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Article

Upper estimates for initial coefficients and Fekete-Szegö functional of a class of bi-univalent functions defined by means of subordination and associated with Horadam polynomials

Atinuke Ayanfe Amao[1,*] and Timothy Oloyede Opoola[1]

[1] Department of Mathematics, Faculty of Physical Sciences, University of Ilorin. PMB 1515, Ilorin, Nigeria.
* Correspondence: tinnuke4praise@gmail.com

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Abstract: In this work, a new class of bi-univalent functions $I^{n+1}_{\Gamma_m,\lambda}(x,z)$ is defined by means of subordination. Upper bounds for some initial coefficients and the Fekete-Szegö functional of functions in the new class were obtained.

Keywords: Analytic functions,subordination, bi-univalent functions,Höradam polynomial,Opoola differential operator.

MSC: 30C45; 30C50; 30C55.

1. Introduction

Let A denote the class of functions that are analytic within the unit disk, defined as

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

Furthermore, let S be a subclass of A, comprising functions that are univalent in U and adhere to the normalization

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

The Koebe one-quarter theorem [1] assures that for every univalent function $f \in S$, its image in U encompasses a disk with a minimum radius of $\frac{1}{4}$. Consequently, every univalent function f possesses an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z \quad (z \in U), \quad f(f^{-1}(w)) = w \quad (|w| < r_0(f)), \quad r_0(f) \geq \frac{1}{4}. \tag{2}$$

Definition 1. A function $f \in S$ is termed bi-univalent in U if f is univalent in U and its inverse f^{-1} extends univalently to U.

The class of bi-univalent functions in U is denoted by Σ. Since each function $f \in \Sigma$ can be expressed by the Maclaurin series (1), it follows that its inverse $g = f^{-1}$ can be expanded as

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \dots. \tag{3}$$

Recent studies have focused on several classes of bi-univalent functions (see, for instance, references [2–4]).

Definition 2. For functions f and g in the class A, f is said to be subordinate to g if there exists a Schwarz function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$.

In particular, if g is univalent in U, subordination implies

$$f(0) = g(0) \quad \text{and} \quad f(U) \subseteq g(U).$$

We denote by

$$S^* = \{f \in S : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in U\}$$

and

$$K = \{f \in S : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, z \in U\}.$$

The classes S^* and K are referred to as the classes of starlike and convex functions, respectively.

A noteworthy concept in the theory of univalent functions, recently revisited in the context of singularity theory[5] and power series with integral coefficients, is the Hankel determinant $H_\delta(n)$ for functions $f(z) \in A$ with the form (1). For $\delta \geq 1$ and $m \geq 1$,

$$H_\delta(n) = \begin{vmatrix} a_m & a_{m+1} & \cdots & a_{m+\delta-1} \\ a_{m+1} & a_{m+2} & \cdots & a_{m+\delta} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m+\delta-1} & a_{m+\delta} & \cdots & a_{m+2(\delta-1)} \end{vmatrix}. \tag{4}$$

This determinant has been extensively investigated in the literature [6–8]. The Hankel determinant $H_2(1) = a_3 - a_2^2$ is well known as the Fekete-Szegö functional. Identifying the upper bound for $\lambda_\sigma(f) = |a_3 - \sigma a_2^2|$ over the class S, a generalization of $H_2(1)$, is recognized as the Fekete-Szegö problem, where σ is a real or complex number.

Using the Loewner method, Fekete and Szegö demonstrated in [9] that

$$\max_{f \in S} \lambda_\sigma(f) = \begin{cases} 1 + 2\exp\left(\frac{-2\sigma}{1-\sigma}\right), & \sigma \in [0,1], \\ 1, & \sigma = 1. \end{cases}$$

In 1969, the Fekete-Szegö problem for the classes of starlike functions S^* and convex functions K was investigated by Koegh and Merkes [10].

Recently, Hörcum and Gokcen [11] considered the Horadam polynomials $J_m(x)$, defined by the recurrence relation

$$J_m(x) = pxJ_{m-1}(x) + qJ_{m-2}(x), \tag{5}$$

where $m \in \mathbb{N} \setminus \{1, 2\}$, $\mathbb{N} = \{1, 2, 3, \ldots\}$, and a, h, p, q are specific real constants.

For particular cases:

1. For $a = h = p = q = 1$, we obtain the Fibonacci polynomial $F_n(x)$.
2. For $a = 1$, $h = p = 2$ and $q = -1$, we acquire the Chebyshev polynomial $U_n(x)$ of the second kind.

Definition 3. [12] Let $f \in A$. Define the operator $D^n(\mu, \beta, t)f(z)$ as follows:

$$\begin{aligned} &D^n(\mu, \beta, t)f(z) : A \to A, \quad n \in N_0 = N \cup \{0\}, \\ &\quad D^0(\mu, \beta, t)f(z) = f(z), \\ &\quad D^1(\mu, \beta, t)f(z) = tzf'(z) - z(\beta - \mu)t + (1 + (\beta - \mu - 1)t)f(z), \\ &\quad D^n(\mu, \beta, t)f(z) = zD_t[D^{n-1}(\mu, \beta, t)f(z)], \end{aligned}$$

where $D_t f(z) = 1 + \sum_{k=2}^{\infty}[1 + (k + \beta - \mu - 1)t]a_k z^k$. If $f(z)$ is expressed as in equation (1), then

$$D^n(\mu, \beta, t)f(z) = z + \sum_{k=2}^{\infty}[1 + (k + \beta - \mu - 1)t]^n a_k z^k,$$

with $0 \leq \mu \leq \beta$ and $t \geq 0$.

Remark 1.

- When $\beta = \mu$ and $t = 1$, $D^n_{\beta,\mu,t}f(z)$ reduces to the $D^n f(z)$ differential operator as defined by Salagean [13].

- For $\beta = \mu$, $D^n(\beta, \mu, t)f(z)$ corresponds to the differential operator $D^n_\lambda f(z)$ as defined by Al-boudi [14].

Definition 4. A function $f(z) \in A$ belongs to the class $I^{n+1}_{\Gamma_m,\lambda}(x,z)$ if

$$\frac{1}{2}\left[\frac{D^{n+1}_{\Gamma_m} f(z)}{D^n_{\Gamma_m} f(z)} + \left(\frac{D^{n+1}_{\Gamma_m} f(z)}{D^n_{\Gamma_m} f(z)}\right)^{\frac{1}{\lambda}}\right] \prec \prod(x,z) + 1 - a, \tag{6}$$

and similarly for $g(w)$,

$$\frac{1}{2}\left[\frac{D^{n+1}_{\Gamma_m} g(w)}{D^n_{\Gamma_m} g(w)} + \left(\frac{D^{n+1}_{\Gamma_m} g(w)}{D^n_{\Gamma_m} g(w)}\right)^{\frac{1}{\lambda}}\right] \prec \prod(x,w) + 1 - a, \tag{7}$$

where $n \in N_0 = N \cup \{0\}$, $\Gamma_m = 1 + (m + \beta - \mu)t$, $m \in N$, $\mu \in [0, \beta]$, $t \geq 0$, $0 < \lambda \leq 1$, and $f(z)$, $g(w)$ are as previously defined. Here, $D^n_{\Gamma_m} f(z)$ denotes the Opoola differential operator, $\prod(x,z) = \sum_{m=1}^{\infty} j_m(x) z^{m-1}$, and $z, w \in U$.

Remark 2.
- For $n = 0$, we obtain the class $W_\Sigma(\mu; x)$ of functions satisfying:

$$\frac{1}{2}\left[\frac{zf'(z)}{f(z)} + \left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\lambda}}\right] \prec \prod(x,z) + 1 - a, \tag{8}$$

  and similarly for $g(w)$. When $n = 0$ and $\lambda = 1$, this class reduces to $W_\Sigma(x)$, comprising functions $f \in A$ satisfying certain criteria, as studied by Srivastava et al. [15].
- For $n = 0$, $a = 1$, and other specific parameters, the class $S_\Sigma(\mu; t)$, as studied by Altinkaya and Yakin [16], is obtained.

Let P denote the class of functions with positive real part. A function $p \in P$ implies that

$$p(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \dots, \quad z \in U, \tag{9}$$

with $\Re p(z) > 0$. It is known that

$$p(z) = \frac{w(z) - 1}{w(z) + 1}, \tag{10}$$

where $w(z)$ is a Schwarz function.

Lemma 1. *[17] If $p(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \dots$ is in P, then $|b_m| \leq 2$.*

Lemma 1 [17]:

Lemma 2. *[17] If $p(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \dots$ is in P, then*

$$\left|b_2 - \sigma b_1^2\right| \leq \begin{cases} 2 - \sigma|b_1|^2, & \sigma \leq \frac{1}{2}, \\ 2 - (1-\sigma)|b_1|^2, & \sigma \geq \frac{1}{2}. \end{cases} \tag{11}$$

2. Main results

Theorem 1. *. Let $f(z) \in I^{n+1}_{\Gamma_m,\lambda}(x,z)$, then*

$$|a_2| \leq \frac{2\lambda\sqrt{|hx|}|hx|}{\sqrt{|[2\lambda(1+\lambda)\Gamma_2^n(\Gamma_2 - 1) + \Gamma_1^{2n}(4\lambda\Gamma_1 - 1 - 3\lambda)](hx)^2 + (hx - (phx^2 + qa))(1+\lambda)^2(\Gamma_1 - 1)^2\Gamma_1^{2n}|}} \tag{12}$$

$$|a_3| \leq \frac{2\lambda|hx|(1+2\lambda)}{\Gamma_2^n(\Gamma_2 - 1)(1+\lambda)} + \frac{4\lambda^2(hx)^2}{(1+\lambda)^2\Gamma_1^{2n}(\Gamma_1 - 1)^2} \tag{13}$$

Proof. Let $f(z) \in I^{n+1}_{\Gamma_m,\lambda}(x,z)$.By definition of Subordination, there exists functions $\Theta(z)$ and $\Phi(z)$, $\Theta(0) = \Phi(0) = 0$ and $|\Theta(z)| < 1$ and $|\Phi(w)| < 1\ \forall z, w \in U$ such that

$$\frac{1}{2}\left[\frac{D^{n+1}_{\Gamma_m}f(z)}{D^{n}_{\Gamma_m}f(z)} + (\frac{D^{n+1}_{\Gamma_m}f(z)}{D^{n}_{\Gamma_m}f(z)})^{\frac{1}{\lambda}}\right] = \prod(x,\Theta(z)) + 1 - a \tag{14}$$

and

$$\frac{1}{2}\left[\frac{D^{n+1}_{\Gamma_m}g(w)}{D^{n}_{\Gamma_m}g(w)} + (\frac{D^{n+1}_{\Gamma_m}g(w)}{D^{n}_{\Gamma_m}g(w)})^{\frac{1}{\lambda}}\right] = \prod(x,\Phi(w)) + 1 - a \tag{15}$$

From 15,

$$\frac{1}{2}\left[\frac{D^{n+1}_{\Gamma_m}f(z)}{D^{n}_{\Gamma_m}f(z)} + (\frac{D^{n+1}_{\Gamma_m}f(z)}{D^{n}_{\Gamma_m}f(z)})^{\frac{1}{\lambda}}\right] = \sum_{m=1}^{\infty} j_m(x)(\Theta(z))^{m-1} + 1 - a$$

$$\frac{1}{2}\left[\frac{D^{n+1}_{\Gamma_m}f(z)}{D^{n}_{\Gamma_m}f(z)} + (\frac{D^{n+1}_{\Gamma_m}f(z)}{D^{n}_{\Gamma_m}f(z)})^{\frac{1}{\lambda}}\right] = j_1(x) + j_2(x)\Theta(z) + j_3(x)[\Theta(z)]^2 + j_4(x)[\Theta(z)]^3 + ... + 1 - a \tag{16}$$

Also, from 16

$$\frac{1}{2}\left[\frac{D^{n+1}_{\Gamma_m}g(w)}{D^{n}_{\Gamma_m}g(w)} + (\frac{D^{n+1}_{\Gamma_m}g(w)}{D^{n}_{\Gamma_m}g(w)})^{\frac{1}{\lambda}}\right] = j_1(x) + j_2(x)\Phi(w) + j_3(x)[\Phi(w)]^2 + j_4(x)[\Phi(w)]^3 + ... + 1 - a \tag{17}$$

From equations 11 and 12

$$\Theta(z) = \frac{p(z) - 1}{p(z) + 1}$$

$$= (b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + ...)(2 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + ...)^{-1}$$

$$\Theta(z) = \frac{1}{2}[b_1 z + (b_2 - \frac{b_1^2}{2})z^2 + (b_3 - b_1 b_2 + \frac{b_1^3}{4})z^3 + ...] \tag{18}$$

$$[\Theta(z)]^2 = \frac{1}{4}[b_1^2 z^2 + 2(b_1 b_2 - \frac{b_1^3}{2})z^3 + (b_1 b_3 - b_1^2 b_2 + \frac{2b_1^4}{4} - b_1^3 b_2 + b_1^2 b_2^2)z^4 + ...] \tag{19}$$

$$[\Theta(z)]^3 = \frac{1}{8}[b_1^3 z^3 + 3(b_1^2 b_2 - \frac{b_1^4}{2})z^4 + ...] \tag{20}$$

$$\Phi(w) = \frac{\rho(w) - 1}{\rho(w) + 1}$$

$$= (d_1 w + d_2 w^2 + d_3 w^3 + b_4 z^4 + ...)(2 + d_1 w + d_2 w^2 + d_3 w^3 + b_4 w^4 + ...)^{-1}$$

$$\Phi(w) = \frac{1}{2}[d_1 w + (d_2 - \frac{d_1^2}{2})w^2 + (d_3 - d_1 d_2 + \frac{d_1^3}{4})w^3 + ...] \tag{21}$$

$$[\Phi(w)]^2 = \frac{1}{4}[d_1^2 w^2 + 2(d_1 d_2 - \frac{d_1^3}{2})w^3 + (d_1 d_3 - d_1^2 d_2 + \frac{2d_1^4}{4} - d_1^3 d_2 + d_1^2 d_2^2)w^4 + ...] \tag{22}$$

$$[\Phi(w)]^3 = \frac{1}{8}[d_1^3 w^3 + 3(d_1^2 d_2 - \frac{d_1^4}{2})w^4 + ...] \tag{23}$$

Substituting 19, 20 and 21 into RHS of 17, we have

$$\frac{1}{2}\left[\frac{D^{n+1}_{\Gamma_m}f(z)}{D^{n}_{\Gamma_m}f(z)} + (\frac{D^{n+1}_{\Gamma_m}f(z)}{D^{n}_{\Gamma_m}f(z)})^{\frac{1}{\lambda}}\right] = 1 + j_2(x)\frac{b_1}{2}z + \frac{1}{2}[j_3(x)\frac{b_1^2}{2} + j_2(x)(b_2 - \frac{b_1^2}{2})]z^2$$
$$+ \frac{1}{2}[j_4(x)\frac{b_1^3}{4} + j_3(x)(b_1 b_2 - \frac{b_1^3}{2}) + j_2(x)(b_3 - b_1 b_2 + \frac{b_1^3}{4})]z^3 + ... \tag{24}$$

Substituting 22, 23 and 24 into RHS of 18, we have

$$\frac{1}{2}\left[\frac{D^{n+1}_{\Gamma_m}g(w)}{D^{n}_{\Gamma_m}g(w)}+(\frac{D^{n+1}_{\Gamma_m}g(w)}{D^{n}_{\Gamma_m}g(w)})^{\frac{1}{\lambda}}\right]=1+j_2(x)\frac{d_1}{2}z+\frac{1}{2}[j_3(x)\frac{d_1^2}{2}+j_2(x)(d_2-\frac{d_1^2}{2})]z^2$$
$$+\frac{1}{2}[j_4(x)\frac{d_1^3}{4}+j_3(x)(d_1d_2-\frac{d_1^3}{2})+j_2(x)(d_3-d_1d_2+\frac{d_1^3}{4})]z^3+... \quad (25)$$

Considering the *LHS* of 17

$$\frac{1}{2}\left[\frac{D^{n+1}_{\Gamma_m}f(z)}{D^{n}_{\Gamma_m}f(z)}+(\frac{D^{n+1}_{\Gamma_m}f(z)}{D^{n}_{\Gamma_m}f(z)})^{\frac{1}{\lambda}}\right]=\frac{1}{2}\left[\frac{D^{n+1}_{\Gamma_m}f(z)(D^{n}_{\Gamma_m}f(z))^{\frac{1}{\lambda}}+(D^{n+1}_{\Gamma_m}f(z))^{\frac{1}{\lambda}}D^{n}_{\Gamma_m}f(z)}{(D^{n}_{\Gamma_m}f(z))^{\frac{1+\lambda}{\lambda}}}\right] \quad (26)$$

$$D^{n+1}_{\Gamma_m}f(z)=z+\sum_{k=2}^{\infty}[1+(k+\beta-\mu-1)t]^{n+1}a_kz^k=z+\Gamma_1^{n+1}a_2z^2+\Gamma_2^{n+1}a_3z^3+\Gamma_3^{n+1}a_4z^4+... \quad (27)$$

$$[D^{n+1}_{\Gamma_m}f(z)]^{\frac{1}{\lambda}}=[z+\Gamma_1^{n+1}a_2z^2+\Gamma_2^{n+1}a_3z^3+\Gamma_3^{n+1}a_4z^4+...]^{\frac{1}{\lambda}} \quad (28)$$

$$D^{n}_{\Gamma_m}f(z)=z+\sum_{k=2}^{\infty}[1+(k+\beta-\mu-1)t]^{n}a_kz^k=z+\Gamma_1^{n}a_2z^2+\Gamma_2^{n}a_3z^3+\Gamma_3^{n}a_4z^4+... \quad (29)$$

$$[D^{n}_{\Gamma_m}f(z)]^{\frac{1}{\lambda}}=[z+\Gamma_1^{n}a_2z^2+\Gamma_2^{n}a_3z^3+\Gamma_3^{n}a_4z^4+...]^{\frac{1}{\lambda}} \quad (30)$$

$$[D^{n}_{\Gamma_m}f(z)]^{\frac{1+\lambda}{\lambda}}=[z+\Gamma_1^{n}a_2z^2+\Gamma_2^{n}a_3z^3+\Gamma_3^{n}a_4z^4+...]^{\frac{1+\lambda}{\lambda}} \quad (31)$$

Equation 28 multiplied by 31, gives

$$[D^{n+1}_{\Gamma_m}f(z)][D^{n}_{\Gamma_m}f(z)]^{\frac{1}{\lambda}}=z^{\frac{1}{\lambda}}+(\frac{1}{\lambda}\Gamma_1^na_2+\Gamma_1^{n+1}a_2)z^{\frac{1+2\lambda}{\lambda}}$$
$$+(\frac{1}{\lambda}\Gamma_2^na_3+\frac{1}{\lambda}\Gamma_1^{2n+1}a_2^2+\frac{1-\lambda}{2\lambda^2}\Gamma_1^{2n}a_2^2+\Gamma_2^{n+1}a_3)z^{\frac{1+3\lambda}{\lambda}}$$
$$+(\frac{1}{\lambda}\Gamma_3^na_4+\frac{1-\lambda}{\lambda^2}\Gamma_1^n\Gamma_2^na_2a_3+\frac{(1-\lambda)(1-2\lambda)\Gamma_1^{3n+3}a_2^3}{6\lambda^3}$$
$$+\frac{1}{\lambda}\Gamma_1^{n+1}\Gamma_2^na_2a_3+\frac{1-\lambda}{2\lambda^2}\Gamma_1^{3n+1}a_2^3$$
$$+\frac{1}{\lambda}\Gamma_1^n\Gamma_2^{n+1}a_2a_3+\Gamma_3n+1a_4)z^{\frac{1+4\lambda}{\lambda}}+... \quad (32)$$

Equation 29 multiplied by 30, gives

$$[D^{n+1}_{\Gamma}mf(z)]^{\frac{1}{\lambda}}[D^{n}_{\Gamma m}f(z)]=z^{\frac{1+\lambda}{\lambda}}+(\frac{1}{\lambda}\Gamma_1^{n+1}a_2+\Gamma_1^na_2)z^{\frac{1+2\lambda}{\lambda}}+$$
$$(\frac{1}{\lambda}\Gamma_2^{n+1}a_3+\frac{1}{\lambda}\Gamma_1^{2n+1}+\frac{1-\lambda}{2\lambda^2}\Gamma_1^{2n+2}a_2^2+\Gamma_2^na_3)z^{\frac{1+3\lambda}{\lambda}}$$
$$+(\frac{1}{\lambda}\Gamma_1^{n+1}a_4+\frac{1-\lambda}{\lambda^2}\Gamma_1^{n+1}\Gamma_2^{n+1}a_2a_3$$
$$+\frac{(1-\lambda)(1-2\lambda)}{6\lambda^3}\Gamma_1^{3n+3}a_2^3+\frac{1}{\lambda}\Gamma_2^{n+1}\Gamma_1^na_2a_3+\frac{1-\lambda}{2\lambda^2}\Gamma_1^{3n+2}a_2^3+\frac{1}{\lambda}\Gamma_1^{n+1}\Gamma_2^na_2a_3+\Gamma_3^na_4)z^{\frac{1+4\lambda}{\lambda}}+.... \quad (33)$$

$$2(D^n_{\Gamma}mf(z))^{\frac{1+\lambda}{\lambda}}=2z^{\frac{1+\lambda}{\lambda}}+\frac{2(1+\lambda)}{\lambda}\Gamma_1^na_2z^{\frac{1+2\lambda}{\lambda}}$$
$$+2\left(\frac{1+\lambda}{\lambda}\Gamma_2^na_3+\frac{1+\lambda}{2\lambda^2}\Gamma_1^{2n}a_2^2\right)z^{\frac{1+3\lambda}{\lambda}} \quad (34)$$
$$+2\left(\frac{1+\lambda}{\lambda}\Gamma_3^na_4+\frac{1+\lambda}{\lambda^2}\Gamma_1^n\Gamma_2^na_2a_3+\frac{(1+\lambda)(1-\lambda)}{6\lambda^3}\Gamma_1^{3n}a_2^3\right)z^{\frac{1+4\lambda}{\lambda}}+\dots$$

Substituting 33, 34 and 35 into 25 and simplifying further, we have

$$1+[\frac{1+\lambda}{2\lambda}\Gamma_1^n(\Gamma_1+1)]a_2z+\frac{1}{2}[(\frac{1+\lambda}{\lambda}\Gamma_2^n(\Gamma_2+1))a_3+\frac{2}{\lambda}\Gamma_1^{2n+1}a_2^2+\frac{1-\lambda}{2\lambda^2}\Gamma_1^{2n}(\Gamma_1+1)a_2^2]z^2+...=1$$
$$+[\frac{1+\lambda}{\lambda}\Gamma_1^n a_2+\frac{j_2(x)}{2}b_1]z$$
$$+[\frac{1+\lambda}{\lambda}\Gamma_2^n a_3+\frac{1+\lambda}{2\lambda^2}\Gamma_1^{2n}a_2^2+\frac{1+\lambda}{2\lambda}j_2(x)b_1+j_3(x)\frac{b_1^2}{4}+\frac{j_2(x)}{2}(b_2-\frac{b_1^2}{2})]z^2+... \quad (35)$$

Also, 18 becomes,

$$1-[\frac{1+\lambda}{2\lambda}\Gamma_1^n(\Gamma_1+1)]a_2z+\frac{1}{2}[(\frac{1+\lambda}{\lambda}\Gamma_2^n(\Gamma_2+1))(2a_2^2-a_3)+\frac{2}{\lambda}\Gamma_1^{2n+1}a_2^2+\frac{1-\lambda}{2\lambda^2}\Gamma_1^{2n}(\Gamma_1+1)a_2^2]z^2+...=$$
$$1+[\frac{1+\lambda}{\lambda}\Gamma_1^n a_2+\frac{j_2(x)}{2}d_1]z$$
$$+[\frac{1+\lambda}{\lambda}\Gamma_2^n a_3+\frac{1+\lambda}{2\lambda^2}\Gamma_1^{2n}a_2^2+\frac{1+\lambda}{2\lambda}j_2(x)d_1+j_3(x)\frac{d_1^2}{4}+\frac{j_2(x)}{2}(d_2-\frac{d_1^2}{2})]z^2+... \quad (36)$$

Comparing the coefficients of z in 36, we have

$$[\frac{1+\lambda}{2\lambda}\Gamma_1^n(\Gamma_1+1)]a_2=\frac{1+\lambda}{\lambda}\Gamma_1^n a_2+\frac{j_2(x)}{2}b_1$$
$$\frac{1+\lambda}{2\lambda}\Gamma_1^n(\Gamma_1-1)a_2=j_2(x)\frac{b_1}{2} \quad (37)$$
$$a_2=\frac{\lambda j_2(x)b_1}{(1+\lambda)\Gamma_1^n(\Gamma_1-1)} \quad (38)$$

$$[\frac{1+\lambda}{\lambda}\Gamma_2^n(\Gamma_2+1)]a_3+\frac{2}{\lambda}\Gamma_1^{2n+1}a_2^2+\frac{1-\lambda}{2\lambda^2}\Gamma_1^{2n}(\Gamma_1+1)a_2^2=[\frac{1+\lambda}{\lambda}\Gamma_2^n a_3+$$
$$\frac{1+\lambda}{2\lambda^2}\Gamma_1^{2n}a_2^2+\frac{1+\lambda}{2\lambda}j_2(x)b_1+j_3(x)\frac{b_1^2}{4}+\frac{j_2(x)}{2}(b_2-\frac{b_1^2}{2})] \quad (39)$$

$$\frac{(\Gamma_2-1)(1+\lambda)}{\lambda j_2(x)}\Gamma_2^n a_3+\frac{4\lambda\Gamma_1-1-3\lambda}{2\lambda^2 j_2(x)}\Gamma_1^{2n}a_2^2=b_2-\left(\frac{j_2(x)-j_3(x)}{j_2(x)}\right)\frac{b_1^2}{2}+\frac{1+\lambda}{\lambda}b_1 \quad (40)$$

Comparing the coefficient of z in 37

$$[\frac{1+\lambda}{2\lambda}\Gamma_1^n(\Gamma_1+1)]a_2=\frac{1+\lambda}{\lambda}\Gamma_1^n a_2+\frac{j_2(x)}{2}d_1$$
$$-[\frac{1+\lambda}{2\lambda}]\Gamma_1^n(\Gamma_1-1)a_2=j_2(x)\frac{d_1}{2} \quad (41)$$
$$-a_2=\frac{\lambda j_2(x)d_1}{(1+\lambda)\Gamma_1^n(\Gamma_1-1)} \quad (42)$$

$$[\frac{1+\lambda}{\lambda}\Gamma_2^n(\Gamma_2+1)](2a_2^2-a_3)+\frac{2}{\lambda}\Gamma_1^{2n+1}a_2^2+\frac{1-\lambda}{2\lambda^2}\Gamma_1^{2n}(\Gamma_1+1)a_2^2=[\frac{1+\lambda}{\lambda}\Gamma_2^n(2a_2^2-a_3)+$$
$$\frac{1+\lambda}{2\lambda^2}\Gamma_1^{2n}a_2^2+\frac{1+\lambda}{2\lambda}j_2(x)d_1+j_3(x)\frac{d_1^2}{4}+\frac{j_2(x)}{2}(d_2-\frac{d_1^2}{2})] \quad (43)$$

$$\frac{(\Gamma_2-1)(1+\lambda)}{\lambda j_2(x)}\Gamma_2^n(2a_2^2-a_3)+\frac{4\lambda\Gamma_1-1-3\lambda}{2\lambda^2 j_2(x)}\Gamma_1^{2n}a_2^2=d_2-\left(\frac{j_2(x)-j_3(x)}{j_2(x)}\right)\frac{d_1^2}{2}+\frac{1+\lambda}{\lambda}d_1 \quad (44)$$

Substituting 39 into 43, gives

$$-\frac{\lambda j_2(x)b_1}{(1+\lambda)\Gamma_1^n(\Gamma_1-1)}=\frac{\lambda j_2(x)d_1}{(1+\lambda)\Gamma_1^n(\Gamma_1-1)} \quad (45)$$

$$\Longrightarrow b_1 = -d_1 \Rightarrow b_1^2 = d_1^2 \Rightarrow b_1^3 = -d_1^3 \tag{46}$$

Square and add 38 and 42, we have

$$2\frac{(1+\lambda)^2}{\lambda^2}\Gamma_1^{2n}(\Gamma_1-1)^2 a_2^2 = j_2^2(x)[b_1+d_1]^2 \tag{47}$$

Adding 66 and 45 with further simplification

$$\frac{2(\Gamma_2-1)(1+\lambda)}{\lambda j_2(x)}\Gamma_2^n a_2^2 + \frac{4\lambda\Gamma_1 - 1 - 3\lambda}{\lambda^2 j_2(x)}\Gamma_1^{2n} a_2^2 = (b_2+d_2) - \left(\frac{j_2(x)-j_3(x)}{j_2(x)}\right)\left(\frac{b_1^2}{2}+\frac{d_1^2}{2}\right) + \frac{1+\lambda}{\lambda}(b_1+d_1) \tag{48}$$

$$\left[\frac{2(\Gamma_2-1)(1+\lambda)}{\lambda j_2(x)}\Gamma_2^n + \frac{4\lambda\Gamma_1 - 1 - 3\lambda}{\lambda^2 j_2(x)}\Gamma_1^{2n}\right] a_2^2 = (b_2+d_2) - \left(\frac{j_2(x)-j_3(x)}{j_2(x)}\right)\left(\frac{b_1^2}{2}+\frac{d_1^2}{2}\right) + \frac{1+\lambda}{\lambda}(b_1+d_1) \tag{49}$$

Using 47 in 50, gives

$$\left[\frac{2\lambda(\Gamma_2-1)(1+\lambda)\Gamma_2^n + (4\lambda\Gamma_1 - 1 - 3\lambda)\Gamma_1^{2n}}{\lambda^2 j_2(x)}\right] a_2^2 = (b_2+d_2) - \left(\frac{j_2(x)-j_3(x)}{j_2(x)}\right)\left(\frac{d_1^2}{2}\right) \tag{50}$$

Using 47 to simplify 48, gives

$$d_1^2 = \frac{(1+\lambda)^2}{j_2^2(x)\lambda^2}\Gamma_1^{2n}(\Gamma_1-1)^2 a_2^2 \tag{51}$$

substituting 52 in 51 with further simplification, we obtain

$$a_2^2 = \frac{\lambda^2 j_2^3(x)(b_2+d_2)}{2\lambda j_2^2(x)(\Gamma_2-1)(1+\lambda)\Gamma_2^n + j_2^2(x)(4\lambda\Gamma_1-1-3\lambda)\Gamma_1^{2n} + (j_2(x)-j_3(x))(1+\lambda)^2\Gamma_1^{2n}(\Gamma_1-1)^2} \tag{52}$$

$$|a_2|^2 = \left|\frac{\lambda^2 j_2^3(x)(b_2+d_2)}{2\lambda j_2^2(x)(\Gamma_2-1)(1+\lambda)\Gamma_2^n + j_2^2(x)(4\lambda\Gamma_1-1-3\lambda)\Gamma_1^{2n} + (j_2(x)-j_3(x))(1+\lambda)^2\Gamma_1^{2n}(\Gamma_1-1)^2}\right| \tag{53}$$

Using Lemma 1 and 2, we have

$$|a_2|^2 = \left|\frac{4\lambda^2|hx|^2|hx|}{|2\lambda(hx)^2(\Gamma_2-1)(1+\lambda)\Gamma_2^n} + (hx)^2(4\lambda\Gamma_1-1-3\lambda)\Gamma_1^{2n} + (hx-(phx^2+qa))(1+\lambda)^2\Gamma_1^{2n}(\Gamma_1-1)^2\right|. \tag{54}$$

Therefore,

$$|a_2| \leq \frac{2\lambda\sqrt{|hx|}|hx|}{\sqrt{[2\lambda(1+\lambda)\Gamma_2^n(\Gamma_2-1) + (4\lambda\Gamma_1-1-3\lambda)](hx)^2 + (hx-(phx^2+qa))(1+\lambda)^2(\Gamma_1-1)^2\Gamma_1^{2n}}} \tag{55}$$

Subtracting 45 from 46 and using 47 for further simplification, we have

$$\frac{2(\Gamma_2-1)(1+\lambda)\Gamma_2^n}{\lambda j_2(x)}(a_3-a_2^2) = (b_2-d_2) - \frac{2(1+\lambda)}{\lambda}d_1 \tag{56}$$

$$a_3 = \frac{\lambda(b_2-d_2)j_2(x)}{2\Gamma_2^n(\Gamma_2-1)(1+\lambda)} - \frac{j_2(x)d_1}{(\Gamma_2-1)\Gamma_2^n} + a_2^2 \tag{57}$$

Obtain a_2^2 from 52 and substitute it into 58, gives

$$a_3 = \frac{\lambda(b_2 - d_2)j_2(x)}{2\Gamma_2^n(\Gamma_2 - 1)(1+\lambda)} - \frac{j_2(x)d_1}{(\Gamma_2 - 1)\Gamma_2^n} + \frac{j_2^2(x)d_1^2\lambda^2}{(1+\lambda)^2\Gamma_1^{2n}(\Gamma_1 - 1)^2} \tag{58}$$

$$|a_3| = |\frac{\lambda(b_2 - d_2)j_2(x)}{2\Gamma_2^n(\Gamma_2 - 1)(1+\lambda)} - \frac{j_2(x)d_1}{(\Gamma_2 - 1)\Gamma_2^n} + \frac{j_2^2(x)d_1^2\lambda^2}{(1+\lambda)^2\Gamma_1^{2n}(\Gamma_1 - 1)^2}| \tag{59}$$

$$|a_3| \leq |\frac{4\lambda(b_2 - d_2)j_2(x))}{2\Gamma_2^n(\Gamma_2 - 1)(1+\lambda)} - \frac{j_2(x)d_1}{(\Gamma_2 - 1)\Gamma_2^n}| + |\frac{j_2^2(x)d_1^2\lambda^2}{(1+\lambda)^2\Gamma_1^{2n}(\Gamma_1 - 1)^2}| \tag{60}$$

$$|a_3| \leq \frac{2\lambda|hx|}{\Gamma_2^n(\Gamma_2 - 1)(1+\lambda)} + \frac{2|hx|}{(\Gamma_2 - 1)\Gamma_2^n} + \frac{4\lambda^2(hx)^2}{(1+\lambda)^2\Gamma_1^{2n}(\Gamma_1 - 1)^2} \tag{61}$$

$$|a_3| \leq \frac{2\lambda|hx|(1+2\lambda)}{\Gamma_2^n(\Gamma_2 - 1)(1+\lambda)} + \frac{4\lambda^2(hx)^2}{(1+\lambda)^2\Gamma_1^{2n}(\Gamma_1 - 1)^2} \tag{62}$$

From 56 and 63, we have the desired result. □ □

Remark 3. It has been observed that when $n = 0$, $t = 1$ and $\beta = \mu$,the result obtained gives an improved estimate of the second coefficient of $W_\Sigma(x;z)$ studied by Srivastava, et al(2018).

Theorem 2. *. Let $f(z) \in I_{\Gamma_m,\lambda}^{n+1}(x,z)$, also let $\sigma \in \Re$ then*

$$|a_3 - \sigma a_2^2| \leq \begin{cases} \frac{4(1+2\lambda)|hx|}{|\Gamma_2^n(\Gamma_2-1)(1+\lambda)|}, \; for \; |1-\sigma| \leq \frac{(1+2\lambda)|hx||\Upsilon+\Psi|}{2\lambda^2|hx|^3|\Gamma_2^n(\Gamma_2-1)(1+\lambda)|} \\ \frac{8|1-\sigma|\lambda^2|hx|^3}{|\Upsilon+\Psi|}, \; for \; |1-\sigma| \geq \frac{1+2\lambda|hx||\Upsilon+\Psi|}{2\lambda^2|hx|^3|\Gamma_2^n(\Gamma_2-1(1+\lambda))|} \end{cases} \tag{63}$$

Where

$$\Upsilon = [2\lambda(1+\lambda)\Gamma_2^n(\Gamma_2 - 1) + (4\lambda\Gamma_1 - 1 - 3\lambda)](hx)^2 \tag{64}$$

and

$$\Psi = (hx - (phx^2 + qa))(1+\lambda)^2(\Gamma_1 - 1)^2\Gamma_1^{2n} \tag{65}$$

Proof. From 53 and 58,

$$a_3 - \sigma a_2^2 = \frac{\lambda(b_2 - d_2)j_2(x)}{2\Gamma_2^n(\Gamma_2 - 1)(1+\lambda)} - \frac{j_2(x)d_1}{(\Gamma_2 - 1)\Gamma_2^n} + a_2^2 - \sigma a_2^2 \tag{66}$$

Let

$$\Upsilon = [2\lambda(1+\lambda)\Gamma_2^n(\Gamma_2 - 1) + (4\lambda\Gamma_1 - 1 - 3\lambda)](hx)^2$$

and

$$\Psi = (hx - (phx^2 + qa))(1+\lambda)^2(\Gamma_1 - 1)^2\Gamma_1^{2n}$$

$$a_3 - \sigma a_2^2 = \frac{\lambda(b_2 - d_2)j_2(x)}{2\Gamma_2^n(\Gamma_2 - 1)(1+\lambda)} - \frac{j_2(x)d_1}{(\Gamma_2 - 1)\Gamma_2^n} + (1-\sigma)\left(\frac{\lambda^2 j_2^3(b_2 + d_2)}{\Upsilon + \Psi}\right) \tag{67}$$

$$|a_3 - \sigma a_2^2| = \left|\frac{\lambda(b_2 - d_2)j_2(x)}{2\Gamma_2^n(\Gamma_2 - 1)(1+\lambda)} - \frac{j_2(x)d_1}{(\Gamma_2 - 1)\Gamma_2^n} + (1-\sigma)\left\{\frac{\lambda^2 j_2^3(b_2 + d_2)}{\Upsilon + \Psi}\right\}\right| \tag{68}$$

$$|a_3 - \sigma a_2^2| \leq \frac{2(1+2\lambda)|hx|}{|\Gamma_2^n(\Gamma_2 - 1)(1+\lambda)|} + |1-\sigma|\frac{4\lambda^2|hx|^3}{|\Upsilon + \Psi|} \tag{69}$$

If

$$|1-\sigma|\frac{4\lambda^2|hx|^3}{|\Upsilon+\Psi|} \leq \frac{2(1+2\lambda)|hx|}{|\Gamma_2^n(\Gamma_2-1)(1+\lambda)|}, \tag{70}$$

then

$$|a_3-\sigma a_2^2| \leq \frac{4(1+2\lambda)|hx|}{|\Gamma_2^n(\Gamma_2-1)(1+\lambda)|} \tag{71}$$

Also, if

$$|1-\sigma|\frac{4\lambda^2|hx|^3}{|\Upsilon+\Psi|} \geq \frac{2(1+2\lambda)|hx|}{|\Gamma_2^n(\Gamma_2-1)(1+\lambda)|}, \tag{72}$$

then

$$|a_3-\sigma a_2^2| \leq \frac{8|1-\sigma|\lambda^2|hx|^3}{|\Upsilon+\Psi|} \tag{73}$$

From 71,72,73 and 74, we have the desired result. □ □

Corollary 1: If $f(z) \in S$ and $I^{n+1}_{\Gamma_m,\lambda}(x,z)$, then

$$|a_3-a_2^2| \leq \frac{4(1+2\lambda)|hx|}{|\Gamma_2^n(\Gamma_2-1)(1+\lambda)|}$$

3. Concclusion

The investigation in this work was driven from Geometric Function Theory(GFT) where several interesting and various special functions and polynomials can be studied. Horadam polynomials $j_n(x)$ and a generalized differential operator were used to define a new class $I^{n+1}_{\Gamma_m,\lambda}(x,z)$ of bi-univalent functions by means of subordination. We obtained upper estimates for initial coefficient and Fekete-szegö functional of the newly defined class of bi-univalent function.

The results obtained have important roles in complex and potential theory, with application in real life like: error analysis in numerical method, controlled growth of analytic function, design of approximation algorithms, electrical engineering, physics and materials science, statistics and probability theory.In essence, these mathematical concepts provide tools and insights that transcend pure mathematics, finding applications in diverse fields where analytical and computational methods are employed for modeling, analysis and design.

The geometric properties of the function class $I^{n+1}_{\Gamma_m,\lambda}(x,z)$ vary when the parameters in the class are changed: when $n=0$,$t=1$, $\beta=\mu$, then we have

$$\frac{1}{2}[\frac{zf'(z)}{f(z)}+(\frac{zf'(z)}{f(z)})^{\frac{1}{\lambda}}] \prec \prod(x,z)+1-a$$

Also, when $n=0$, $t=1$, $\beta=\mu$, $a=p=x$, $\lambda=1$ and $q=0$, then we have

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}$$

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