrow[t\to\infty]{} 0,$$

where $\rho = \int_0^{\infty} n^0(x)\phi(0,x)dx$.

This theorem implies that under periodic and positive coefficients, and suitable assumptions, the solution n to equation (1) tends asymptotically towards $N(t,x)$ times a time-exponential $e^{\lambda_{\text{per}}t}$. In biological terms, this indicates a balance between growth and division, maintaining the population at finite sizes.

The equation (1) can be written as an evolution equation

$$\begin{cases} \frac{\partial}{\partial t}n = An + Ln \\ n(0,x) = n^0(x) \end{cases}$$

with the operator

$$An = -\frac{\partial}{\partial x}n - \beta n$$

and

$$Ln = 2\int_x^{\infty} \beta(t,y)n(t,y)\frac{dy}{y}$$

defined on the space $E = \mathcal{D}'((0,\infty) \times (0,\infty))$. It is T-periodic when β is. This allows us to apply the Floquet theory for the linear differential equation on a Banach space with a T-periodic operator [9,10].

2. Floquet Eigenvalue Problem

In this section, we will prove Theorem 1 which is a consequence of the following theorem and the Floquet theory on Banach space.

Theorem 3. *Let $\Lambda > 0$ and $\sup_{t\in[0,T]} \frac{\beta(t,x)}{x}$ be bounded. Then there is a unique solution $n \in C(\mathbb{R}_+, L^1(\mathbb{R}_+; dx))$ to the equation*

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + (\Lambda + \beta(t,x))n(t,x) = 2\int_x^\infty \beta(t,y)n(t,y)\frac{dy}{y} \\ n(t, x = 0) = 0 \\ n(t = 0, x) = n^0(x) \in L^1(\mathbb{R}_+; dx). \end{cases}$$

Proof. Let us consider the T-periodic space $X = C([0,T], L^1(\mathbb{R}_+; dx))$, which is a Banach space endowed with the supremum norm $\|n\|_X = \sup_{t\in[0,T]} \|n(t,.)\|_{L^1(\mathbb{R}_+)}$. We will prove that $n(t,x)$ is a fixed point of a contraction operator and conclude the result by the Banach fixed point theorem. To do this, define the operator U as follows:

$$\begin{aligned} U : X &\to X \\ m &\mapsto n = U(m) \end{aligned}$$

where n is a solution of the following partial differential equation

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + (\Lambda + \beta(t,x))n(t,x) = 2\int_x^\infty \beta(t,y)m(t,y)\frac{dy}{y} \\ n(t, x = 0) = 0 \\ n(t = 0, x) = n^0(x) \end{cases}$$

Let $m_1, m_2 \in X$ and $n_i = U(m_i), i = 1, 2$. Then the difference $n = n_1 - n_2$ satisfies

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + (\Lambda + \beta(t,x))n(t,x) = 2\int_x^\infty \beta(t,y)m(t,y)\frac{dy}{y} \\ n(t, x = 0) = 0 \\ n(t = 0, x) = 0 \end{cases}$$

where $m = m_1 - m_2$. By the characteristics method, we have for $x < t$

$$n(t,x) = \int_0^x \int_y^\infty \beta(t - x + y', y')m(t - x + y', y')\frac{dy'}{y'} e^{-\int_y^x (\Lambda+\beta)(t-x+z,z)dz} dy$$

It follows that

$$\begin{aligned} &\|n(t,.)\|_{L^1(\mathbb{R}_+)} \\ &\leq \int_0^t \left| \int_0^x \int_y^\infty \beta(t - x + y', y')m(t - x + y', y')\frac{dy'}{y'} e^{-\int_y^x (\Lambda+\beta)(t-x+z,z)dz} dy \right| dx \\ &\leq M \int_0^t \int_0^t \|m(t,.)\|_{L^1(\mathbb{R}_+)} dy dx \\ &\leq t^2 M \|m(t,.)\|_{L^1(\mathbb{R}_+)}. \end{aligned}$$

Hence

$$\|n\|_X = \sup_{t\in[0,T]} \|n(t,.)\|_{L^1(\mathbb{R}_+)} \leq \sup_{t\in[0,T]} t^2 M \|m(t,.)\|_{L^1(\mathbb{R}_+)} = T^2 M \|m\|_X.$$

This implies that $U : X \to X$. Choose T so that $T^2 M \leq \frac{1}{2}$, then we have

$$\|U(m_1) - U(m_2)\|_X \leq \frac{1}{2}\|m_1 - m_2\|_X.$$

This means that U is contraction in the Banach space X, which proves the existence of the fixed point. This process can be iterated on the intervals $[T, 2T], [2T, 3T], \ldots$ and such solution can be built in $C(\mathbb{R}_+, L^1(\mathbb{R}_+; dx))$. Next, the density argument is used to complete the proof. To do this, let $n^0 \in L^1(\mathbb{R}_+; \phi(0,x)dx), \exists n_k^0 \in$

$L^1(\mathbb{R}_+;dx)$ such that $n_k^0 \to n^0$ in $L^1(\mathbb{R}_+;\phi(0,x)dx)$, and $\tilde{n}_k$ be a solution of the following partial differential equation

$$\begin{cases} \frac{\partial}{\partial t}\tilde{n}_k(t,x) + \frac{\partial}{\partial x}\tilde{n}_k(t,x) + (\mu + \beta(t,x))\tilde{n}_k(t,x) = 2\int_x^\infty \beta(t,y)\tilde{n}_k(t,y)\frac{dy}{y} \\ \tilde{n}_k(t,x=0) = 0. \end{cases}$$

If $\tilde{n} = \tilde{n}_k - \tilde{n}_l$, then

$$\begin{cases} \frac{\partial}{\partial t}\left(\tilde{n}(t,x)\phi(t,x)\right) + \frac{\partial}{\partial x}\left(\tilde{n}(t,x)\phi(t,x)\right) \\ \qquad\qquad = 2\phi(t,x)\int_x^\infty \beta(t,y)\tilde{n}(t,y)\frac{dy}{y} - 2\tilde{n}(t,x)\frac{\beta(t,x)}{x}\int_0^x \phi(t,y)dy \\ \tilde{n}(t,x=0)\phi(0,x) = 0. \end{cases}$$

It also holds that

$$\begin{cases} \frac{\partial}{\partial t}\left(|\tilde{n}(t,x)|\phi(t,x)\right) + \frac{\partial}{\partial x}\left(|\tilde{n}(t,x)|\phi(t,x)\right) \\ \qquad\qquad \leq 2\phi(t,x)\int_x^\infty \beta(t,y)|\tilde{n}(t,y)|\frac{dy}{y} - 2|\tilde{n}(t,x)|\frac{\beta(t,x)}{x}\int_0^x \phi(t,y)dy \\ |\tilde{n}(t,x=0)|\phi(0,x) = 0. \end{cases}$$

Integrating with respect to x gives

$$\frac{d}{dt}\int_0^\infty |\tilde{n}(t,x)|\phi(t,x)dx \leq 0.$$

This implies that

$$\int_0^\infty |\tilde{n}_k - \tilde{n}_l|\phi(t,x)dx \leq \int_0^\infty |n_k^0 - n_l^0|\phi(0,x)dx.$$

Thus, $\tilde{n}_k$ is a Cauchy sequence in a Banach space $C(\mathbb{R}_+;L^1(\mathbb{R}_+;\phi(.,x)dx))$. So $\tilde{n}_k$ converges in the space to a solution in the distribution sense. □

We are now ready to prove the existence and uniqueness of the solution of Floquet eigenvalue and its adjoint. Let us restate Theorem 1 as follows.

Theorem 4. *With the assumptions on β. There is a unique Floquet exponent $\lambda_{per} > 0$ and $N,\phi \in C(\mathbb{R}_+,L^1(\mathbb{R}_+;\phi(.,x)dx))$ of the Floquet eigenvalue problem (2) and its adjoint eigenvalue problem (3).*

Proof. Let $\Lambda = \lambda_{\text{per}} > 0$. It follows from Theorem 3 that there exists a unique solution $N(t,x) \in C(\mathbb{R}_+;L^1(\mathbb{R}_+;dx))$ satisfying

$$\frac{\partial}{\partial t}N(t,x) + \frac{\partial}{\partial x}N(t,x) + (\lambda_{\text{per}} + \beta(t,x))N(t,x) = 2\int_x^\infty \beta(t,y)N(t,y)\frac{dy}{y}.$$

In the same manner, its adjoint is given by

$$-\frac{\partial}{\partial t}\phi(t,x) - \frac{\partial}{\partial x}\phi(t,x) + (\lambda_{\text{per}} + \beta(t,x))\phi(t,x) = 2\frac{\beta(t,x)}{x}\int_0^x \phi(t,y)dy,$$

where $\phi(t,x) \in C(\mathbb{R}_+;L^1(\mathbb{R}_+;dx))$. In addition, the operator U defined in Theorem 3 is T-periodic, strictly positive as soon as K is. The operator U is compact as a result of Arzela-Ascoli theorem since $\sup\left\{\|U(n)\|_X;\|n\|_X \leq 1\right\}$ is uniformly bounded; hence equicontinuous. Thus by Corollary 1.11 and Corollary 1.14 in [9] with $\Lambda = \lambda_{\text{per}}$ such that the spectral radius of U, $r(\Lambda) = 1$ and up to renormalization N,ϕ is unique. To end the proof, Λ need to be found such that $r(\Lambda) = 1$. Since r is decreasing function and vanishing at infinity and

$$r(0) \geq \inf_{t\in(0,T)}\int_0^\infty\int_0^\infty \beta(t,y)\frac{dy}{y}e^{-\int_0^x \beta(t-x+z,z)dz}dx > 1.$$

Thus, there exists a unique λ_{per} such that $r(\lambda_{\text{per}}) = 1$. □

3. Long term behavior by entropy method

In this section, the long run asymptotic decay of the solution of the growth-fragmentation equation is established by entropy method [1–3]. We first derive the following relative entropy inequality. Then with appropriate entropy dissipation together with convex function $H(s) = s^2$ in the relativity entropy inequality and passing to the weak limits, we obtain the long run asymptotic behavior of the solution of the growth-fragmentation equation.

Theorem 5. *With the assumptions on β and for all convex functions H and for all $t > 0$; it holds true that*

$$\frac{d}{dt}\int_0^\infty \phi(t,x)N(t,x)H\left(\frac{\tilde{n}(t,x)}{N(t,x)}\right)dx = -D_H(\tilde{n})(t) \leq 0$$

where $\tilde{n}(t,x) = n(t,x)e^{-\lambda_{pert}t}$ and

$$D_H(\tilde{n})(t) = 2\int_0^\infty\int_x^\infty \phi(t,x)\frac{\beta(t,y)}{y}N(t,y)\Bigg[H\bigg(\frac{\tilde{n}(t,y)}{N(t,y)}\bigg) - H\bigg(\frac{\tilde{n}(t,x)}{N(t,x)}\bigg) \\ - H'\bigg(\frac{\tilde{n}(t,x)}{N(t,x)}\bigg)\bigg(\frac{\tilde{n}(t,y)}{N(t,y)} - \frac{\tilde{n}(t,x)}{N(t,x)}\bigg)\Bigg]dydx.$$

Proof. The equations (1),(2) and (3) yield

$$\frac{\partial}{\partial t}\left(\frac{\tilde{n}(t,x)}{N(t,x)}\right) + \frac{\partial}{\partial x}\left(\frac{\tilde{n}(t,x)}{N(t,x)}\right) = \frac{2}{N(t,x)}\int_x^\infty \frac{\beta(t,y)}{y}N(t,y)\left[\frac{\tilde{n}(t,y)}{N(t,y)} - \frac{\tilde{n}(t,x)}{N(t,x)}\right]dy$$

and

$$\frac{\partial}{\partial t}H\bigg(\frac{\tilde{n}(t,x)}{N(t,x)}\bigg) + \frac{\partial}{\partial x}H\bigg(\frac{\tilde{n}(t,x)}{N(t,x)}\bigg) \\ = \frac{2}{N(t,x)}H'\bigg(\frac{\tilde{n}(t,x)}{N(t,x)}\bigg)\int_x^\infty \frac{\beta(t,y)}{y}N(t,y)\left[\frac{\tilde{n}(t,y)}{N(t,y)} - \frac{\tilde{n}(t,x)}{N(t,x)}\right]dy.$$

It follows that

$$\frac{\partial}{\partial t}\left[\phi(t,x)N(t,x)H\left(\frac{\tilde{n}(t,x)}{N(t,x)}\right)\right] + \frac{\partial}{\partial x}\left[\phi(t,x)N(t,x)H\left(\frac{\tilde{n}(t,x)}{N(t,x)}\right)\right] \\ = -2\int_0^x \frac{\beta(t,x)}{x}\phi(t,y)N(t,x)H\left(\frac{\tilde{n}(t,x)}{N(t,x)}\right)dy \\ + 2\int_x^\infty \frac{\beta(t,y)}{y}\phi(t,x)N(t,y)H\left(\frac{\tilde{n}(t,x)}{N(t,x)}\right)dy \\ + 2\int_x^\infty \phi(t,x)\frac{\beta(t,y)}{y}N(t,y)H'\left(\frac{\tilde{n}(t,x)}{N(t,x)}\right)\left[\frac{\tilde{n}(t,y)}{N(t,y)} - \frac{\tilde{n}(t,x)}{N(t,x)}\right]dy.$$

Then integrating in x to get

$$\frac{d}{dt}\int_0^\infty \phi(t,x)N(t,x)H\left(\frac{\tilde{n}(t,x)}{N(t,x)}\right)dx \\ = -2\int_0^\infty\int_0^x \frac{\beta(t,x)}{x}\phi(t,y)N(t,x)H\left(\frac{\tilde{n}(t,x)}{N(t,x)}\right)dydx \\ + 2\int_0^\infty\int_x^\infty \frac{\beta(t,y)}{y}\phi(t,x)N(t,y)H\left(\frac{\tilde{n}(t,x)}{N(t,x)}\right)dydx \\ + 2\int_0^\infty\int_x^\infty \phi(t,x)\frac{\beta(t,y)}{y}N(t,y)H'\left(\frac{\tilde{n}(t,x)}{N(t,x)}\right)\left[\frac{\tilde{n}(t,y)}{N(t,y)} - \frac{\tilde{n}(t,x)}{N(t,x)}\right]dydx \\ = -2\int_0^\infty\int_x^\infty \phi(t,x)\frac{\beta(t,y)}{y}N(t,y)\Bigg[H\bigg(\frac{\tilde{n}(t,y)}{N(t,y)}\bigg) - H\bigg(\frac{\tilde{n}(t,x)}{N(t,x)}\bigg) \\ + H'\bigg(\frac{\tilde{n}(t,x)}{N(t,x)}\bigg)\bigg(\frac{\tilde{n}(t,y)}{N(t,y)} - \frac{\tilde{n}(t,x)}{N(t,x)}\bigg)\Bigg]dydx \\ = -D_H(\tilde{n})(t).$$

Finally, by the convexity of H; it can be concluded that $D_H(\tilde{n})(t)$ is nonpositive. □

Theorem 6. *Under the assumptions on β and $n^0 \in L^1(\mathbb{R}_+, \phi(0,x)dx)$. Then it holds*

$$\int_0^\infty |\tilde{n}(t,x) - \rho N(t,x)|\phi(t,x)dx \xrightarrow[t\to\infty]{} 0$$

where $\rho = \int_0^\infty n^0(x)\phi(0,x)dx$.

Proof. Let $h(t,x) = \tilde{n}(t,x) - \rho N(t,x)$; then h satisfies the following partial differential equation

$$\begin{cases} \frac{\partial}{\partial t}h(t,x) + \frac{\partial}{\partial x}h(t,x) + (\lambda_{\text{per}} + \beta(t,x))h(t,x) = 2\int_x^\infty \beta(t,y)h(t,y)\frac{dy}{y} \\ h(t,x=0) = 0. \end{cases} \tag{4}$$

It also holds

$$\frac{\partial}{\partial t}(|h(t,x)|\phi(t,x)) + \frac{\partial}{\partial x}(|h(t,x)|\phi(t,x)) \\ \leq 2\phi(t,x)\int_x^\infty \beta(t,y)|h(t,y)|\frac{dy}{y} - 2|h(t,x)|\frac{\beta(t,x)}{x}\int_0^x \phi(t,y)dy.$$

Integrating with respect to x gives

$$\frac{d}{dt}\int_0^\infty |h(t,x)|\phi(t,x)dx \leq 0.$$

It follows that $\int_0^\infty |h(t,x)|\phi(t,x)dx$ is decaying and it is positive, so it converges to some value $L \geq 0$. It remains to prove that $L = 0$. Now let define the sequence of functions $h_k(t,x) = h(t+k,x)$; it also satisfies (4). Then $h_k(t,x)$ is bounded in $L^1(\mathbb{R}_+; \phi(.,x)dx)$. So up to subsequence $h_k \rightharpoonup g$ weakly. The entropy dissipation of $h(t,x)$ can be worked on and the property of relative entropy for a convex function $H(s) = s^2$ gives

$$\int_0^\infty \int_0^\infty \int_x^\infty \phi(t,x)\frac{\beta(t,y)}{y}N(t,y)\left(\frac{h(t,y)}{N(t,y)} - \frac{h(t,x)}{N(t,x)}\right)^2 dydxdt \leq C.$$

Therefore, as $k \to \infty$,

$$\int_0^\infty \int_0^\infty \int_x^\infty \phi(t,x)\frac{\beta(t,y)}{y}N(t,y)\left(\frac{h_k(t,y)}{N(t,y)} - \frac{h_k(t,x)}{N(t,x)}\right)^2 dydxdt \\ = \int_k^\infty \int_0^\infty \int_x^\infty \phi(t,x)\frac{\beta(t,y)}{y}N(t,y)\left(\frac{h(t,y)}{N(t,y)} - \frac{h(t,x)}{N(t,x)}\right)^2 dydxdt \to 0.$$

Passing to the weak limits yields

$$\int_0^\infty \int_0^\infty \int_x^\infty \phi(t,x)\frac{\beta(t,y)}{y}N(t,y)\left(\frac{g(t,y)}{N(t,y)} - \frac{g(t,x)}{N(t,x)}\right)^2 dydxdt = 0.$$

That is, $\frac{g(t,y)}{N(t,y)} = \frac{g(t,x)}{N(t,x)}$ almost everywhere on the support of β. On the other hand, in the limits the entropy dissipation for g/N vanishes, so we obtain

$$\frac{\partial}{\partial t}\left(\frac{g(t,x)}{N(t,x)}\right) + \frac{\partial}{\partial x}\left(\frac{g(t,x)}{N(t,x)}\right) = 0.$$

Using Lemma 4.5 in [3, p.100], it follows that $g(t,x) =$ constant and by the condition $\int_0^\infty g(t,x)\phi(t,x)dx = 0$ it can be concluded that $g = 0$ and thus $L = 0$. □

Author Contributions: All authors contributed equally in this paper. All authors read and approved the final version of this paper.

Conflicts of Interest: The author declares no conflict of interest.

Data Availability: All data required for this research is included within this paper.

Acknowledgments: The authors would like to thank the referee for his/her valuable comments that resulted in the present improved version of the article.

References

[1] Michel, P., Mischler, S., & Perthame, B. (2004). General entropy equations for structured population models and scattering. *Comptes Rendus Mathematique, 338*(9), 697-702.

[2] Michel, P., Mischler, S., & Perthame, B. (2005). General relative entropy inequality: an illustration on growth models. *Journal de mathématiques pures et appliquées, 84*(9), 1235-1260.

[3] Perthame, B. (2007). *Transport equations in biology. Frontiers in Mathematics*. Birkhäuser Verlag, Basel.

[4] Doumic Jauffret, M., & Gabriel, P. (2010). Eigenelements of a general aggregation-fragmentation model. *Mathematical Models and Methods in Applied Sciences 20*(5), 757-783.

[5] Clairambault, J., Gaubert, S., & Lepoutre, T. (2009). Comparison of Perron and Floquet eigenvalues in age structured cell division cycle models. *Mathematical Modelling of Natural Phenomena, 4*(3), 183-209.

[6] Laurencot, P., & Perthame, B. (2009). Exponential decay for the growth-fragmentation/cell-division equations. *Communications in Mathematical Sciences, 7*(2), 503-510.

[7] Perthame, B., & Ryzhik, L. (2005). Exponential Decay for the Fragmentation or Cell Division Equation. *Journal of Differential Equations, 210*(1), 155-177.

[8] Clairambault, J., Michel, P., & Perthame, B. (2006). Circadian rhythm and tumour growth. *Comptes Rendus Biologies, 342*(1), 17-22.

[9] Meas, L. (2021). Spectral Analysis of Banach Spaces and their Application to Age-Structured Equations. *Cambodia Journal of Basic and Applied Research (CJBAR), 3*(1), 32-69.

[10] Meas, L. (2023). Asymptotic Behavior of the Solution to the Equal Mitosis Equation. *Open Journal of Mathematical Science*, (submitted).