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Asymptotic stability of stochastic single-species model under regime switching

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Abstract: This paper investigates the stationary probability distribution of the well-known stochastic logistic equation under regime switching. Sufficient conditions for the asymptotic stability of both the zero solution and the positive equilibrium are derived. The stationary distribution of the logistic equation under Markovian switching is obtained by computing the weighted mean of the stationary distributions of its subsystems. The weights correspond to the limiting distribution of the underlying Markov chain.

Keywords: logistic model, markov chains, regime switching, asymptotic stability.

MSC: 60H10; 92B05.

1. Introduction: Background and Research Aims

Logistic equation is one of the most important models in mathematical ecology. The classical logistic equation can be expressed as follows

$$\mathrm{d}x(t) = rx(t)\Big(1 - \frac{x(t)}{K}\Big)\mathrm{d}t. \tag{1}$$

For model (1), a famous result is that if $r < 0$, then $\lim_{t\to\infty} x(t) = 0$; if $r > 0$, then $\lim_{t\to\infty} x(t) = K$ (see e.g., Murray [1]).

In recent years, random systems have received more and more attentions, and many people have studied this (see e.g., [2–8]). In [9], Mao pointed out that small environmental noise may have different effects on the growth rate of species, that is, white noise can be used to simulate environmental disturbance, which is the most common method, such as [10–18]. Suppose that the growth rate r is affected by environmental noise with

$$r \to r + \sigma\dot{B}(t).$$

From (1), we can obtain the Itô type stochastic model

$$\mathrm{d}x(t) = x(t)(1 - \frac{x(t)}{K})\Big[r\mathrm{d}t + \sigma \mathrm{d}B(t)\Big]. \tag{2}$$

Consider the natural growth of many populations vary with t, Liu and Wang in [19] studied the stochastic non-autonomous logistic equation

$$\mathrm{d}x(t) = x(t)(1 - \frac{x(t)}{K})\Big[r(t)\mathrm{d}t + \sigma(t)\mathrm{d}B(t)\Big]. \tag{3}$$

They investigated the effect of white noise on the stability of these two equilibria a: 0 and K for (3).

However, large and sudden environmental disturbance are unavoidable, such as earthquakes, tsunamis, hurricanes, floods, or droughts may have important consequences on the system. Therefore, in addition to the small disturbances described by the white noise, there are also some environmental noises that will obviously change the population growth at random times, making the population growth switch from one state to another. It cannot be represented by the stochastic differential equation driven by the standard Brownian motion, but needs to be modeled by the continuous time Markov chain. Many researches such as [20–26] and the references therein show that this regime switching can be described by a right-continuous Markov chain taking value in a finite state space. Suppose $\xi(t)$ represents a right continuous Markov chain in state space $S = \{1, 2, ..., N\}$, which is independent of $B(t)$. Thus it is reasonable and important to study the following logistic equation and stochastic logistic equation with Markovian switching

$$\mathrm{d}x(t) = r(\xi(t), t)x(t)\Big(1 - \frac{x(t)}{K}\Big)\mathrm{d}t \tag{4}$$

and

$$\mathrm{d}x(t) = x(t)\Big(1 - \frac{x(t)}{K}\Big)\Big[r(\xi(t), t)\mathrm{d}t + \sigma(\xi(t), t)\mathrm{d}B(t)\Big]. \tag{5}$$

Note that Eq. (4) has two equilibria: 0 and K, so does Eq. (5). The aim of this paper is to investigate the effect of Markovian switching noise and white noise on the stability of these two equilibria. For (5), we shall show that

- $\lim_{t\to\infty} x(t) = 0, a.s.$ if $b^* =: \limsup_{t\to\infty} \frac{1}{t}\int_0^t [r(\xi(s), s) + \frac{1}{2}\sigma^2(\xi(s), s)]\mathrm{d}s < 0.$
- $\lim_{t\to\infty} x(t) = K, a.s.$ if $b_* =: \liminf_{t\to\infty} \frac{1}{t}\int_0^t [r(\xi(s), s) - \frac{1}{2}\sigma^2(\xi(s), s)]\mathrm{d}s > 0.$

For the Richards model ([27,28]) with Markovian switching

$$\mathrm{d}x(t) = x(t)(1 - \frac{x^\theta(t)}{K})\Big[r(\xi(t))\mathrm{d}t + \sigma(\xi(t))\mathrm{d}B(t)\Big]. \tag{6}$$

The similar results are obtained as

- $\lim_{t\to\infty} x(t) = 0, a.s.$ if $D^* =: \limsup_{t\to\infty} \frac{1}{t}\int_0^t [r(\xi(s), s) + \frac{\theta}{2}\sigma^2(\xi(s), s)]\mathrm{d}s < 0.$
- $\lim_{t\to\infty} x(t) = \sqrt[\theta]{K}, a.s.$ if $D_* =: \liminf_{t\to\infty} \frac{1}{t}\int_0^t [r(\xi(s), s) - \frac{\theta}{2}\sigma^2(\xi(s), s)]\mathrm{d}s > 0.$

2. Main Results

Lemma 1. *For any initial value $x_0 > 0$, Eq. (5) has a unique and positive solution $x(t)$ on $t \geq 0$ a.s..*

Proof. The proof is similar to Mao et al. [29] by defining $V(x) = 4\sqrt{x} - 4 - 2\ln x, x > 0$, and hence is omitted. □

Lemma 2. *For all $t > 0$, the solution of Eq. (5) obeys that $x(t) < K$ under $0 < x(0) < K$.*

Proof. Define $U(x) = \ln\left|\frac{x}{K-x}\right|$, by using generalised Itô formula, we find that

$$\begin{aligned}\mathrm{d}U(x(t)) &= \frac{K}{x(t)(K - x(t))}\mathrm{d}x(t) - \frac{1}{2}\cdot\frac{K(K-2x(t))}{x^2(t)(K-x(t))^2}\cdot[x(t)\sigma(\xi(t), t)(1 - \frac{x(t)}{K})]^2\mathrm{d}t \\ &= [r(\xi(t), t) - \frac{1}{2}\sigma^2(\xi(t), t) + \frac{x(t)}{K}\sigma^2(\xi(t), t)]\mathrm{d}t + \sigma(\xi(t), t)\mathrm{d}B(t).\end{aligned} \tag{7}$$

Calculate the integral from 0 to t on both sides of the above equation, we get that

$$\ln\left|\frac{x(t)}{K - x(t)}\right| = \ln\left|\frac{x(0)}{K - x(0)}\right| + \int_0^t [r(\xi(s), s) - \frac{1}{2}\sigma^2(\xi(s), s) + \frac{x(s)}{K}\sigma^2(\xi(s), s)]\mathrm{d}s + M_1(t). \tag{8}$$

In other words

$$\frac{x(t)}{K - x(t)} = \frac{x(0)}{K - x(0)}\exp\Big\{\int_0^t [r(\xi(s), s) - \frac{1}{2}\sigma^2(\xi(s), s) + \frac{x(s)}{K}\sigma^2(\xi(s), s)]\mathrm{d}s + M_1(t)\Big\}.$$

Where $M_1(t) = \int_0^t \sigma(\xi(s))\mathrm{d}B(s)$, therefore

$$x(t) = \frac{K}{\frac{K-x(0)}{x(0)}\exp\left\{-\int_0^t [r(\xi(s),s) - \frac{1}{2}\sigma^2(\xi(s),s) + \frac{x(s)}{K}\sigma^2(\xi(s),s)]\mathrm{d}s - M_1(t)\right\} + 1}. \tag{9}$$

So we can get that $x(t) < K$ for all $t > 0$ when $x(0) < K$. $\square$

Theorem 1. *Let* $0 < x(0) = x_0 < K$, *and*

$$b^* =: \limsup_{t\to\infty} \frac{1}{t}\int_0^t [r(\xi(s),s) + \frac{1}{2}\sigma^2(\xi(s),s)]\mathrm{d}s < 0$$

then (5) has globally asymptotically stable zero solution, i.e

$$\lim_{t\to\infty} x(t) = 0, \ \ a.s..$$

Proof. By (8) and Lemma 2, we can get that

$$\begin{aligned}
\ln\left|\frac{x(t)}{K-x(t)}\right| &= \ln|\frac{x(0)}{K-x(0)}| + \int_0^t [r(\xi(s),s) - \frac{1}{2}\sigma^2(\xi(s),s) + \frac{x(s)}{K}\sigma^2(\xi(s),s)]\mathrm{d}s + M_1(t) \\
&\le \ln|\frac{x(0)}{K-x(0)}| + \int_0^t [r(\xi(s),s) - \frac{1}{2}\sigma^2(\xi(s),s) + \sigma^2(\xi(s),s)]\mathrm{d}s + M_1(t) \\
&= \ln|\frac{x(0)}{K-x(0)}| + \int_0^t [r(\xi(s),s) + \frac{1}{2}\sigma^2(\xi(s),s)]\mathrm{d}s + M_1(t),
\end{aligned} \tag{10}$$

where $M_1(t) = \int_0^t \sigma(\xi(s),s)\mathrm{d}B(s)$. Through a series of calculations, we can obtain that

$$\begin{aligned}
x(t) &\le \frac{K}{\frac{K-x(0)}{x(0)}\exp\left\{-\int_0^t [r(\xi(s),s) + \frac{1}{2}\sigma^2(\xi(s),s)]\mathrm{d}s - M_1(t)\right\} + 1} \\
&\le \frac{K}{\frac{K-x(0)}{x(0)}\exp\left\{-\int_0^t [r(\xi(s),s) + \frac{1}{2}\sigma^2(\xi(s),s)]\mathrm{d}s - M_1(t)\right\}}.
\end{aligned} \tag{11}$$

Calculating the logarithmic function on both sides of the inequality (11) together, and we can get that

$$\begin{aligned}
\ln x(t) &\le \ln K - \ln\frac{K-x(0)}{x(0)} + \int_0^t [r(\xi(s),s) + \frac{1}{2}\sigma^2(\xi(s),s)]\mathrm{d}s + M_1(t) \\
&= \ln\frac{Kx(0)}{K-x(0)} + \int_0^t [r(\xi(s),s) + \frac{1}{2}\sigma^2(\xi(s),s)]\mathrm{d}s + M_1(t).
\end{aligned} \tag{12}$$

Note that $M_1(t)$ is a martingale with quadratic variation

$$\langle M_1(t), M_1(t)\rangle = \int_0^t \sigma^2(\xi(s),s)\mathrm{d}s \le \max_{1\le i\le N}\widehat{\sigma_i^2}t,$$

where $\widehat{\sigma_i^2} = \sup_{t\ge 0}\sigma_i(t)$. By the strong law of large numbers for local martingales (see, e.g., [30,31]),

$$\lim_{t\to\infty}\frac{M_1(t)}{t} = 0 \ \ a.s.. \tag{13}$$

Therefore

$$\limsup_{t\to\infty}\frac{\ln x(t)}{t} \le \limsup_{t\to\infty}\frac{1}{t}\int_0^t [r(\xi(s),s) + \frac{1}{2}\sigma^2(\xi(s),s)]\mathrm{d}s = b^*.$$

The required assertion
$$\lim_{t\to\infty} x(t) = 0 \ \ a.s.$$
follows from $b^* < 0$. □

Theorem 2. *Let $0 < x(0) = x_0 < K$, and*
$$b_* =: \liminf_{t\to\infty} \frac{1}{t}\int_0^t [r(\xi(s),s) - \frac{1}{2}\sigma^2(\xi(s),s)]\mathrm{d}s > 0$$
then Eq. (5) has globally asymptotically stable positive equilibrium K, i.e
$$\lim_{t\to\infty} x(t) = K, \ a.s..$$

Proof. Define
$$\eta(t) = \frac{K}{\frac{K-x(0)}{x(0)}\exp\Big\{-\int_0^t [r(\xi(s),s) - \frac{1}{2}\sigma^2(\xi(s),s)]\mathrm{d}s - M_1(t)\Big\} + 1}.$$
In the light of (9), $\eta(t) \le x(t)$. In addition, $\eta(t)$ can also be expressed as
$$\eta(t) = \frac{K}{\frac{K-x(0)}{x(0)}\exp\Big\{-t\Big(\frac{1}{t}\int_0^t [r(\xi(s),s) - \frac{1}{2}\sigma^2(\xi(s),s)]\mathrm{d}s + \frac{1}{t}M_1(t)\Big)\Big\} + 1}.$$
Since $b_* =: \liminf_{t\to\infty} \frac{1}{t}\int_0^t [r(\xi(s),s) - \frac{1}{2}\sigma^2(\xi(s),s)]\mathrm{d}s > 0$ and (13), so
$$\lim_{t\to\infty} \exp\Big\{-t\Big(\frac{1}{t}\int_0^t [r(\xi(s),s) - \frac{1}{2}\sigma^2(\xi(s),s)]\mathrm{d}s + \frac{1}{t}M_1(t)\Big)\Big\} = 0.$$
Therefore
$$\lim_{t\to\infty} \eta(t) = K \ \ a.s.,$$
This, along with $x(t) < K$, imply that
$$\lim_{t\to\infty} x(t) = K \ \ a.s..$$
□

Let Markov chain $\xi(t)$ is irreducible, so it has a unique stationary (probability) distribution π_1. By the ergodic property of the irreducible Markov chain, we can get the results for Eq. (5) with the special case.

Corollary 1. *For $r(\xi(t),t) = r(\xi(t))$, $\sigma(\xi(t),t) = \sigma(\xi(t))$. Let $0 < x(0) = x_0 < K$. Then*

(i) If $\sum_{i\in\mathbb{S}} \pi_i(r_i + \frac{1}{2}\sigma_i^2) < 0$, then the zero solution of Eq. (5) is globally asymptotically stable a.s., that is,
$$\lim_{t\to+\infty} x(t) = 0, \quad a.s..$$

(ii) If $\sum_{i\in\mathbb{S}} \pi_i(r_i - \frac{1}{2}\sigma_i^2) > 0$, then the positive equilibrium K of Eq. (5) is globally asymptotically stable a.s., that is,
$$\lim_{t\to+\infty} x(t) = K, \quad a.s..$$

Corollary 2. *For $r(i,t+T) = r(i,t)$, $\sigma(i,t+T) = \sigma(i,t)$. Let $0 < x(0) = x_0 < K$. Then*

(i) If $\sum_{i\in\mathbb{S}} \frac{\pi_i}{T}\int_0^T (r_i(s) + \frac{1}{2}\sigma_i^2(s))\mathrm{d}s < 0$, then $\lim_{t\to+\infty} x(t) = 0, \quad a.s.$;

(ii) If $\sum_{i\in\mathbb{S}} \frac{\pi_i}{T}\int_0^T (r_i(s) - \frac{1}{2}\sigma_i^2(s))\mathrm{d}s > 0$, then $\lim_{t\to+\infty} x(t) = K, \quad a.s..$

For Eq. (4), we have the following results.

Theorem 3. *Let* $0 < x(0) = x_0 < K$.

(i) If $r^* =: \limsup_{t\to\infty} \frac{1}{t}\int_0^t r(\xi(s),s)\mathrm{d}s < 0$*, then the zero solution of Eq. (4) is globally asymptotically stable a.s., that is,*

$$\lim_{t\to+\infty} x(t) = 0, \quad a.s..$$

(ii) If $r_* =: \liminf_{t\to\infty} \frac{1}{t}\int_0^t r(\xi(s),s)\mathrm{d}s > 0$*, then the positive equilibrium* K *of Eq. (4) is globally asymptotically stable a.s., that is,*

$$\lim_{t\to+\infty} x(t) = K, \quad a.s..$$

Theorem 4. *Let* $0 < x(0) = x_0 < K$*. If* $r^* = r_* = r$ *which is a constant, then Eq. (4) has the properties that*

(i) If $r < 0$*, then* $\lim_{t\to+\infty} x(t) = 0, \quad a.s.;$

(ii) If $r > 0$*, then* $\lim_{t\to+\infty} x(t) = K, \quad a.s..$

Following we introduce stochastic Richards model ([27,28]) with Markovian switching which can be expressed as

$$\mathrm{d}x(t) = x(t)(1 - \frac{x^\theta(t)}{K})[r(\xi(t),t)\mathrm{d}t + \sigma(\xi(t),t)\mathrm{d}B(t)], \tag{14}$$

where θ is a positive constant. Note that model (14) is obtained from the generalized hybrid logistic model

$$\mathrm{d}x(t) = x(t)r(\xi(t),t)(1 - \frac{x^\theta(t)}{K})\mathrm{d}t.$$

by changing $r(\xi(t),t)$ to $r(\xi(t),t) + \sigma(\xi(t),t)\dot{B}(t)$, it is worth mentioning that model (14) become to Eq. (5) if $\theta = 1$. Here we use a result of Theorem 2 from [32] to Eq. (14) which reads

$$\boxed{\text{if } 0 < x_0 < \sqrt[\theta]{K}, \text{ then } 0 < x(t) < \sqrt[\theta]{K} \text{ for } t > 0, \text{ a.s..}}$$

Theorem 5. *If*

$$D^* =: \limsup_{t\to\infty} \frac{1}{t}\int_0^t [r(\xi(s),s) + \frac{\theta}{2}\sigma^2(\xi(s),s)]\mathrm{d}s < 0$$

and $0 < x(0) < \sqrt[\theta]{K}$*, then the zero solution of Eq. (14) is asymptotically stable a.s., that is,* $\lim_{t\to\infty} x(t) = 0$ *a.s..*

Proof. Define $V = \ln\left|\frac{x^\theta(t)}{K - x^\theta(t)}\right|$, by using generalised Itô formula, we find that

$$\mathrm{d}V(t) = [\theta r(\xi(t),t) - \frac{\theta}{2}\sigma^2(\xi(t),t) + \frac{\theta(\theta+1)}{2K}x^\theta(t)\sigma^2(\xi(t),t)]\mathrm{d}t + \theta\sigma(\xi(t),t)\mathrm{d}B(t). \tag{15}$$

Therefore we have that

$$\mathrm{d}V(t) \le [\theta r(\xi(t),t) - \frac{\theta}{2}\sigma^2(\xi(t),t) + \frac{\theta(\theta+1)}{2}\sigma^2(\xi(t),t)]\mathrm{d}t + \theta\sigma(\xi(t),t)\mathrm{d}B(t).$$

Integrating both sides from 0 to t implies that

$$\ln\left|\frac{x^\theta(t)}{K - x^\theta(t)}\right| \le \ln\left|\frac{x^\theta(0)}{K - x^\theta(0)}\right| + \int_0^t [\theta r(\xi(s),s) + \frac{\theta^2}{2}\sigma^2(\xi(s),s)\mathrm{d}s + M_2(t).$$

Therefore

$$\begin{aligned} x^\theta(t) &\leq \frac{K}{\frac{K-x^\theta(0)}{x^\theta(0)}\exp\Big\{-\int_0^t[\theta r(\xi(s),s)+\frac{\theta^2}{2}\sigma^2(\xi(s),s)]\mathrm{d}s-M_2(t)\Big\}+1} \\ &\leq \frac{K}{\frac{K-x^\theta(0)}{x^\theta(0)}\exp\Big\{-\int_0^t[\theta r(\xi(s),s)+\frac{\theta^2}{2}\sigma^2(\xi(s),s)]\mathrm{d}s-M_2(t)\Big\}}. \end{aligned} \tag{16}$$

The logarithm of both sides for the above equation

$$\ln x^\theta(t) \leq \ln\frac{Kx^\theta(0)}{K-x^\theta(0)} + \int_0^t[\theta r(\xi(s),s)+\frac{\theta^2}{2}\sigma^2(\xi(s),s)]\mathrm{d}s + M_2(t), \tag{17}$$

where

$$M_2(t) = \int_0^t \theta\sigma(\xi(s),s)\mathrm{d}B(s).$$

Note that $M_2(t)$ is a martingale with quadratic variation

$$\langle M_2(t), M_2(t)\rangle = \int_0^t \theta^2\sigma^2(\xi(s),s)\mathrm{d}s.$$

For $\sigma(\xi(t))$ is a bounded function on $[0,+\infty)$, we have that

$$\limsup_{t\to\infty}\frac{\langle M_2(t), M_2(t)\rangle}{t} < +\infty.$$

By virtue of the strong law of large numbers for martingales, we can see that

$$\lim_{t\to\infty}\frac{M_2(t)}{t} = 0. \tag{18}$$

For arbitrary $\varepsilon > 0$, there exists $T > 0$ such that for $t \geq T$,

$$\frac{M_2(t)+\ln\frac{Kx^\theta(0)}{K-x^\theta(0)}}{t} < \frac{\theta\varepsilon}{2},\quad \frac{1}{t}\int_0^t[\theta r(\xi(s),s)+\frac{\theta^2}{2}\sigma^2(\xi(s),s)]\mathrm{d}s \leq \theta(D^*+\frac{\varepsilon}{2}).$$

Using these inequalities in (17), one can obtain that

$$\frac{\ln x^\theta(t)}{t} \leq \frac{\theta\varepsilon}{2} + \theta(D^*+\frac{\varepsilon}{2}).$$

Then,

$$\frac{\ln x(t)}{t} \leq (D^*+\varepsilon) < 0.$$

Therefore we have $\lim_{t\to\infty} x(t) = 0$, a.s.. □

Theorem 6. *If*

$$D_* =: \liminf_{t\to\infty}\frac{1}{t}\int_0^t[r(\xi(s),s)-\frac{\theta}{2}\sigma^2(\xi(s),s)]\mathrm{d}s > 0$$

and $0 < x(0) < \sqrt[\theta]{K}$, *then the positive equilibrium* $\sqrt[\theta]{K}$ *of* (14) *is asymptotically stable a.s., that is,* $\lim_{t\to\infty} x(t) = \sqrt[\theta]{K}$.

Proof. Define

$$\eta^\theta(t) = \frac{K}{\frac{K-x^\theta(0)}{x\theta(0)}\exp\Big\{-\int_0^t[\theta r(\xi(s),s)-\frac{\theta^2}{2}\sigma^2(\xi(s),s)]\mathrm{d}s-M_2(t)\Big\}+1}.$$

In the light of the expression of $x^\theta(t)$, $\eta^\theta(t) \le x^\theta(t)$. In addition, $\eta^\theta(t)$ can also be expressed as

$$\eta^\theta(t) = \frac{K}{\frac{K - x^\theta(0)}{x^\theta(0)} \exp\Big\{ -t\Big(\frac{1}{t}\int_0^t [\theta r(\xi(s),s) - \frac{\theta^2}{2}\sigma^2(\xi(s),s)]\mathrm{d}s + \frac{1}{t}M_2(t)\Big)\Big\} + 1}.$$

Since $D_* =: \liminf_{t\to\infty} \frac{1}{t}\int_0^t [r(\xi(s),s) - \frac{\theta}{2}\sigma^2(\xi(s),s)]\mathrm{d}s > 0$ and (18), so

$$\lim_{t\to\infty} \exp\Big\{ -t\Big(\frac{1}{t}\int_0^t [\theta r(\xi(s),s) - \frac{\theta^2}{2}\sigma^2(\xi(s),s)]\mathrm{d}s + \frac{1}{t}M_2(t)\Big)\Big\} = 0.$$

Therefore

$$\lim_{t\to\infty} \eta(t) = \sqrt[\theta]{K} \;\; a.s.,$$

This, along with $x(t) < \sqrt[\theta]{K}$, imply that

$$\lim_{t\to\infty} x(t) = \sqrt[\theta]{K} \;\; a.s..$$

This completes the proof. □

Remark 1. The same results as Corollary 1 and Corollary 2 can be obtained for model (14) under the condition that Markov chain $\xi(\cdot)$ is irreducible.

3. Conclusion

This paper investigates the stochastic logistic equation under regime switching. We establish sufficient conditions for the global asymptotic stability of both the zero solution and the positive equilibrium. Furthermore, we derive an explicit expression for the limiting behavior of hybrid models. Our findings reveal several significant and biologically relevant insights: both white noise and switching noise can profoundly influence population dynamics.

While our study addresses key aspects of the stochastic logistic equation, several intriguing questions remain open for future research. For instance, exploring the dynamics of state-dependent or infinite-state Markov chains presents a promising direction for further investigation.

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