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Best approximate inversion formulas for the Hartley-Bessel-Stockwell transform

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Abstract: The primary objective of this paper is to introduce a novel integral transform, referred to as the *Hartley-Bessel-Stockwell transform*, and to establish several fundamental results associated with it. Specifically, we derive generalized versions of Parseval's identity, Plancherel's theorem, the inversion formula, and Calderon's reproducing formula for this transform. Furthermore, we investigate the concentration properties of the Hartley-Bessel-Stockwell transform on sets of finite measure and present an uncertainty principle for orthonormal sequences. Finally, leveraging the theory of reproducing kernels and best approximation methods, we examine the extremal functions associated with this transform. We provide their integral representations and derive optimal estimates for these functions within weighted Sobolev spaces.

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1. Introduction

Time-frequency analysis plays a fundamental role in mathematics and physics, appearing prominently in harmonic analysis and signal theory. This field encompasses various methods that involve not only the signal and its Fourier transform $\hat{f}$ but also every representation of a signal in the time-frequency domain.

One of the primary objectives of Fourier analysis is the study of time-frequency analysis. This theory, significantly advanced by Gröchenig [1], introduced innovative ways to examine the local frequency spectrum of signals. Through representations such as the short-time Fourier transform, the wavelet transform, and the Wigner distribution, this approach enables the simultaneous representation of spatial and frequency variables in a unified framework called the time-frequency plane. However, the short-time Fourier transform has a notable limitation: the fixed width of its analyzing window. In many practical applications, the high-frequency components of a signal are more time-localized than the low-frequency ones. This rigidity in the window function motivated the development of the wavelet transform [2].

The Stockwell transform, often referred to as the S-transform in the literature, was first introduced by geophysicist Stockwell [3]. It provides a solution to the limitations of fixed window width, offering an adaptable representation for analyzing signals.

The **Hartley transform** is a linear operator defined for a suitable function $\psi(x)$ as:

$$\mathscr{H}(\psi)(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\psi(x)\operatorname{cas}(\lambda x)dx, \tag{1}$$

where $\operatorname{cas}(x)$, the cas function, is given by:

$$\operatorname{cas}(x)=\sum_{n=0}^{\infty}\frac{(-1)^{\binom{n+1}{2}}}{n!}x^n, \tag{2}$$

with $\binom{n}{2}=\frac{n(n-1)}{2}$ denoting the binomial coefficient.

The cas function, as defined in (2), can be interpreted as a generalization of the exponential function exp. A straightforward computation reveals that $\mathrm{cas}(x)$ is the unique C^{∞} solution of the following differential-reflection problem [4]:

$$\begin{cases} R\partial_x u(x) = \lambda u(x), \\ u(0) = 0, \end{cases} \tag{3}$$

where ∂_x denotes the first-order derivative, and R is the reflection operator acting on functions $f(x)$ as:

$$R(f)(x) = f(-x). \tag{4}$$

The cas function is multiplicative on $\mathbb{R}$, satisfying:

$$\mathrm{cas}(x)\,\mathrm{cas}(y) = \frac{1}{2}\big(\mathrm{cas}(x+y) - \mathrm{cas}(-x-y) + \mathrm{cas}(x-y) + \mathrm{cas}(y-x)\big). \tag{5}$$

Inspired by the relation (4), the author in [4] generalized it for the Hartley-Bessel function and introduced a generalized convolution product. This paper focuses on the generalized Hartley transform, referred to as the Hartley-Bessel transform, introduced in [4–6]. Specifically, we consider the differential-reflection operator Δ_α defined by:

$$\Delta_\alpha = R\left(\partial_x + \frac{\alpha}{x}\right) + \frac{\alpha}{x}, \quad \alpha \geq 0, \tag{6}$$

where R is the reflection operator given in (3).

The operator Δ_α is closely linked to Dunkl's theory [4,7]. Moreover, its eigenfunctions are related to Bessel functions and satisfy a product formula, enabling the development of a novel harmonic analysis associated with this operator [4].

The Stockwell transform has been successfully employed in diverse applications, such as seismic recording, ground vibration analysis, geophysics, medical imaging, hydrology, gravitational wave detection, and power system analysis [8–11]. Given its significance, the mathematical theory of this transform is evolving in multiple directions, with numerous extensions proposed recently, see [1,7,12,13].

Since harmonic analysis associated with the Hartley-Bessel operator (5) has seen remarkable development, it is natural to explore whether a time-frequency analysis equivalent for the Stockwell transform exists in the Hartley-Bessel setting.

The primary aim of this paper is twofold. First, we introduce the Stockwell transform in the Hartley-Bessel setting and present new results related to this transform. Second, we analyze the concentration of this transform on sets of finite measure and establish uncertainty principles for orthonormal sequences. Finally, using best approximations and reproducing kernel theory, we investigate extremal functions related to this transform, deriving their integral representation and optimal estimates on weighted Sobolev spaces. The remainder of this paper is structured as follows. Section 2 reviews the main results related to harmonic analysis associated with the Hartley-Bessel operator (5). In Section 3, we define the Stockwell transform in the Hartley-Bessel setting and present new findings related to this transform. Section 4 focuses on uncertainty principles associated with the Hartley-Bessel-Stockwell transform. Finally, Section 5 examines extremal functions linked to this transform in weighted Sobolev spaces.

2. Harmonic Analysis Associated with the Hartley-Bessel Transform

In this section, we recall key results in harmonic analysis related to the Hartley-Bessel transform. For further details, we refer the reader to [4].

2.1. Weighted Lebesgue Measure and Function Spaces

For $\alpha \geq 0$, the weighted Lebesgue measure μ_α on $\mathbb{R}$ is defined as:

$$d\mu_\alpha(x) := \frac{|x|^{2\alpha}}{2^{\alpha+\frac{1}{2}}\Gamma\left(\alpha+\frac{1}{2}\right)}\,dx, \tag{7}$$

where Γ denotes the Gamma function.

The weighted Lebesgue space $L^p_\alpha(\mathbb{R})$, $1 \leq p \leq \infty$, consists of measurable functions f satisfying:

$$\|f\|_{p,\mu_\alpha} := \begin{cases} \left(\int_{\mathbb{R}} |f(x)|^p \, d\mu_\alpha(x)\right)^{1/p} < \infty, & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)| < \infty, & p = \infty. \end{cases} \tag{8}$$

In particular, for $p = 2$, the space $L^2_\alpha(\mathbb{R})$ is a Hilbert space with the inner product:

$$\langle f, g\rangle_{\mu_\alpha} = \int_{\mathbb{R}} f(x)\overline{g(x)}\, d\mu_\alpha(x). \tag{9}$$

2.2. Eigenfunctions of the Differential-Reflection Operator Δ_α

For $\lambda \in \mathbb{C}$, consider the Cauchy problem:

$$(S) : \begin{cases} \Delta_\alpha u(x) = \lambda u(x), \\ u(0) = 1. \end{cases} \tag{10}$$

As shown in [4], this problem admits a unique solution $B_\alpha(\lambda\cdot)$ given by:

$$B_\alpha(\lambda x) = j_{\alpha-\frac{1}{2}}(\lambda x) + \frac{\lambda x}{2\alpha+1} j_{\alpha+\frac{1}{2}}(\lambda x), \tag{11}$$

where j_α denotes the normalized Bessel function of order α (see [4]). The function $B_\alpha(\lambda\cdot)$ is infinitely differentiable on $\mathbb{R}$, and it satisfies the following bound:

$$\forall \lambda, x \in \mathbb{R}, \quad |B_\alpha(\lambda x)| \leq \sqrt{2}. \tag{12}$$

Furthermore, from [4], the Hartley-Bessel kernel exhibits the multiplicative property:

$$\forall \lambda \in \mathbb{R},\ x, y \in \mathbb{R}^*, \quad B_\alpha(\lambda x)B_\alpha(\lambda y) = \int_{\mathbb{R}} B_\alpha(\lambda z) K_\alpha(x, y, z)\, d\mu_\alpha(z), \tag{13}$$

where K_α is the Bessel kernel explicitly provided in [4]. The product formula (9) generalizes classical relations and facilitates the definition of a translation operator, convolution product, and the development of harmonic analysis associated with Δ_α.

2.3. The Hartley-Bessel Transform

Definition 1 ([4])**.** The Hartley-Bessel transform $\mathscr{H}_\alpha$ is defined on $L^1_\alpha(\mathbb{R})$ as:

$$\mathscr{H}_\alpha(f)(\lambda) = \int_{\mathbb{R}} B_\alpha(\lambda x) f(x)\, d\mu_\alpha(x), \quad \lambda \in \mathbb{R}. \tag{14}$$

The Hartley-Bessel transform satisfies the following key properties (see [4] for proofs):

Proposition 2. 1. ***Boundedness:*** *For every $f \in L^1_\alpha(\mathbb{R})$,*

$$\|\mathscr{H}_\alpha(f)\|_{\infty,\mu_\alpha} \leq \sqrt{2}\|f\|_{1,\mu_\alpha}. \tag{15}$$

2. ***Inversion Formula:*** *For $f \in (L^1_\alpha \cap L^2_\alpha)(\mathbb{R})$ such that $\mathscr{H}_\alpha(f) \in L^1_\alpha(\mathbb{R})$, we have:*

$$f(x) = \int_{\mathbb{R}} B_\alpha(\lambda x)\mathscr{H}_\alpha(f)(\lambda)\, d\mu_\alpha(\lambda), \quad \text{a.e. } x \in \mathbb{R}. \tag{16}$$

3. ***Plancherel Theorem:*** *The Hartley-Bessel transform $\mathscr{H}_\alpha$ extends to an isometric isomorphism on $L^2_\alpha(\mathbb{R})$:*

$$\|f\|_{2,\mu_\alpha} = \|\mathscr{H}_\alpha(f)\|_{2,\mu_\alpha}. \tag{17}$$

2.4. Translation Operator and Generalized Convolution

The product formula (9) enables the definition of the translation operator.

Definition 3. Let f be a measurable function on $\mathbb{R}$. For $x, y \in \mathbb{R}$, the translation operator $\mathcal{T}_\alpha^x$ is defined as:

$$\mathcal{T}_\alpha^x f(y) = \int_{\mathbb{R}} f(z) K_\alpha(x,y,z)\, d\mu_\alpha(z). \tag{18}$$

The translation operator satisfies the following properties ([4]):

Proposition 4. *For all $x, y \in \mathbb{R}$:*

1. *Symmetry:*
$$\mathcal{T}_\alpha^x f(y) = \mathcal{T}_\alpha^y f(x). \tag{19}$$
2. *Preservation of Integrals:*
$$\int_{\mathbb{R}} \mathcal{T}_\alpha^x f(y)\, d\mu_\alpha(y) = \int_{\mathbb{R}} f(y)\, d\mu_\alpha(y). \tag{20}$$
3. *Norm Preservation: For $f \in L_\alpha^p(\mathbb{R})$, $p \in [1,\infty]$, the translation operator preserves norms:*
$$\|\mathcal{T}_\alpha^x f\|_{p,\mu_\alpha} \leq \|f\|_{p,\mu_\alpha}. \tag{21}$$

Using the translation operator, the generalized convolution product of f, g is defined as:

$$(f *_\alpha g)(x,t) = \int_{\mathbb{R}} \mathcal{T}_\alpha^x f(y) g(y)\, d\mu_\alpha(y). \tag{22}$$

The generalized convolution satisfies properties such as Young's inequality, Plancherel's theorem, and the convolution theorem. For further details, see [4].

3. Stockwell Transform Associated with The Hartley-Bessel operator

The main purpose of this section is to introduce the Hartley-Bessel-Stockwell transform and to give some new results related to this transform.
Notation : we denote by

- $L_\alpha^p\left(\mathbb{R}^2\right), 1 \leq p \leq +\infty$ the space of measurable functions on $\mathbb{R} \times \mathbb{R}$ satisfying

$$\|f\|_{p,\mu_\alpha \otimes \mu_\alpha} := \begin{cases} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x,y)|^p \,\mathrm{d}\mu_\alpha(x) \otimes \mathrm{d}\mu_\alpha(y)\right)^{\frac{1}{p}}, & \text{if } p \in [1,+\infty[; \\ \operatorname{ess\,sup} |f(x,y)| & \text{if } p = +\infty. \\ (x,y) \in \mathbb{R} \times \mathbb{R} & \end{cases}$$

Let ψ in $L_\alpha^2(\mathbb{R})$ and $y \in \mathbb{R}$, we recall that the modulation operator of ψ is given by

$$\mathcal{M}^y(\psi) := \mathscr{H}_\alpha\left(\sqrt{\tau_\alpha^y \left|\mathscr{H}_\alpha(\psi)\right|^2}\right).$$

By using Plancherel's formula (12) and the relation (15) we find that $\mathcal{M}^y(\psi) \in L_\alpha^2(\mathbb{R})$ and

$$\|\mathcal{M}^y(\psi)\|_{2,\alpha} = \|\psi\|_{2,\alpha}. \tag{23}$$

Definition 5. Let $y \in \mathbb{R}$. We define the dilation operator $\mathscr{D}_y$ of a measurable function ψ by

$$\forall x \in \mathbb{R}, \quad \mathscr{D}_y(\psi)(x) := y^{\alpha+1}\psi(xy).$$

For all $\psi \in L_\alpha^2(\mathbb{R})$ we have $\mathscr{D}_y(\psi) \in L_\alpha^2(\mathbb{R})$ and

$$\|\mathscr{D}_a(\psi)\|_{2,\alpha} = \|\psi\|_{2,\alpha}. \tag{24}$$

Now, for every non-zero window function ψ in $L^2_\alpha(\mathbb{R})$, we consider the family $\psi^{x,y}$ defined by

$$\psi^{x,y}(z) = \tau^x_\alpha\left(\mathcal{M}^y(\mathscr{D}_y(\psi))\right)(z), \quad \forall (x,y) \in \mathbb{R}\times\mathbb{R}. \tag{25}$$

Definition 6. For every f and ψ in $L^2_\alpha(\mathbb{R})$ we define the Hartley-Bessel-Stockwell transform by

$$S^\alpha_\psi(f)(x,y) := \int_{\mathbb{R}} f(z)\overline{\psi^{x,y}(z)}\mathrm{d}\mu_\alpha(z), \tag{26}$$

Remark 1. 1- The Hartley-Bessel-Stockwell transform (26) can be also expressed in the form

$$S^\alpha_\psi(f)(x,y) = \left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi})) *_\alpha f\right)(x). \tag{27}$$

By using Hölder's inequality and the relations (23),(24)and (25) we find that $S^\alpha_\psi(f) \in L^\infty_\alpha\left(\mathbb{R}^2\right)$ and we have

$$\left\|S^\alpha_\psi(f)\right\|_{\infty,\mu_\alpha\otimes\mu_\alpha} \le \|f\|_{2,\alpha}\|\psi\|_{2,\alpha} \tag{28}$$

Definition 7. Let ψ_1, ψ_2 be non-zero functions in $L^2_\alpha(\mathbb{R})$, we say that the pair (ψ_1, ψ_2) is admissible if for almost all $\lambda \in \mathbb{R}$ we have

$$0 < C_{\psi_1,\psi_2} = \int_{\mathbb{R}} \mathscr{H}_\alpha\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_1}))\right)(\lambda)\overline{\mathscr{F}_\alpha\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_2}))\right)(\lambda)}\mathrm{d}\mu_\alpha(y) < \infty. \tag{29}$$

In the following we have generalized Parseval's formula for S^α_ψ.

Theorem 8. *Let (ψ_1, ψ_2) be an admissible pair then for all $f, g \in L^2_\alpha(\mathbb{R})$ we have*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} S^\alpha_{\psi_1}(f)(x,y)\overline{S^\alpha_{\psi_2}(g)(x,y)}\mathrm{d}\mu_\alpha(x)\otimes\mathrm{d}\mu_\alpha(y) = C_{\psi_1,\psi_2}\int_{\mathbb{R}} f(x)\overline{g(x)}\mathrm{d}\mu_\alpha(x) \tag{30}$$

Proof. By using Fubini's theorem and the relations (12), (18), and (27) we find that

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} S^\alpha_{\psi_1}(f)(x,y)\overline{S^\alpha_{\psi_2}(g)(x,y)}\mathrm{d}\mu_\alpha(x)\otimes\mathrm{d}\mu_\alpha(y)\\
&= \int_{\mathbb{R}}\left[\int_{\mathbb{R}}\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_1})) *_\alpha f\right)(x)\overline{\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_2})) *_\alpha g\right)(x)}\mathrm{d}\mu_\alpha(x)\right]\mathrm{d}\mu_\alpha(y)\\
&= \int_{\mathbb{R}}\left[\int_{\mathbb{R}}\mathscr{H}_\alpha\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_1}))\right)(\lambda)\overline{\mathscr{H}_\alpha\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_2}))\right)(\lambda)}\mathscr{H}_\alpha(f)(\lambda)\overline{\mathscr{F}_\alpha(g)(\lambda)}\mathrm{d}\mu_\alpha(\lambda)\right]\mathrm{d}\mu_\alpha(y)\\
&= C_{\psi_1,\psi_2}\int_{\mathbb{R}} f(x)\overline{g(x)}\mathrm{d}\mu_\alpha(x).
\end{aligned}$$

The proof is complete. □

Corollary 9 (Plancherel's formula for S^α_ψ). *If $\psi = \psi_1 = \psi_2$ and $f = g$ we find that*

$$\left\|S^\alpha_\psi(f)\right\|_{2,\mu_\alpha\otimes\mu_\alpha} = \sqrt{C_\psi}\|f\|_{2,\mu_\alpha}. \tag{31}$$

where

$$C_\psi = C_{\psi,\psi} = \int_{\mathbb{R}} |\mathscr{F}_\alpha\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi}))\right)(\lambda)|^2\mathrm{d}\mu_\alpha(y) \tag{32}$$

We have the following result.

Theorem 10. *Let ψ be an admissible function in $L^2_\alpha(\mathbb{R})$, for every $f \in L^2_\alpha(\mathbb{R})$ the function $S^\alpha_\psi(f)$ belongs to $L^p_\alpha(\mathbb{R}^2))$, $p \in [2, +\infty]$ and we have*

$$\left\|S^\alpha_\psi(f)\right\|_{p,\mu_\alpha\otimes\mu_\alpha} \le C_\psi^{\frac{1}{p}}\|\psi\|_{2,\alpha}^{1-\frac{2}{p}}\|f\|_{2,\alpha}. \tag{33}$$

Proof. By using the relations (28) and (31), the relation (33) follows from the Riesz-Thorin interpolation theorem see [2]. □

In the following, we establish a generalized inversion formula for the Hartley-Bessel-Stockwell transform S^{α}_{ψ}.

Theorem 11. *Let (ψ_1, ψ_2) be an admissible pair in $L^2_{\alpha}(\mathbb{R})$, then for all $f \in L^2_{\alpha}\left(\mathbb{R}^d_+\right)$ we have*

$$f(.) = \frac{1}{C_{\psi_1,\psi_2}} \int_{\mathbb{R}} \int_{\mathbb{R}} S^{\alpha}_{\psi_1}(f)(x,y) {\psi_2}^{x,y}(.) \mathrm{d}\mu_{\alpha}(x) \otimes \mathrm{d}\mu_{\alpha}(y),$$

weakly in $L^2_{\alpha}(\mathbb{R})$.

Proof. By using the relations (26),(30) and Fubini's theorem we find that

$$\begin{aligned}
\int_{\mathbb{R}^d_+} f(z)\overline{h(z)} \mathrm{d}\mu_{\alpha}(z) &= \frac{1}{C_{\psi_1,\psi_2}} \int_{\mathbb{R}} \int_{\mathbb{R}} S^{\alpha}_{\psi_1}(f)(x,y) \overline{S^{\alpha}_{\psi_2}(g)(x,y)} \mathrm{d}\mu_{\alpha}(x) \otimes \mathrm{d}\mu_{\alpha}(y) \\
&= \frac{1}{C_{\psi_1,\psi_2}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} S^{\alpha}_{\psi_1}(f)(x,y) {\psi_2}^{x,y}(z) \mathrm{d}\mu_{\alpha}(x) \otimes \mathrm{d}\mu_{\alpha}(y) \right) \overline{h(z)} \mathrm{d}\mu_{\alpha}(z),
\end{aligned}$$

which gives the result. □

The reproducing kernels for Hilbert space play an important role in harmonic analysis [14]. In this context, we have the following result.

Theorem 12. *The space $S^{\alpha}_{\psi}\left(L^2_{\alpha}(\mathbb{R})\right)$ is a reproducing kernel Hilbert space in $L^2_{\alpha}\left(\mathbb{R}^{2d}_+\right)$ with kernel function $\mathcal{K}_{\psi}$ defined by*

$$\mathcal{K}_{\psi}\left((x',y');(x,y)\right) = \frac{1}{C_{\psi}} \left(\mathcal{M}^{y'}(\mathscr{D}_{y'}(\overline{\psi})) *_{\alpha} \psi^{x,y} \right)(x'),$$

where C_{ψ} is given by the relation (32).

Furthermore, the kernel is pointwise bounded

$$\left|\mathcal{K}_{\psi}\left((x',y');(x,y)\right)\right| \leq \frac{\|\psi\|^2_{2,\alpha}}{C_{\psi}}, \quad \forall (x,y);(x',y') \in \mathbb{R}^2.$$

Proof. By using the relations (27) and (30) we find that

$$\begin{aligned}
S^{\alpha}_{\psi}(f)(x,y) &= \frac{1}{C_{\psi}} \int_{\mathbb{R}} \int_{\mathbb{R}} S^{\alpha}_{\psi}(f)\left(x',y'\right) \overline{S^{\alpha}_{\psi}\left(\psi^{x,y}\right)\left(x',y'\right)} \mathrm{d}\mu_{\alpha}\left(x'\right) \otimes \mathrm{d}\mu_{\alpha}\left(y'\right) \\
&= \left\langle S^{\alpha}_{\psi}(f) \mid \mathcal{K}_{\psi}((.);(x,y)) \right\rangle_{\mu_{\alpha}\otimes\mu_{\alpha}},
\end{aligned}$$

where

$$\mathcal{K}_{\psi}\left((x',y');(x,y)\right) = \frac{1}{C_{\psi}} \left(\mathcal{M}^{y'}(\mathscr{D}_{y'}(\overline{\psi})) *_{\alpha} \psi^{x,y} \right)(x'),$$

Finally by the Cauchy-Schwarz inequality, we get

$$\left|\mathcal{K}_{u}\left((x',y');(x,y)\right)\right| \leq \frac{1}{C_{\psi}} \int_{\mathbb{R}} \left|\psi^{x,y}(z)\right| \left|\psi^{x',y'}(z)\right| \mathrm{d}\mu_{\alpha}(z) \leq \frac{\|\psi\|^2_{2,\alpha}}{C_{\psi}}.$$

□

The rest of this section is devoted to give Calderón's type reproducing formula for the Hartley-Bessel-Stockwell transform, to do this we need the help of the following result.

Proposition 13. *Let $0 < \gamma < \delta < +\infty$ and (ψ_1, ψ_2) be an admissible pair such that $\mathscr{H}_\alpha\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_1}))\right)$ and $\mathscr{H}_\alpha\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_2}))\right)$ belongs to $L^\infty_\alpha(\mathbb{R})$ for all $y \in \mathbb{R}$. We put*

$$G_{\gamma,\delta}(x) := \frac{1}{C_{\psi_1,\psi_2}} \int_{D(\gamma,\delta)} \left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_1}))\right) *_\alpha \overline{\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_2}))\right)}(x) \mathrm{d}\mu_\alpha(y) \tag{34}$$

and

$$K_{\gamma,\delta}(\lambda) := \frac{1}{C_{\psi_1,\psi_2}} \int_{D(\gamma,\delta)} \mathscr{H}_\alpha\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_1}))\right)(\lambda) \overline{\mathscr{H}_\alpha\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_2}))\right)(\lambda)} \mathrm{d}\mu_\alpha(y) \tag{35}$$

where

$$D(\gamma,\delta) = \{x \in \mathbb{R} : \gamma \leq x \leq \delta\}.$$

Then we have $G_{\gamma,\delta}$ belongs to $L^2_\alpha\left(\mathbb{R}^d_+\right)$ and

$$\mathscr{H}_\alpha(G_{\gamma,\delta})(\lambda) = K_{\gamma,\delta}(\lambda). \tag{36}$$

Proof. By using Hölder's inequality and the relations (23) and (24) we find that

$$\left|G_{\gamma,\delta}(x)\right|^2 \leq \frac{\mu_\alpha(D(\gamma,\delta))}{C^2_{\psi_1,\psi_2}} \int_{D(\gamma,\delta)} \left|\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_1}))\right) *_\alpha \overline{\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_2}))\right)}(x)\right|^2 \mathrm{d}\mu_\alpha(y)$$

So

$$\begin{aligned}\|G_{\gamma,\delta}\|^2_{2,\alpha} &\leq \frac{\mu_\alpha(D(\gamma,\delta))}{C^2_{\psi_1,\psi_2}} \int_{D(\gamma,\delta)} \left(\int_{\mathbb{R}} |\mathscr{H}_\alpha\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_1}))\right)(\lambda)|^2 |\mathscr{H}_\alpha\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_2}))\right)(\lambda)|^2 \mathrm{d}\mu_\alpha(\lambda)\right) \mathrm{d}\mu_\alpha(y) \\ &\leq \left(\frac{\mu_\alpha(D(\gamma,\delta))}{C_{\psi_1,\psi_2}}\right)^2 \left\|\mathscr{H}_\alpha\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_1}))\right)\right\|^2_{\infty,\alpha} \|\psi_2\|^2_{2,\mu_\alpha} < \infty.\end{aligned}$$

Which proves that $G_{\gamma,\delta}$ belongs to $L^2_\alpha(\mathbb{R})$, furthermore by using Parseval's relation (14) and (20) we find that

$$\begin{aligned}\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_1}))\right) *_\alpha \overline{\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_2}))\right)}(x) &= \int_{\mathbb{R}} \mathscr{H}_\alpha\left(\tau^x_\alpha(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_1})))\right)(\lambda) \overline{\mathscr{H}_\alpha\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_2}))\right)(\lambda)} \mathrm{d}\mu_\alpha(\lambda) \\ &= \int_{\mathbb{R}} B_\alpha(\lambda x) \mathscr{H}_\alpha\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_1}))\right)(\lambda) \overline{\mathscr{H}_\alpha\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_2}))\right)(\lambda)} \mathrm{d}\mu_\alpha(\lambda).\end{aligned}$$

Now, by using Fubini's theorem we find that

$$\begin{aligned}G_{\gamma,\delta}(x) &= \frac{1}{C_{\psi_1,\psi_2}} \int_{\mathbb{R}} B_\alpha(\lambda x) \left(\int_{D(\gamma,\delta)} \mathscr{H}_\alpha\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_1}))\right)(\lambda) \overline{\mathscr{H}_\alpha\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_2}))\right)(\lambda)} \mathrm{d}\mu_\alpha(y)\right) \mathrm{d}\mu_\alpha(\lambda) \\ &= \int_{\mathbb{R}} B_\alpha(\lambda x) K_{\gamma,\delta}(\lambda) \mathrm{d}\mu_\alpha(\lambda).\end{aligned}$$

Inversion formula (11) gives the relation (36). □

In the following we establish generalized reproducing inversion formula of Calderón's type for the Hartley-Bessel-Stockwell transform S^α_ψ which is more general than that which is proved in [13].

Theorem 14. *Let $0 < \gamma < \delta < +\infty$ and (ψ_1, ψ_2) be an admissible pair such that $\mathscr{H}_\alpha\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_1}))\right)$ and $\mathscr{H}_\alpha\left(\mathcal{M}^y(\mathscr{D}_y(\overline{\psi_2}))\right)$ belongs to $L^\infty_\alpha(\mathbb{R})$ for all $y \in \mathbb{R}$, the function $f_{\gamma,\delta}$ defined for all $z \in \mathbb{R}$ by:*

$$f_{\gamma,\delta}(z) = \frac{1}{C_{\psi_1,\psi_2}} \int_{D(\gamma,\delta)} \int_{\mathbb{R}} S^\alpha_{\psi_1}(f)(x,y) {\psi_2}^{x,y}(z) \mathrm{d}\mu_\alpha(x) \otimes \mathrm{d}\mu_\alpha(y), \tag{37}$$

belongs to $L^2_\alpha(\mathbb{R})$ and satisfies

$$\lim_{(\gamma,\delta)\to(0,+\infty)} \left\|f_{\gamma,\delta} - f\right\|_{2,\mu_\alpha} = 0. \tag{38}$$

Proof. It is easy to see that for all $f \in L^2_\alpha(\mathbb{R})$ we have $f_{\gamma,\delta} = f *_\alpha G_{\gamma,\delta}$, where $G_{\gamma,\delta}$ is the function given by the relation (34), by using the relations (12),(36) we find that

$$\|f_{\gamma,\delta} - f\|^2_{2,\alpha} = \int_{\mathbb{R}} |\mathscr{H}_\alpha(f)(\lambda)|^2 \left(1 - K_{\gamma,\delta}(\lambda)\right)^2 \, \mathrm{d}\mu_\alpha(\lambda).$$

By using the relations (29),(35), the relation (38) follows from the dominated convergence theorem. □

4. Uncertainty Principles Associated with the Hartley-Bessel-Stockwell Transform

In this section, we estimate the concentration of $S^\alpha_\psi(f)$ on subset of $\mathbb{R} \times \mathbb{R}$ of finite measure, similar results have been checked in [15] and we establish the uncertainty principle for orthonormal sequences associated with the Hartley-Bessel-Stockwell transform, first we consider the following orthogonal projections

(1) Let P_ψ be the orthogonal projection from $L^2_\alpha(\mathbb{R}^2)$onto $S^\alpha_\psi\left(L^2_\alpha(\mathbb{R})\right)$ and Im P_ψ denotes the range of P_ψ.

(2) Let P_E be the orthogonal projection on $L^2_\alpha(\mathbb{R}^2)$ defined by

$$P_E F = \chi_E F, \quad F \in L^2_\alpha\left(\mathbb{R}^2\right), \tag{39}$$

where $E \subset \mathbb{R} \times \mathbb{R}$ and Im P_E is the range of P_E. Also, we define

$$\|P_E P_\psi\| = \sup\left\{\|P_E P_\psi(F)\|_{2,\mu_\alpha \otimes \mu_\alpha} : F \in L^2_\alpha\left(\mathbb{R}^2\right), \|F\|_{2,\mu_\alpha\otimes\mu_\alpha} = 1\right\}.$$

We first need the following result.

Theorem 15. *Let ψ be an admissible function in $L^2_\alpha(\mathbb{R})$. Then for any $E \subset \mathbb{R} \times \mathbb{R}$ of finite measure $\mu_\alpha \otimes \mu_\alpha(E) < \infty$, the operator $P_E P_\psi$ is a Hilbert-Schmidt operator. Moreover, we have the following estimation*

$$\|P_E P_\psi\| \le \frac{\|\psi\|^2_{2,\alpha}}{C_\psi} \sqrt{\mu_\alpha \otimes \mu_\alpha(E)}.$$

Proof. Since P_ψ is a projection onto a reproducing karnel Hilbert space, for any function $F \in L^2_\alpha(\mathbb{R}^2)$, the orthogonal projection P_ψ can be expressed as

$$P_\psi(F)(x,\xi) = \iint_{\mathbb{R}^2} F\left(x',\xi'\right) K_u\left(\left(x',\xi'\right);(x,\xi)\right) d\mu_\alpha(x') \otimes d\mu_\alpha(\xi'),$$

where $\mathcal{K}_\psi\left(\left(x',\xi'\right);(x,\xi)\right)$ is same as already defined, using the relation (39), we find that

$$P_E P_\psi(F)(x,\xi) = \iint_{\mathbb{R}^2} \chi_E(x,\xi) F\left(x',\xi'\right) \mathcal{K}_\psi\left(\left(x',\xi'\right);(x,\xi)\right) d\mu_\alpha(x') \otimes d\mu_\alpha(\xi').$$

This shows that the operator $P_E P_\psi$ is an integral operator with kernel $K((x',\xi');(x,\xi)) = \chi_E(x,\xi)\mathcal{K}_\psi\left(\left(x',\xi'\right);(x,\xi)\right)$. Using the relation (28) and Fubini's theorem, we find that

$$\begin{aligned}\|P_\Sigma P_\psi\|^2_{HS} &= \iint_{\mathbb{R}^2} \iint_{\mathbb{R}^2} |\chi_\Sigma(x,\xi)|^2 \left|\mathcal{K}_u\left(\left(x',\xi'\right);(x,\xi)\right)\right|^2 d\mu_\alpha(x') \otimes d\mu_\alpha(\xi') d\mu_\alpha(x) \otimes d\mu_\alpha(\xi) \\ &\le \frac{\|\psi\|^2_{2,\alpha}}{C_\psi} \sqrt{\mu_\alpha \otimes \mu_\alpha(E)} < \infty. \end{aligned} \tag{40}$$

Thus, the operator $P_E P_u$ is a Hilbert-Schmidt operator. Now, the proof follows from the fact that $\|P_E P_\psi\| \le \|P_E P_\psi\|_{HS}$. □

In the following, we obtain the uncertainty principle for orthonormal sequences associated with the Hartley-Bessel-Stockwell transform.

Theorem 16. *Let ψ be an admissible function in $L^2_\alpha(\mathbb{R})$ and $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in $L^2_\alpha(\mathbb{R})$. Then for any subset $E \subset \mathbb{R}\times\mathbb{R}$ of finite measure $\mu_\alpha \otimes \mu_\alpha(E) < \infty$, we have*

$$\sum_{n=1}^{N}\left(1-\left\|\chi_{E^c} S^\alpha_\psi(\phi_n)\right\|_{2,\mu_\alpha\otimes\mu_\alpha}\right) \leq \frac{\|\psi\|^2_{2,\alpha}}{C_\psi}\sqrt{\mu_\alpha\otimes\mu_\alpha(E)},$$

for every $N \in \mathbb{N}$.

Proof. Proof. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $L^2_\alpha\left(\mathbb{R}^{2d}_+\right)$. Since $P_E P_\psi$ is a Hilbert-Schmidt operator, and satisfied the relation (40) and we have

$$\sum_{n\in\mathbb{N}}\langle P_\psi P_E P_\psi e_n, e_n\rangle_{\mu_\alpha\otimes\mu_\alpha} = \|P_E P_\psi\|^2_{HS} \leq \frac{\|\psi\|^2_{2,\alpha}}{C_\psi}\sqrt{\mu_\alpha\otimes\mu_\alpha(E)} < \infty.$$

According to the paper [16], the positive operator $P_\psi P_E P_\psi$ is a trace class operator and we have

$$\operatorname{tr}\left(P_\psi P_E P_\psi\right) = \|P_E P_\psi\|^2_{HS} \leq \frac{\|\psi\|^2_{2,\alpha}}{C_\psi}\sqrt{\mu_\alpha\otimes\mu_\alpha(E)} < \infty,$$

where $\operatorname{tr}\left(P_\psi P_E P_\psi\right)$ denotes the trace of the operator $P_\psi P_E P_\psi$. Since $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in $L^2_\alpha(\mathbb{R}^d_+)$, from the orthogonality relation (31), we obtain that $\left\{S^\alpha_\psi(\phi_n)\right\}_{n\in\mathbb{N}}$ is also an orthonormal sequence in $L^2_\alpha\left(\mathbb{R}^{2d}_+\right)$ thus

$$\sum_{n=1}^{N}\left\langle P_E S^\alpha_\psi(\phi_n), S^\alpha_\psi(\phi_n)\right\rangle_{\mu_\alpha\otimes\mu_\alpha} = \sum_{n=1}^{N}\left\langle P_\psi P_\Sigma P_\psi S^\alpha_\psi(\phi_n), S^\alpha_\psi(\phi_n)\right\rangle_{\mu_\alpha\otimes\mu_\alpha} \leq \operatorname{tr}\left(P_\psi P_E P_\psi\right).$$

Hence, we find that

$$\sum_{n=1}^{N}\left\langle P_E S^\alpha_\psi(\phi_n), S^\alpha_\psi(\phi_n)\right\rangle_{\mu_\alpha\otimes\mu_\alpha} \leq \frac{\|\psi\|^2_{2,\alpha}}{C_\psi}\sqrt{\mu_\alpha\otimes\mu_\alpha(E)} < \infty.$$

Moreover, for any n with $1 \leq n \leq N$, using the Cauchy-Schwarz inequality, we get

$$\left\langle P_E S^\alpha_\psi(\phi_n), S^\alpha_\psi(\phi_n)\right\rangle_{\mu_\alpha\otimes\mu_\alpha} = 1-\left\langle P_{E^c} S^\alpha_\psi(\phi_n), S^\alpha_\psi(\phi_n)\right\rangle_{\mu_\alpha\otimes\mu_\alpha} \geq 1-\left\|\chi_{E^c} S^\alpha_\psi(\phi_n)\right\|_{2,\mu_\alpha\otimes\mu_\alpha}.$$

Thus, we obtain

$$\sum_{n=1}^{N}\left(1-\left\|\chi_{E^c} S^\alpha_\psi(\phi_n)\right\|_{2,\mu_\alpha\otimes\mu_\alpha}\right) \leq \sum_{n=1}^{N}\left\langle P_E S^\alpha_\psi(\phi_n), S^\alpha_\psi(\phi_n)\right\rangle_{\mu_\alpha\otimes\mu_\alpha} \leq \frac{\|\psi\|^2_{2,\alpha}}{C_\psi}\sqrt{\mu_\alpha\otimes\mu_\alpha(E)} < \infty.$$

This completes the proof of the theorem. □

5. Extremal Functions Associated with the Hartley-Bessel-Stockwell Transform

By using the theory of reproducing kernels [3,14], the main purpose of this section is to study the extremal functions associated with the Hartley-Bessel-Stockwell transform and to give an integral representation and best estimate of these functions on weighted Sobolev spaces.

5.1. Sobolev type spaces Associated with the Hartley-Bessel Transform

Definition 17. Let $s \in \mathbb{R}$, we define the Hartley-Bessel-Sobolev space of order s that will be denoted by

$$H^s_\alpha(\mathbb{R}) := \left\{f \in L^2_\alpha(\mathbb{R}) / \left(1+|\lambda|^2\right)^{s/2}\mathscr{H}_\alpha(f) \in L^2_\alpha(\mathbb{R})\right\}.$$

We provide $H^s_\alpha(\mathbb{R}^{d+1}_+)$ with the inner product given by

$$\langle f,g\rangle_{H^s_\alpha} := \int_{\mathbb{R}} \left(1+|\lambda|^2\right)^s \mathscr{H}_\alpha(f)(\lambda)\overline{\mathscr{H}_\alpha(g)(\lambda)}d\mu_\alpha(\lambda), \tag{41}$$

and the norm

$$\|f\|^2_{H^s_\alpha} := \langle f,f\rangle_{H^s_\alpha} = \int_{\mathbb{R}} \left(1+|\lambda|^2\right)^s |\mathscr{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda). \tag{42}$$

Definition 18. Let ψ be a admissible function in $L^2_\alpha(\mathbb{R})$, we introduce the inner product in the Hilbert space $H^s_\alpha(\mathbb{R})$ for any fixed $\beta > 0$ by

$$\langle f,g\rangle_{H^s_{\beta,\psi}} := \beta\langle f,g\rangle_{H^s_\alpha} + \left\langle S^\alpha_\psi(f), S^\alpha_\psi(g)\right\rangle_{\mu_\alpha\otimes\mu_\alpha}. \tag{43}$$

The norm associated to this inner product is defined by

$$\|f\|^2_{H^s_{\beta,\psi}} := \beta\|f\|^2_{H^s_\alpha} + \left\|S^\alpha_\psi(f)\right\|^2_{2,\mu_\alpha\otimes\mu_\alpha}. \tag{44}$$

We have the following result.

Proposition 19. *For $s > \alpha+1$ and ψ be a admissible function in $L^2_\alpha(\mathbb{R})$ and $\beta > 0$ then we have*

$$f \in H^s_{\beta,\psi}(\mathbb{R}) \quad \Rightarrow \quad \mathscr{H}_\alpha(f) \in L^1_\alpha(\mathbb{R}). \tag{45}$$

Proof. By using the relations (12),(31),(41) and (43) we find that

$$\|f\|^2_{H^s_{\beta,\psi}} = \int_{\mathbb{R}} \left[\beta\left(1+|\lambda|^2\right)^s + C_\psi\right] |\mathscr{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda), \tag{46}$$

by using Hölder's inequality and the fact that $s > \alpha+1$ we find that

$$\|\mathscr{H}_\alpha(f)\|_{1,\mu_\alpha} \leq \|f\|_{H^s_{\beta,\psi}} \left(\int_{\mathbb{R}} \frac{d\mu_\alpha(\lambda)}{\left[\beta\left(1+|\lambda|^2\right)^s + C_\psi\right]}\right)^{\frac{1}{2}} < \infty,$$

which gives the result. □

Theorem 20. *Let $s > \alpha+1$ ψ be an admissible function in $L^2_\alpha(\mathbb{R})$ and $\beta > 0$ then the space $\left(H^s_{\beta,\psi}(\mathbb{R}), \langle,\rangle_{H^s_{\beta,\psi}}\right)$ is a reproducing kernel Hilbert space with kernel given by*

$$\mathscr{K}_{\beta,\psi}(x,y) = \int_{\mathbb{R}} \frac{B_\alpha(\lambda x)B_\alpha(\lambda y)}{\beta\left(1+|\lambda|^2\right)^s + C_\psi} d\mu_\alpha(\lambda),$$

that is for every $y \in \mathbb{R}$;

(i) The function $x \to \mathscr{K}_{\beta,u}(x,y) \in H^s_{\beta,\psi}(\mathbb{R})$.

(ii) For every $f \in H^s_{\beta,\psi}(\mathbb{R})$ and $y \in \mathbb{R}$ we have

$$f(y) = \langle f, \mathscr{K}_{\beta,\psi}(\cdot,y)\rangle_{H^s_{\beta,\psi}}.$$

Proof. Let $y \in \mathbb{R}$, by using the fact that $s > \alpha+1$ and the relation (2.4), the function

$$\lambda \to \frac{B_\alpha(\lambda y)}{\beta\left(1+|\lambda|^2\right)^s + C_\psi}$$

belongs to $L^1_\alpha(\mathbb{R}) \cap L^2_\alpha(\mathbb{R})$, by using Plancherel's theorem for the Hartley-Bessel transform, there exist a unique function in $L^2_\alpha(\mathbb{R})$, which we denote by $\mathscr{K}_{\beta,u}(\cdot, y)$ such that

$$\mathscr{H}_\alpha\left(\mathscr{K}_{\beta,\psi}(\cdot,y)\right)(\lambda) = \frac{B_\alpha(\lambda y)}{\beta\left(1+|\lambda|^2\right)^s + C_\psi}, \tag{47}$$

by using inversion formula (11) we find that

$$\mathscr{K}_{\beta,\psi}(x,y) = \int_{\mathbb{R}} \frac{B_\alpha(\lambda x)B_\alpha(\lambda y)}{\beta\left(1+|\lambda|^2\right)^s + C_\psi} d\mu_\alpha(\lambda).$$

Furthermore, by using the relations (10),(45)and (47) we find that

$$\|\mathscr{K}_{\beta,\psi}(\cdot,y)\|^2_{H^s_{\beta,\psi}} \leq \int_{\mathbb{R}} \frac{d\mu_\alpha(\lambda)}{\beta\left(1+|\lambda|^2\right)^s + C_\psi} < \infty,$$

which proves that $\mathscr{K}_{\beta,\psi}(\cdot,y) \in H^s_{\beta,\psi}(\mathbb{R})$, now let $f \in H^s_{\beta,u}(\mathbb{R})$ by using the relations (46) and (50) we find that

$$\langle f, \mathscr{K}_{\beta,\psi}(\cdot,y))\rangle_{H^s_{\beta,\psi}} = \int_{\mathbb{R}} \mathscr{H}_\alpha(f)(\lambda)B_\alpha(\lambda y)\mathrm{d}\mu_\alpha(\lambda).$$

Inversion formula (11) gives the desired result. □

In the following we give the main result of this section.

Theorem 21. *Let $s > \alpha + 1$, ψ be an admissible function in $L^2_\alpha(\mathbb{R})$ and $g \in L^2_\alpha(\mathbb{R}^2)$, $\beta > 0$ then the infimum*

$$\inf_{f \in H^s_\alpha(\mathbb{R})} \left\{ \beta\|f\|^2_{H^s_\alpha} + \left\|g - S^\alpha_\psi(f)\right\|^2_{2,\mu_\alpha\otimes\mu_\alpha} \right\}, \tag{48}$$

is attained by a unique function $f^*_{g,\psi,\beta}$ given by

$$f^*_{g,\psi,\beta}(y) = \iint_{\mathbb{R}^2} g(x,z)\phi_{\psi,\beta}(x,y,z)d\mu_\alpha(x)\otimes d\mu_\alpha(z), \tag{49}$$

where $\phi_{\psi,\beta}$ is given by

$$\phi_{u,\beta}(x,y,z) = \int_{\mathbb{R}} \frac{B_\alpha(\lambda x)B_\alpha(\lambda y)\mathscr{H}_\alpha\left(\mathcal{M}^z(\mathscr{D}_z(\psi))\right)(\lambda)}{\beta\left(1+|\lambda|^2\right)^s + C_\psi} d\mu_\alpha(\lambda). \tag{50}$$

Proof. The existence and unicity of the extremal function $f^*_{g,\psi,\beta}$, satistfying the relation (48) is given in [3] and this function is given by the following relation

$$f^*_{g,\psi,\beta}(y) = \left\langle g, S^\alpha_\psi(\mathscr{K}_{\beta,\psi}(\cdot,y))\right\rangle_{\mu_\alpha\otimes\mu_\alpha}, \tag{51}$$

where $\mathscr{K}_{\beta,\psi}$ is the kernel function given the relation (47), on the other hand, by using the relations (12),(25) and (26) we find that

$$S^\alpha_\psi(\mathscr{K}_{\beta,\psi}(\cdot,y))(x,z) = \int_{\mathbb{R}} \mathscr{H}_\alpha(\mathscr{K}_{\beta,\psi}(\cdot,y))(\lambda)\overline{\mathscr{H}_\alpha(\psi^{x,z})(\lambda)}\mathrm{d}\mu_\alpha(\lambda).$$

Using the relations (25), (47) and (51) we find the result. □

We have the following results.

Theorem 22. *Let $s > \alpha + 1$, ψ be an admissible function in $L^2_\alpha(\mathbb{R})$ and $g \in L^2_\alpha(\mathbb{R}^2)$, $\beta > 0$ then we have*

$$(i) \quad f^*_{g,\psi,\beta}(y) = \iint_{\mathbb{R}^2} \frac{B_\alpha(\lambda y)\mathscr{H}_\alpha\left(g(\cdot,z)\right)(\lambda)\mathscr{H}_\alpha\left(\mathcal{M}^z(\mathscr{D}_z(\psi))\right)(\lambda)}{\beta\left(1+|\lambda|^2\right)^s + C_\psi} d\mu_\alpha(\lambda)\otimes d\mu_\alpha(z). \tag{52}$$

$$(ii) \quad \mathscr{H}_\alpha(f^*_{g,u,\beta})(\lambda) = \int_{\mathbb{R}} \frac{\mathscr{H}_\alpha\left(g(\cdot,z)\right)(\lambda)\mathscr{H}_\alpha\left(\mathcal{M}^z(\mathscr{D}_z(\psi))\right)(\lambda)}{\beta\left(1+|\lambda|^2\right)^s + C_\psi}\, d\mu_\alpha(z). \tag{53}$$

$$(iii) \quad \|f^*_{g,\psi,\beta}\|_{H^s_\alpha} \leq \frac{\|g\|_{2,\mu_\alpha\otimes\mu_\alpha}\,\|\psi\|_{2,\mu_\alpha}}{\beta}. \tag{54}$$

Proof. (i) Is a consequence of (49), (50) and Fubini's theorem.

(ii) Is a consquence of Fubini's theorem and the relation (52).

(iii)By using the relation (42) we find that

$$\|f^*_{g,\psi,\beta}\|^2_{H^s_\alpha} = \int_{\mathbb{R}} \left(1+|\lambda|^2\right)^s \left|\mathscr{H}_\alpha(f^*_{g,\psi,\beta})(\lambda)\right|^2 d\mu_\alpha(\lambda).$$

By using Hölder's inequality, we find that

$$\left|\mathscr{H}_\alpha(f^*_{g,\psi,\beta})(\lambda)\right|^2 \leq \frac{\|\psi\|^2_{2,\alpha}}{(\beta\left(1+|\lambda|^2\right)^s + C_\psi} \int_{\mathbb{R}} |g(\lambda,z)|^2\, d\mu_\alpha(z),$$

so we find that

$$\|f^*_{g,\psi,\beta}\|^2_{H^s_\alpha} \leq \frac{\left(1+|\lambda|^2\right)^s \|\psi\|^2_{2,\alpha}\|g\|^2_{2,\mu_\alpha\otimes\mu_\alpha}}{(\beta\left(1+|\lambda|^2\right)^s + C_\psi)^2} \leq \frac{\|g\|^2_{2,\mu_\alpha\otimes\mu_\alpha}\,\|\psi\|^2_{2,\mu_\alpha}}{\beta^2},$$

which gives the desired result. □

Corollary 23. *Let $s > \alpha+1$, ψ be an admissible function in $L^2_\alpha(\mathbb{R})$ and $\beta > 0$, for all $f \in H^s_\alpha(\mathbb{R})$ and $g = S^\alpha_\psi(f)$, the extremal function $f^*_{S^\alpha_\psi(f),\psi,\beta}$ satisfies the following properties*

$$(i) \quad \mathscr{H}_\alpha(f^*_{S^\alpha_\psi(f),\psi,\beta})(\lambda) = \frac{\mathscr{H}_\alpha(f)(\lambda)C_\psi}{\beta\left(1+|\lambda|^2\right)^s + C_\psi}. \tag{55}$$

$$(ii) \quad \|f^*_{S^\alpha_\psi(f),\psi,\beta}\|_{H^s_\alpha} \leq \frac{\|f\|_{2,\mu_\alpha}\,\|\psi\|_{2,\mu_\alpha}\,\sqrt{C_\psi}}{\beta^2}. \tag{56}$$

Proof. (i) By using the relations (25) and (26) we find that

$$\mathscr{H}_\alpha(S^\alpha_\psi(f)(\cdot,z))(\lambda) = \mathscr{H}_\alpha(f)(\lambda)\overline{\mathscr{H}_\alpha\left(\mathcal{M}^z(\mathscr{D}_z(\psi))\right)(\lambda)}. \tag{57}$$

Using the relations (53) and (57) we find the relation (55).

(ii) Is a consequence of (31) and (54). □

Theorem 24 (Second Calderón Reproducing Formula)**.** *Let $s > \alpha+1$, ψ be an admissible function in $L^2_\alpha(\mathbb{R})$, and $\beta > 0$. For all $f \in H^s_\alpha(\mathbb{R})$, the extremal function $f^*_{S^\alpha_\psi(f),\psi,\beta}$ satisfies:*

$$\lim_{\beta\to 0^+} \left\| f^*_{S^\alpha_\psi(f),\psi,\beta} - f \right\|_{H^s_\alpha} = 0.$$

Moreover, $f^*_{S^\alpha_\psi(f),\psi,\beta}$ converges uniformly to f as $\beta \to 0^+$.

Proof. Using the relation (55), we obtain:

$$\mathscr{H}_\alpha\left(f^*_{S^\alpha_\psi(f),\psi,\beta} - f\right)(\lambda) = \frac{-\beta\left(1+|\lambda|^2\right)^s \mathscr{H}_\alpha(f)(\lambda)}{\beta\left(1+|\lambda|^2\right)^s + C_\psi}. \tag{58}$$

Consequently, the H^s_α-norm of the difference is:

$$\left\| f^*_{S^\alpha_\psi(f),\psi,\beta} - f \right\|^2_{H^s_\alpha} = \int_{\mathbb{R}^d_+} \frac{\beta^2 \left(1+|\lambda|^2\right)^{3s} |\mathscr{H}_\alpha(f)(\lambda)|^2}{\left(\beta\,(1+|\lambda|^2)^s + C_\psi\right)^2} \, d\mu_\alpha(\lambda).$$

By applying the dominated convergence theorem and noting that:

$$\frac{\beta^2 \left(1+|\lambda|^2\right)^{3s} |\mathscr{H}_\alpha(f)(\lambda)|^2}{\left(\beta\,(1+|\lambda|^2)^s + C_\psi\right)^2} \leq \left(1+|\lambda|^2\right)^s |\mathscr{H}_\alpha(f)(\lambda)|^2,$$

we conclude that:

$$\lim_{\beta \to 0^+} \left\| f^*_{S^\alpha_\psi(f),\psi,\beta} - f \right\|_{H^s_\alpha} = 0.$$

Next, using the inversion formula (11) and relation (58), we find:

$$f^*_{S^\alpha_\psi(f),\psi,\beta}(y) - f(y) = \int_{\mathbb{R}} \mathscr{H}_\alpha \left(f^*_{S^\alpha_\psi(f),\psi,\beta} - f \right)(\lambda) B_\alpha(\lambda y) \, d\mu_\alpha(\lambda).$$

Substituting (58), we get:

$$f^*_{S^\alpha_\psi(f),\psi,\beta}(y) - f(y) = \int_{\mathbb{R}} \frac{-\beta \left(1+|\lambda|^2\right)^s \mathscr{H}_\alpha(f)(\lambda) B_\alpha(\lambda y)}{\beta\,(1+|\lambda|^2)^s + C_\psi} \, d\mu_\alpha(\lambda).$$

Applying the dominated convergence theorem again and observing that:

$$\left| \frac{-\beta \left(1+|\lambda|^2\right)^s \mathscr{H}_\alpha(f)(\lambda) B_\alpha(\lambda y)}{\beta\,(1+|\lambda|^2)^s + C_\psi} \right| \leq |\mathscr{H}_\alpha(f)(\lambda)|,$$

we deduce that:

$$\lim_{\beta \to 0^+} \left\| f^*_{S^\alpha_\psi(f),\psi,\beta} - f \right\|_{\infty,\alpha} = 0.$$

This establishes that $f^*_{S^\alpha_\psi(f),\psi,\beta}$ converges uniformly to f as $\beta \to 0^+$, completing the proof. □

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References

[1] Gröchenig, K. (2013). *Foundations of time-frequency analysis*. Springer Science & Business Media.

[2] Wilczok, E. (2000). New uncertainty principles for the continuous Gabor transform and the continuous wavelet transform. *Documenta Mathematica*, *5*, 207–226.

[3] Stockwell, R. G., Mansinha, L., & Lowe, R. P. (1996). Localization of the complex spectrum: The S transform. *IEEE Transactions on Signal Processing*, *44*(4), 998–1001.

[4] Elmonser, H., & Brahim, K. (2023). Local price uncertainty principle and time-frequency localization operators for the q-Hankel–Stockwell transform. *Asian-European Journal of Mathematics*, *16*(03), 2350049.

[5] Donoho, D. L., & Stark, P. B. (1989). Uncertainty principles and signal recovery. *SIAM Journal on Applied Mathematics*, *49*(3), 906–931.

[6] Eramian, M. G., Schincariol, R. A., Mansinha, L., & Stockwell, R. G. (1999). Generation of aquifer heterogeneity maps using two-dimensional spectral texture segmentation techniques. *Mathematical Geology*, *31*, 327–348.

[7] Osler, J. C., & Chapman, D. M. (1996). Seismo-acoustic determination of the shear-wave speed of surficial clay and silt sediments on the Scotian shelf. *Canadian Acoustics*, *24*(4), 11–22.

[8] Gohberg, I., Goldberg, S., & Krupnik, N. (2012). *Traces and determinants of linear operators* (Vol. 116). Birkhäuser.

[9] Ricaud, B., & Torrésani, B. (2014). A survey of uncertainty principles and some signal processing applications. *Advances in Computational Mathematics, 40*(3), 629–650.

[10] Saitoh, S., & Sawano, Y. (2016). *Theory of reproducing kernels and applications*. Springer Singapore.

[11] Assous, S., Humeau, A., Tartas, M., Abraham, P., & L'Huillier, J. P. (2006). S-transform applied to laser Doppler flowmetry reactive hyperemia signals. *IEEE Transactions on Biomedical Engineering, 53*(6), 1032–1037.

[12] Mejjaoli, H. (2021). Dunkl–Stockwell transform and its applications to the time–frequency analysis.*Journal of Pseudo-Differential Operators and Applications, 12*(2), 32.

[13] Reynolds, R. W., & Smith, T. M. (1995). A high-resolution global sea surface temperature climatology. *Journal of Climate, 8*(6), 1571–1583.

[14] Stein, E. M. (1956). Interpolation of linear operators. *Transactions of the American Mathematical Society, 83*(2), 482–492.

[15] Ghobber, S., & Mejjaoli, H. (2024). Generalized Stockwell transforms: Spherical mean operators and applications. *Georgian Mathematical Journal*.

[16] Hamadi, N. B., & Hafirassou, Z. (2020). Local price uncertainty principle and time-frequency localization operators for the Hankel–Stockwell transform. *International Journal of Wavelets, Multiresolution and Information Processing, 18*(06), 2050050.