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Multiplicity results for a class of nonlinear singular differential equation with a parameter

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Abstract: This paper gives sufficient conditions for the existence of positive periodic solutions to general indefinite singular differential equations. Furthermore, under some assumptions we show the existence of two positive periodic solutions. The methods used are Krasnoselskiǐ's-Guo fixed point theorem and the positivity of the associated Green's function.

Keywords: indefinite singularity; positive periodic solution; Krasnoselskiǐ's-Guo fixed point; Green's function.

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1. Introduction

Singular differential equations help solve problems in biology, engineering and economics, and it is of great relevance. The study of the singularity periodic problem only began to receive more scholarly attention in 1987 when Lazer and Solimini [1] opened new doors for the singularity problems. In this paper we study the existence of positive periodic solutions to a class of indefinite singular differential equation with a parameter

$$y''(t) + a(t)y(t) = \mu \frac{q(t)}{y^{\rho}(t)} + \mu b(t)y^{\delta}(t) + \mu c(t), \tag{1}$$

where μ is a positive parameter, ρ and δ are two positive constants and $0 < \delta < 1$, $a \in L^p(\mathbb{R}/\omega\mathbb{Z})$, $q,\ c \in L^1(\mathbb{R}/\omega\mathbb{Z})$, $b \in L^1(\mathbb{R}/\omega\mathbb{Z})$ is positive, here ω is a constant and $1 \leq p \leq +\infty$.

At present, scholars are more concerned about the periodic problems with singularity of attractive and repulsive type [2–6]. However, it becomes relatively difficult to study the periodic problems when considering indefinite singularities, which means that there is still plenty of scope for studying the periodic problems with indefinite singularities. In 2010, Bravo and Torres [7] first studied the following class of second-order differential equation with an indefinite singularity

$$y''(t) = \frac{q(t)}{y^{\rho}(t)}, \tag{2}$$

where $q \in C(\mathbb{R}, \mathbb{R})$ and $\rho = 3$. They gave a sufficient condition for the existence of a positive periodic solution of equation (2) as $\int_0^{\omega} q(t)dt < 0$. Hakl and Zamora [8] in 2017 used Leray-Schauder degree theory to prove the existence of positive periodic solutions of the equation (2) with strong singularity ($\rho \geq 1$). In the same year, Zamora and Godoy [9] studied the existence of positive periodic solutions to equation (2) with weak singularity ($0 < \rho < 1$).

In 2021, Cheng and Cui [10] applied the fixed point theorem in cones to prove the existence of positive periodic solutions to the following indefinite differential equation

$$y''(t) + a(t)y(t) = \frac{q(t)}{y^{\rho}(t)} + c(t), \tag{3}$$

where $a,\ c \in C(\mathbb{R}, (0, +\infty))$ and ρ is a positive constant. Han and Cheng [11] used the Krasnoselskiǐ's-Guo fixed point theorem in 2022 to consider the existence of positive periodic solutions to equation (3).

Based on their study, we will discuss the existence of positive periodic solutions to the equation (1) by the methods of the Krasnoselskiǐ's-Guo fixed point and the positivity of the associated Green's function. It is worth noting that, according to the range of values taken for μ, we can obtain one and two positive periodic solutions to equation (1), respectively. First, we introduce the Krasnoselskiǐ's-Guo fixed point.

Lemma 1. *([12, P. 94]) Let Y be a Banach space and $\mathcal{K}$ is a cone in Y. Assume that S_1 and S_2 are open subsets of Y with $0 \in S_1$, $\overline{S}_1 \subset S_2$. Let*

$$\Psi : \mathcal{K} \cap (\overline{S}_2 \backslash S_1) \to \mathcal{K}$$

be a completely continuous operator such that one of the following conditions holds:
(i) $\|\Psi y\| \geq \|y\|$ for $y \in \mathcal{K} \cap \partial S_1$ and $\|\Psi y\| \leq \|y\|$ for $y \in \mathcal{K} \cap \partial S_2$;
(ii) $\|\Psi y\| \leq \|y\|$ for $y \in \mathcal{K} \cap \partial S_1$ and $\|\Psi y\| \geq \|y\|$ for $y \in \mathcal{K} \cap \partial S_2$.
Then Ψ has a fixed point in the set $\mathcal{K} \cap (\overline{S}_2 \backslash S_1)$.

Before applying the Krasnoselskiǐ's-Guo fixed point, we have to write the periodic problem as an equivalent fixed point problem using the concept of Green's function. A general construction of the Green's function is described in [13]. Next we give another Lemma to be used.

Lemma 2. *(see [14, Corollary 2.3])Define*

$$\mathcal{Q}(\alpha) = \begin{cases} \dfrac{2\pi}{\alpha \omega^{1+2/\alpha}} \left(\dfrac{2}{2+\alpha}\right)^{1-2/\alpha} \left(\dfrac{\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{\alpha}\right)}\right)^2, & 1 \leq \alpha < \infty, \\ \dfrac{4}{\omega}, \quad \alpha = \infty, \end{cases}$$

where Γ is the Gamma function, i.e., $\Gamma(t) = \int_0^{+\infty} y^{t-1} e^{-u} du$. Assume that $a(t) \geq 0$ for almost every $t \in [0, \omega]$ and $a \in L^p(\mathbb{R}/\omega\mathbb{Z})$. If

$$\|a\|_p := \left(\int_0^\omega |a(t)|^p dt\right)^{\frac{1}{p}} < \mathcal{Q}(2p^*), \tag{4}$$

where $p^* = \frac{p}{p-1}$ if $1 \leq p < \infty$ and $p^* = 1$ if $p = +\infty$, then the Green's function $\mathcal{H}(t,s) > 0$ for all $(t,s) \in [0,\omega] \times [0,\omega]$.

Define

$$(\Psi_\mu y)(t) := \mu \int_0^\omega \mathcal{H}(t,s) \left(\frac{q(t)}{y^\rho(t)} + b(t) y^\delta(t) + c(t)\right) ds. \tag{5}$$

Here, a fixed point of the map Ψ defined by (5) is a positive periodic solution of the equation (1). Besides, from Lemma 2, we know that $\mathcal{H}(t,s) > 0$ for all $(t,s) \in [0,\omega] \times [0,\omega]$.

Remark 1. (see [?]) In the special case $a(t) \equiv \varsigma^2$ with $\varsigma > 0$, the Green's function has the following form

$$\mathcal{H}(t,s) = \begin{cases} \dfrac{\cos \varsigma(t - s - \frac{\omega}{2})}{2\varsigma \sin \frac{\varsigma\omega}{2}}, & 0 \leq s \leq t \leq \omega, \\ \dfrac{\cos \varsigma(t - s + \frac{\omega}{2})}{2\varsigma \sin \frac{\varsigma\omega}{2}}, & 0 \leq s \leq t \leq \omega. \end{cases}$$

If $\varsigma < \frac{\pi}{\omega}$, then the Green's function $\mathcal{H}(t,s) > 0$ for all $(t,s) \in [0,\omega] \times [0,\omega]$.

For convenience, we have given some symbolic definitions

$$\mathcal{H}_* := \min_{0 \leq s,t \leq \omega} \mathcal{H}(t,s), \quad \mathcal{H}^* := \max_{0 \leq s,t \leq \omega} \mathcal{H}(t,s), \quad \sigma := \frac{\mathcal{H}_*}{\mathcal{H}^*}. \tag{6}$$

According to (6), it is easy to know that $0 < \mathcal{H}_* < \mathcal{H}^*$ and $0 < \sigma \leq 1$.

Furthermore, we give some information about $q(t)$ and $b(t)$

$$q^+(t) := \max\{q(t), 0\}, \quad q^-(t) := -\min\{q(t), 0\}, \quad \bar{q} := \frac{1}{\omega}\int_0^\omega q(t) dt, \quad b^* := \max_{t \in [0,\omega]} g(t), \quad g_* := \min_{t \in [0,\omega]} b(t).$$

Finally, we give our main conclusion.

Theorem 1. *Assume that equation (4) holds. Then the following one of conclusions holds.*
(i) There exists $\mu_0 > 0$ such that equation (1) has a positive periodic solution for $\mu > \mu_0$;
(ii) For all sufficiently small $\mu > 0$, equation (1) has two positive periodic solutions.

2. Proof of Theorem 1.1

First, define

$$\mathcal{K} := \{y \in C_\omega : \min_{t\in\mathbb{R}} y(t) \geq \sigma\|y\|\},$$

$$S_{r_1} := \{y \in C_\omega : \|y\| < r_1\} \quad \text{and} \quad S_{r_2} := \{y \in C_\omega : \|y\| < r_2\},$$

where r_1 and r_2 are two positive constants, and $C_\omega := \{y \in C(\mathbb{R}, \mathbb{R}),\ y(t+\omega) \equiv y(t),\ \text{for all } t \in \mathbb{R}\}$ with norm $\|y\| := \max_{t\in\mathbb{R}} |y(t)|$. It is easy to verify that $\mathcal{K}$ is cone in C_ω.

Lemma 3. *Assume that (4) holds. Besides, the follows inequality is satisfied*

$$r_2 > r_1 > \frac{1}{\sigma} \max\left\{ \left(\frac{2\|q^-\|}{b_*}\right)^{\frac{1}{\delta+\rho}}, \left(\frac{2\|c^-\|}{b_*}\right)^{\frac{1}{\delta}} \right\} := \xi.$$

Then $\Psi_\mu : \mathcal{K} \cap (\overline{S}_{r_2} \backslash S_{r_1}) \to \mathcal{K}$ is a completely continuous operator.

Proof. First, we prove that $\Psi_\mu(\mathcal{K} \cap (\overline{S}_{r_2} \backslash S_{r_1})) \subset \mathcal{K}$. Obviously, we have

$$\sigma r_1 < y(t) \leq r_2,\ \forall\, y \in \mathcal{K} \cap (\overline{S}_{r_2} \backslash S_{r_1}),\ \forall\, t \in \mathbb{R}.$$

Because $r_1 > \xi$, we obtain

$$\begin{aligned} \frac{q(t)}{y^\rho(t)} + b(t)y^\delta(t) + c(t) =& \frac{q^+(t)}{y^\rho(t)} - \frac{q^-(t)}{y^\rho(t)} + b(t)y^\delta(t) + c^+(t) - c^-(t) \\ >& -\frac{\|q^-\|}{(\sigma r_1)^\rho} + b_*(\sigma r_1)^\delta - \|c^-\| \\ >& 0, \end{aligned} \tag{7}$$

for all $t \in \mathbb{R}$. It follows from (6) and (7) that

$$\begin{aligned} \min_{t\in\mathbb{R}} (\Psi_\mu y)(t) =& \mu \min_{t\in\mathbb{R}} \int_0^\omega \mathcal{H}(t,s) \left(\frac{q(s)}{y^\rho(s)} + b(s)y^\delta(s) + c(s)\right) ds \\ \geq& \mu\mathcal{H}_* \int_0^\omega \left(\frac{q(s)}{y^\rho(s)} + b(s)y^\delta(s) + c(s)\right) ds \\ =& \mu\sigma\mathcal{H}^* \int_0^\omega \left(\frac{q(s)}{y^\rho(s)} + b(s)y^\delta(s) + c(s)\right) ds \\ \geq& \mu\sigma \max_{t\in\mathbb{R}} \int_0^\omega \mathcal{H}(t,s) \left(\frac{q(s)}{y^\rho(s)} + b(s)y^\delta(s) + c(s)\right) ds \\ =& \mu\sigma\|\Psi_\mu y\|, \end{aligned}$$

which implies $\Psi_\mu(\mathcal{K} \cap (\overline{S}_{r_2} \backslash S_{r_1})) \subset \mathcal{K}$. Besides, applying the Arzela-Ascoli theorem, it is easy to prove that $\Psi_\mu : \mathcal{K} \cap (\overline{S}_{r_2} \backslash S_{r_1}) \to \mathcal{K}$ is a completely continuous operator. □

The proof of Theorem 1

(i) Our proof relies on Lemma 1. First, define

$$S_{\mathcal{R}_1} := \{y \in C_\omega : \|y\| < \mathcal{R}_1\} \quad \text{and} \quad S_{\mathcal{R}_2} := \{y \in C_\omega : \|y\| < \mathcal{R}_2\},$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are two positive constants. Besides, $\mathcal{R}_2 > \mathcal{R}_1 > \xi$ and (4) is satisfied. We can obtain $\Psi_\mu : \mathcal{K} \cap (\overline{S}_{\mathcal{R}_2} \backslash S_{\mathcal{R}_1}) \to \mathcal{K}$ is a completely continuous operator according to Lemma 3.

Then we prove that

$$\|\Psi_\mu y\| \geq \|y\|, \ \text{for} \ y \in \mathcal{K} \cap \partial S_{\mathcal{R}_1}. \tag{8}$$

Obviously, we can get $\|y\| = \mathcal{R}_1$ and

$$\sigma\mathcal{R}_1 \leq y(t) \leq \mathcal{R}_1, \ \forall\, y \in \mathcal{K} \cap \partial S_{\mathcal{R}_1}, \ \forall\, t \in \mathbb{R}.$$

According to (7), it follows that

$$\begin{aligned}
(\Psi_\mu y)(t) =& \mu \int_0^\omega \mathcal{H}(t,s) \left(\frac{q(s)}{y^\rho(s)} + b(s)y^\delta(s) + c(s) \right) ds \\
=& \mu \int_0^\omega \mathcal{H}(t,s) \left(\frac{q^+(s)}{y^\rho(s)} - \frac{q^-(s)}{y^\rho(s)} + b(s)y^\delta(s) + c^+(s) - c^-(s) \right) ds \\
>& \mu \int_0^\omega \mathcal{H}(t,s) \frac{q^+(s)}{y^\rho(s)} ds \\
>& \mu \frac{\mathcal{H}_* \omega \overline{q^+}}{\mathcal{R}_1^\rho}.
\end{aligned}$$

For $\mu > \mu_0$, the existence of $\mu_0 > \frac{\mathcal{R}_1^{\rho+1}}{\mathcal{H}_*\omega\overline{q^+}} > 0$ satisfies (8).

On the other hand, we prove that

$$\|\Psi_\mu y\| \leq \|y\|, \ \text{for} \ y \in \mathcal{K} \cap \partial S_{\mathcal{R}_2}. \tag{9}$$

Obviously, we can get $\|y\| = R_2$ and

$$\sigma\mathcal{R}_2 \leq y(t) \leq \mathcal{R}_2, \ \forall\, y \in \mathcal{K} \cap \partial S_{\mathcal{R}_2}, \ \forall\, t \in \mathbb{R}.$$

From (7) we get

$$\begin{aligned}
(\Psi_\mu y)(t) =& \mu \int_0^\omega \mathcal{H}(t,s) \left(\frac{q(s)}{y^\rho(s)} + b(s)y^\delta(s) + c(s) \right) ds \\
=& \mu \int_0^\omega \mathcal{H}(t,s) \left(\frac{q^+(s)}{y^\rho(s)} - \frac{q^-(s)}{y^\rho(s)} + b(s)y^\delta(s) + c^+(s) - c^-(s) \right) ds \\
\leq& \mu \left(\frac{\mathcal{H}^* \omega \overline{q^+}}{(\sigma\mathcal{R}_2)^\rho} - \frac{\mathcal{H}_* \omega \overline{q^+}}{\mathcal{R}_2^\rho} + \mathcal{H}^* \omega \bar{b} \mathcal{R}_2^\delta + \mathcal{H}^* \omega \overline{c^+} - \mathcal{H}_* \omega \overline{c^-} \right) \\
\leq& \mu \left(\frac{\mathcal{H}^* \omega \overline{q^+}}{(\sigma\mathcal{R}_2)^\rho} + \mathcal{H}^* \omega \bar{b} \mathcal{R}_2^\delta + \mathcal{H}^* \omega \overline{c^+} \right).
\end{aligned}$$

It is obvious that we can choose $\mathcal{R}_2$ large enough such that

$$(\Psi_\mu y)(t) \leq \mu \left(\frac{\mathcal{H}^* \omega \overline{q^+}}{(\sigma\mathcal{R}_2)^\rho} + \mathcal{H}^* \omega \bar{b} \mathcal{R}_2^\delta + \mathcal{H}^* \omega \overline{c^+} \right) < \mathcal{R}_2.$$

Therefore, (9) is satisfied. According to Lemma 1, we get that Ψ_μ has a fixed point and equation (1) has a positive periodic solution.

(ii) First, define

$$S_{\mathcal{R}_3} := \{y \in C_\omega : \|y\| < \mathcal{R}_3\} \quad \text{and} \quad S_{\mathcal{R}_4} := \{y \in C_\omega : \|y\| < \mathcal{R}_4\},$$

where $\mathcal{R}_3$ and $\mathcal{R}_4$ are two positive constants and $\mathcal{R}_4 > \left(\frac{\mathcal{H}_*\omega\overline{q^+}}{\eta}\right)^{\frac{1}{\rho+1}} > \mathcal{R}_3 > \xi$, and (4) is satisfied, here $\eta > 0$ is a constant and $\mu\eta > 1$.

According to Lemma 3, we can know that $\Psi_\mu : \mathcal{K} \cap (\bar{S}_{\mathcal{R}_4} \backslash S_{\mathcal{R}_3}) \to \mathcal{K}$ is a completely continuous operator. Afterward, let us prove that

$$\|\Psi_\mu y\| \geq \|y\|, \ \text{for} \ y \in \mathcal{K} \cap \partial S_{\mathcal{R}_3}. \tag{10}$$

Obviously, we can get $\|y\| = \mathcal{R}_3$ and

$$\sigma\mathcal{R}_3 \le y(t) \le \mathcal{R}_3,\ \forall\, y \in \mathcal{K} \cap \partial S_{\mathcal{R}_3},\ \forall\, t \in \mathbb{R}.$$

According to (7), it follows that

$$\begin{aligned}(\Psi_\mu u)(t) =&\mu \int_0^\omega \mathcal{H}(t,s)\left(\frac{q(s)}{y^\rho(s)} + b(s)y^\delta(s) + c(s)\right) ds\\ =&\mu \int_0^\omega \mathcal{H}(t,s)\left(\frac{q^+(s)}{y^\rho(s)} - \frac{q^-(s)}{y^\rho(s)} + b(s)y^\delta(s) + c^+(s) - c^-(s)\right) ds\\ >&\mu \int_0^\omega \mathcal{H}(t,s)\frac{q^+(s)}{y^\rho(s)} ds\\ >&\mu \frac{\mathcal{H}_*\omega\overline{q^+}}{\mathcal{R}_3^\rho}\\ \ge&\mu\eta\mathcal{R}_3.\end{aligned}$$

Because $\mu\eta > 1$, $(\Psi_\mu u)(t) \ge R_3$, then (10) holds.

Then we prove that

$$\|\Psi_\mu y\| \le \|y\|,\ \text{ for } y \in \mathcal{K} \cap \partial S_{\mathcal{R}_4}. \tag{11}$$

Obviously, we can get $\|y\| = \mathcal{R}_4$ and

$$\sigma\mathcal{R}_4 \le y(t) \le \mathcal{R}_4,\ \forall\, y \in \mathcal{K} \cap \partial S_{\mathcal{R}_4},\ \forall\, t \in \mathbb{R}.$$

From (7) we get

$$\begin{aligned}(\Psi_\mu y)(t) =&\mu \int_0^\omega \mathcal{H}(t,s)\left(\frac{q(s)}{y^\rho(s)} + b(s)y^\delta(s) + c(s)\right) ds\\ =&\mu \int_0^\omega G(t,s)\left(\frac{q^+(s)}{y^\rho(s)} - \frac{q^-(s)}{y^\rho(s)} + b(s)y^\delta(s) + c^+(s) - c^-(s)\right) ds\\ \le&\mu\left(\frac{\mathcal{H}^*\omega\overline{q^+}}{(\sigma\mathcal{R}_4)^\rho} - \frac{\mathcal{H}_*\omega\overline{q^+}}{\mathcal{R}_4^\rho} + \mathcal{H}^*\omega\bar{b}\mathcal{R}_4^\delta + \mathcal{H}^*\omega\overline{c^+} - \mathcal{H}_*\omega\overline{c^-}\right)\\ \le&\mu\left(\frac{\mathcal{H}^*\omega\overline{h^+}}{(\sigma R_4)^\rho} + \mathcal{H}^*\omega\bar{b}\mathcal{R}_4^\delta + \mathcal{H}^*\omega\overline{c^+}\right).\end{aligned}$$

There exists $\mu_1 > 0$ such that

$$\mu_1 < \frac{\sigma^\rho\mathcal{R}_4^{\rho+1}}{\mathcal{H}^*\omega\overline{q^+} + \sigma^\rho\mathcal{R}_4^{\delta+\rho}\mathcal{H}^*\omega\bar{b} + \sigma^\rho\mathcal{R}_4^\rho\mathcal{H}^*\omega\overline{c^+}}.$$

For $\mu < \mu_1$, (11) holds.

It follows from Lemma 1 that Ψ_μ has a fixed point $y_1 \in \mathcal{K} \cap (\overline{S}_{\mathcal{R}_4} \backslash S_{\mathcal{R}_3})$, which is a positive periodic solution of equation (1) for $\mu < \mu_2$ and satisfies $\mathcal{R}_3 < \|y_1\| < \mathcal{R}_4$.

On the other hand, define

$$S_{\mathcal{R}_5} := \{y \in C_\omega : \|y\| < \mathcal{R}_5\} \quad \text{and} \quad S_{\mathcal{R}_6} := \{y \in C_\omega : \|y\| < \mathcal{R}_6\},$$

where $\mathcal{R}_5$ and $\mathcal{R}_6$ are two positive constants and $\mathcal{R}_6 > \left(\frac{\mathcal{H}_*\omega\overline{q^+}}{\eta'}\right)^{\frac{1}{\rho+1}} > \mathcal{R}_5 > \mathcal{R}_4 > \xi$, and (4) is satisfied, here $\eta' > 0$ is a constant, $\mu\eta' > 1$ and $\eta' < \eta$.

Similarly, according to Step 1 of (i), we can know that $\Psi_\mu(\mathcal{K} \cap (\overline{S}_{\mathcal{R}_6} \backslash S_{\mathcal{R}_5})) \subset \mathcal{K}$ and $\Psi_\mu : \mathcal{K} \cap (\overline{S}_{\mathcal{R}_6} \backslash S_{\mathcal{R}_5}) \to \mathcal{K}$ is a completely continuous operator.

Afterward, let us prove that

$$\|\Psi_\mu y\| \ge \|y\|,\ \text{ for } y \in \mathcal{K} \cap \partial S_{\mathcal{R}_5}. \tag{12}$$

Obviously, we can get $\|y\| = \mathcal{R}_5$ and

$$\sigma\mathcal{R}_5 \le y(t) \le \mathcal{R}_5, \ \forall\, y \in \mathcal{K} \cap \partial S_{\mathcal{R}_5}, \ \forall\, t \in \mathbb{R}.$$

From (7) we get

$$\begin{aligned}
(\Psi_\mu y)(t) =& \mu \int_0^\omega \mathcal{H}(t,s) \left(\frac{q(s)}{y^\rho(s)} + b(s)y^\delta(s) + c(s) \right) ds \\
=& \mu \int_0^\omega \mathcal{H}(t,s) \left(\frac{q^+(s)}{y^\rho(s)} - \frac{q^-(s)}{y^\rho(s)} + b(s)y^\delta(s) + c^+(s) - c^-(s) \right) ds \\
>& \mu \int_0^\omega \mathcal{H}(t,s) \frac{q^+(s)}{y^\rho(s)} ds \\
>& \mu \frac{\mathcal{H}_* \omega \overline{q^+}}{\mathcal{R}_5^\rho} \\
\ge& \mu \eta' \mathcal{R}_5.
\end{aligned}$$

Because $\mu\eta' > 1$, $(\Psi_\mu y)(t) \ge \mathcal{R}_5$, then (12) holds.

Then we prove that

$$\|\Psi_\mu y\| \le \|y\|, \ \text{ for } \ y \in \mathcal{K} \cap \partial S_{\mathcal{R}_6}. \tag{13}$$

Obviously, we can get $\|u\| = R_6$ and

$$\sigma\mathcal{R}_6 \le y(t) \le \mathcal{R}_6, \ \forall\, y \in \mathcal{K} \cap \partial S_{\mathcal{R}_6}, \ \forall\, t \in \mathbb{R}.$$

According to (7), it follows that

$$\begin{aligned}
(\Psi_\mu y)(t) =& \mu \int_0^\omega \mathcal{H}(t,s) \left(\frac{q(s)}{y^\rho(s)} + b(s)u^\delta(s) + c(s) \right) ds \\
=& \mu \int_0^\omega \mathcal{H}(t,s) \left(\frac{q^+(s)}{y^\rho(s)} - \frac{q^-(s)}{y^\rho(s)} + b(s)y^\delta(s) + c^+(s) - c^-(s) \right) ds \\
\le& \mu \left(\frac{\mathcal{H}^* \omega \overline{q^+}}{(\sigma\mathcal{R}_6)^\rho} - \frac{\mathcal{H}_* \omega \overline{q^+}}{\mathcal{R}_6^\rho} + \mathcal{H}^* \omega \bar{b} \mathcal{R}_6^\delta + \mathcal{H}^* \omega \overline{c^+} - \mathcal{H}_* \omega \overline{c^-} \right) \\
\le& \mu \left(\frac{\mathcal{B} \omega \overline{q^+}}{(\sigma\mathcal{R}_6)^\rho} + \mathcal{H}^* \omega \bar{b} \mathcal{R}_6^\delta + \mathcal{H}^* \omega \overline{c^+} \right).
\end{aligned}$$

There exists $\mu_2 > 0$ satisfying

$$\mu_2 < \min \left\{ \frac{\sigma^\rho \mathcal{R}_6^{\rho+1}}{\mathcal{H}^* \omega \overline{q^+} + \sigma^\rho \mathcal{R}_6^{\delta+\rho} \mathcal{H}^* \omega \bar{b} + \sigma^\rho \mathcal{R}_6^\rho \mathcal{H}^* \omega \overline{c^+}}, \mu_1 \right\}.$$

Therefore, for $\mu < \mu_2$, we know (13) holds.

It follows from Lemma 1 that Ψ_μ has a fixed point $y_2 \in \mathcal{K} \cap (\overline{S}_{\mathcal{R}_6} \backslash S_{\mathcal{R}_5})$, which is a positive periodic solution of equation (1) for $\mu < \mu_2$ and satisfying $\mathcal{R}_5 < \|y_2\| < \mathcal{R}_6$. Noting that

$$\mathcal{R}_3 < \|y_1\| < \mathcal{R}_4 < \mathcal{R}_5 < \|y_2\| < \mathcal{R}_6,$$

we can deduce that y_1 and y_2 are two desired distinct positive periodic solutions of equation (1) for $\mu < \mu_2$.

□

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